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Painlevé-Kuratowski Convergence of Solution Sets for Generalized Strong Vector Quasi-Equilibrium Problem with Domination Structure

Thanatporn Bantaojai[†] **and Rabian Wangkeeree**[‡],[1]

[†]Mathematics English Program, Faculty of Education,
Valaya Alongkorn Rajabhat University under the Royal Patronage
e-mail : thanatpornmaths@gmail.com
[‡]Department of Mathematics, Faculty of Science, Naresuan University
e-mail : rabianw@nu.ac.th

Abstract : This paper is concerned with the stability of the solution set for a generalized strong vector quasi-equilibrium problem with domination structure (GSVQEP), that is, we study Painlevé-Kuratowski convergence of the solution sets with a sequence of mappings converging continuously and sequence of set converging in the sense of Painlevé-Kuratowski. Our main results are new and different from the existing ones in the literature. Illustrating examples are provied.

Keywords : generalized strong vector quasi-equilibrium problem; domination structure, Painlevé-Kuratowski convergence, continuous convergence, gap function
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1 Introduction

The vector equilibrium problems contain many problems as special cases, including vector variational inequality problems, vector optimization problems, vector complementarity problems, vector Nash equilibrium problems, etc. Because

[0]Thanks! Valaya Alongkorn Rajabhat University under the Royal Patronage
[1]Corresponding author email: rabianw@nu.ac.th (R. Wangkeeree)

of the general form of this problem, in fact it was investigated earlier under other terminologies (see, [4, 8, 20]). Recently, there has been an increasing interest in the study to stability of vector equilibrium problems see, e.g., [1, 2, 24, 25, 26].

As for the stable results investigated on the convergence of the sequence of mappings, there are some results for the vector optimization, vector variational inequality problems and vector equilibrium problems with a sequence of sets converging in the sense of Painlevé-Kuratowski (see e.g., [9, 10, 13, 16, 21]. In [13], Huang discussed the convergence of the approximate efficient sets to the efficient sets of vector-valued and set-valued optimization problems in the sense of Painlevé-Kuratowski and Mosco. In [10], Fang et al. investigated the Painlevé-Kuratowski convergence of the solution sets of the perturbed set-valued weak vector variational inequality problems. In [16], Lalitha and Chatterjee investigated the Painlevé-Kuratowski set convergence of the solution sets of a nonconvex vector optimization problem. In [21], Peng and Yang investigated the Painlevé-Kuratowski set convergence of the solution sets of the perturbed vector equilibrium problems without monotonicity in real linear metric spaces. Very recently, Li et al. [17] concerned with the stability for a generalized Ky Fan inequality when it is perturbed by vector-valued bifunction sequence and set sequence. By continuous convergence of the bifunction sequence and Painlevé-Kuratowski convergence of the set sequence, they established the Painlevé-Kuratowski convergence of the approximate solution mappings of a family of perturbed problems to the corresponding solution mapping of the original problem.

On the other hand, J.Y. Fu et al. in [11] presented a generalized strong vector quasi-equilibrium problem with set-valued mapping and domination structure (GSVQEP). It is better than other solutions, such as weak efficient solution, efficient solution, and proper efficient solution. Therefore, it is interesting to discuss the (GSVQEP) and the properties of its solution set.

Motivated by the work reported in above, this paper aims to establish some results for the solution set of a generalized strong vector quasi-equilibrium problem with set-valued mapping and domination structure. We first discuss the Painlevé-Kuratowski upper convergence of the solution sets. We consider a generalized nonlinear scalarization function, which will be used to construct a gap function for such problem on set-valued mapping concerning with set-valued mapping and domination structure and we also introduce a key assumption (H_g). Thus, the Painlevé-Kuratowski lower convergence of the solution set is established under the main assumption (H_g).

The rest of the paper is organized as follows. In section 2, we introduce the generalized strong vector quasi-equilibrium problem with domination structure (for short, GSVQEP) and $(\text{GSVQEP})_n$, recall some definitions and important properties. In section 3, we establish Painlevé-Kuratowski convergence of the solution sets, and provide some examples to illustrate that our main results are new and different from the existing ones in literature.

2 Preliminaries

Throughout this paper, unless specified otherwise, we always suppose that X, Y and Z are metric linear spaces. Let $A \subset X$ be a nonempty, compact, closed convex subset, $K : A \to 2^A$ be a set-valued mapping, $E \subset Y$ be a nonempty compact subset, and set-valued mapping $P : A \to 2^Z$ be a closed mapping such that for all $x \in A, P(x) \subset Z$ be a proper, closed convex cone with the apex at the origin 0 of Z. The family $\{P(x) : x \in A\}$ is called a domination structure on Z. Let set-valued mappings $f : E \times A \times A \to 2^Z$ and closed mapping $T : A \to 2^E$.

We consider the following generalized strong vector quasi-equilibrium problem with set-valued mapping and domination structure:

$$\text{(GSVQEP)} \begin{cases} \text{Find } x \in K(x) \text{ such that for each } y \in K(x), \text{ there is some } t \in T(x) \\ \text{such that } f(t, x, y) \subset P(x). \end{cases}$$

For each $n \in \mathbb{N}$, let $f_n : E \times A \times A \to 2^Z$, $K_n : A \to 2^A$ and $T_n : A \to 2^E$ be three set-valued mappings. We consider the following sequence of the generalized strong vector quasi-equilibrium problems :

$$\text{(GSVQEP)}_n \begin{cases} \text{Find } x_n \in K_n(x_n) \text{ such that for each } y \in K_n(x_n), \text{ there is some} \\ t_n \in T_n(x_n) \text{ such that } f_n(t_n, x_n, y) \subset P(x_n). \end{cases}$$

We denote the solution sets of problems (GSVQEP) by $S(T, K, f)$ and (GSVQEP)_n by $S(T_n, K_n, f_n)$. We mainly analyze the behavior of $S(T, K, f)$ and $S(T_n, K_n, f_n)$. Thus, we always assume that $S(T, K, f)$ and $S(T_n, K_n, f_n)$ are not equal empty sets.

Remark 2.1. If $K(x) \equiv K$ where K be a nonempty, closed convex subset of X, then the problem (GSVQEP) reduces to the generalized strong vector quasiequilibrium problem with set-valued mapping and domination structure (for short, GSVQEP) studied in [11].

Definition 2.1. Let X and Y be two metric spaces and $G : X \to 2^Y$ be a set-valued mapping.

(i) G is said to be *lower semicontinuous* at $x_0 \in X$, if $G(x_0) \cap U \neq \emptyset$ for some open set $U \subset Y$ implies the existence of a neighborhood N of x_0 such that $G(x) \cap U \neq \emptyset, \forall x \in N$. G is said to be lower semicontinuous in X if it is lower semicontinuous at each $x_0 \in X$.

(ii) G is said to be *upper semicontinuous* at $x_0 \in X$, if for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq G(x), \forall x \in N$. G is said to be upper semicontinuous in X if it is upper semicontinuous at each $x_0 \in X$.

(iii) G is said to be *continuous* at $x_0 \in X$, if it is both lower semicontinuous and upper semicontinuous at x_0. G is said to be continuous in X if it is both lower semicontinuous and upper semicontinuous at each $x_0 \in X$.

(vi) G is said to be *closed* at x_0, if for each sequence $\{(x_n, y_n)\} \subset \text{graph}G := \{(x, y)|y \in G(x)\}, (x_n, y_n) \to (x_0, y_0)$, it follows that $(x_0, y_0) \in \text{graph}G$. G is said to be closed in X if it is closed at each $x_0 \in X$.

Lemma 2.2. *([3]) Let X and Y be two metric spaces and $G : X \to 2^Y$ be a set-valued mapping. If G has compact values, then G is upper semicontinuous at x_0 if and only if, for each sequence $\{x_n\} \subset X$ which converges to x_0 and for each sequence $\{y_n\} \subset G(x_n)$, there are $y \in G(x)$ and a subsequence $\{y_m\}$ of $\{y_n\}$ such that $y_m \to y$.*

Lemma 2.3. *([3]) Let X and Y be topological spaces. If a set-valued mapping $T : X \to 2^Y$ is upper semicontinuous with compact values, then for every compact set $K \subset X$, the set $T(K) = \cup_{x\in K}T(x)$ is compact.*

Definition 2.2. *([22])* Let $\{C_n\}$ be a sequence of sets of $\mathbb{R}^m$ and C be a subset of $\mathbb{R}^m$.

(i) $\limsup_{n\to\infty} C_n := \{x \in \mathbb{R}^m|\exists x_{n_k} \in C_{n_k}, x_{n_k} \to x\}$ is its outer limit;

(ii) $\liminf_{n\to\infty} C_n := \{x \in \mathbb{R}^m|\exists x_n \in C_n, x_n \to x\}$ is its inner limit;

(iii) $\{C_n\}$ is said to be Painlevé-Kuratowski convergent to C, denoted by $C_n \xrightarrow{P.K.} C$, if and only if $\limsup_{n\to\infty} C_n \subset C \subset \liminf_{n\to\infty} C_n$.

The relations $\limsup_{n\to\infty} C_n \subset C$ and $C \subset \liminf_{n\to\infty} C_n$ are, respectively, referred as the upper part and the lower part of the convergence. Clearly,

$$\liminf_{n\to\infty} C_n \subset \limsup_{n\to\infty} C_n.$$

Definition 2.3. *([22])* Let $S : X \to 2^Y$ be a set-valued mapping.

(i) S is said to be *outer semicontinuous (osc)* at $\bar{x}$ if $\limsup_{x\to\bar{x}} S(x) \subset S(\bar{x})$ with
$\limsup_{x\to\bar{x}} S(x) := \cup_{x_n\to\bar{x}} \limsup_{n\to\infty} S(x_n)$.

(ii) S is said to be *inner semicontinuous (isc)* at $\bar{x}$ if $S(\bar{x}) \subset \liminf_{x\to\bar{x}} S(x)$ with
$\liminf_{x\to\bar{x}} S(x) := \cap_{x_n\to\bar{x}} \liminf_{n\to\infty} S(x_n)$.

(iii) S is said to be *continuous* at $\bar{x}$, written as $S(x) \to S(\bar{x})$ as $x \to \bar{x}$ if it is both outer semicontinuous and inner semicontinuous.

Definition 2.4. *([22])* Let $f_n : X \to 2^Y$ be a sequence of set-valued mappings and
$f : X \to 2^Y$ be a set-valued mapping.

(i) $\{f_n\}$ is said to be *outer converges continuously* to f at x_0 if

$$\limsup_{n\to\infty} f_n(x_n) \subset f(x_0), \forall x_n \to x_0;$$

(ii) $\{f_n\}$ is said to be *inner converges continuously* to f at x_0 if

$$f(x_0) \subset \liminf_{n\to\infty} f_n(x_n)), \forall x_n \to x_0;$$

(iii) $\{f_n\}$ is said to be *converges continuously* to f at x_0 if

$$\limsup_{n\to\infty} f_n(x_n) \subset f(x_0) \subset \liminf_{n\to\infty} f_n(x_n),\ \forall x_n \to x_0.$$

(iv) $\{f_n\}$ is said to be *converges continuously* to f on X if $\{f_n\}$ converges continuously to f at every $x_0 \in X$.

Next, we recall the concept of the nonlinear scalarization function which can be found in [6, 7, 28].

Definition 2.5. [6, 7, 28] Let $e : X \to Y$ be a vector-valued mapping and for any $x \in X$, $e(x) \in \text{int}C(x)$. The nonlinear scalarization function $\xi_e : X \times Y \to \mathbb{R}$ defined by

$$\xi_e(x, y) = \inf\{r \in \mathbb{R} : y \in re(x) - C(x)\}.$$

In the special case where $Y = \mathbb{R}^p$ and for any $x \in X, P(x) \equiv \mathbb{R}^p_+$ and $e(x) = e$, let $e = (1, 1, 1, \ldots, 1)^T \in int\mathbb{R}^p_+$, the nonlinear scalarization function can be expressed in the following equivalent form [7]:

$$\xi_e(y) = \max_{1\le i\le p}\{y_i\},\ \forall y := (y_1, y_2, \ldots, y_p)^T \in \mathbb{R}^p. \tag{2.1}$$

Proposition 2.6. *For each $r \in \mathbb{R}$ and $x, y \in X$, the following statements are satisfied.*

(i) $\xi_e(x, y) < r \Leftrightarrow y \in re(x) - intC(x)$;

(ii) $\xi_e(x, y) \le r \Leftrightarrow y \in re(x) - C(x)$;

(iii) $\xi_e(x, y) \ge r \Leftrightarrow y \notin re(x) - intC(x)$;

(iv) $\xi_e(x, y) > r \Leftrightarrow y \notin re(x) - C(x)$;

(v) $\xi_e(x, y) = r \Leftrightarrow y \in re(x) - \partial C(x)$, *where $\partial C(x)$ is the topological boundary of $C(x)$.*

Lemma 2.4. *([6]) Let X and Y be two locally convex Hausdorff topological vector spaces, and let $C : X \to 2^Y$ be a set-valued mapping such that, for each $x \in X, C(x)$ is a proper, closed, convex cone in Y with $intC(x) \ne \emptyset$. Furthermore, let $e : X \to Y$ be the continuous selection of the set-valued map $intC(\cdot)$. Define a set-valued mapping $V : X \to 2^Y$ by $V(x) = Y\backslash intC(x)$ for $x \in X$. We have*

(i) *If $V(\cdot)$ is usc in X, then $\xi_e(\cdot, \cdot)$ is upper semicontinuous in $X \times Y$;*

(ii) *If $C(\cdot)$ is usc in X, then $\xi_e(\cdot, \cdot)$ is lower semicontinuous in $X \times Y$.*

Remark 2.5. *From Lemma 2.4, we know that if $V(\cdot)$ and $C(\cdot)$ are both usc in X, then $\xi_e(\cdot,\cdot)$ is continuous in $X \times Y$.*

A subset $B \subset X$ is said to be balanced if $\alpha B \subset B$ for every $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$. The following result is a well-known fact.

Lemma 2.6. *For each neighborhood U of 0_X, there exists a balanced open neighborhood V of 0_X such that $V + V + V \subset U$.*

3 Main Results

In this section, we investigate the upper Painlevé-Kuratowski convergence of the sequence sets $S(T_n, K_n, f_n)$. In this section, our focus is on the Painlevé-Kuratowski upper convergence of the solution sets for $(\text{GSVQEP})_n$.

Theorem 3.1. *For $(\text{GSVQEP})_n$, assume the following conditions are satisfied:*

1. *$\{K_n\}$ converges continuously to K;*
2. *$\{T_n\}$ outer converges continuously to T;*
3. *$\{f_n\}$ converges continuously to f;*
4. *P is closed.*

Then, $\limsup_{n\to\infty} S(T_n, K_n, f_n) \subset S(T, K, f)$.

Proof. Let $x_0 \in \limsup_{n\to\infty} S(T_n, K_n, f_n)$ be any given. So, there exists a subsequence $\{x_{n_k}\}$ in $S(T_{n_k}, K_{n_k}, f_{n_k})$ converging to x_0. Then, for each $y \in K_{n_k}(x_{n_k})$, there exists $t_{n_k} \in T_{n_k}(x_{n_k})$ such that

$$f_{n_k}(t_{n_k}, x_{n_k}, y) \subset P(x_{n_k}), \text{ for all } k \in \mathbb{N}. \tag{3.1}$$

Since $x_{n_k} \in K_{n_k}(x_{n_k})$ and $x_{n_k} \to x_0$, one implies that $x_0 \in \limsup_{n\to\infty} K_n(x_n)$. As $\{K_n\}$ outer converges continuously to K, we have $\limsup_{n\to\infty} K_n(x_n) \subset K(x_0)$. Hence, $x_0 \in K(x_0)$. Further, since $T(x_{n_k}) \subset T(A)$ and $T(A)$ is compact, thus $\{t_{n_k}\}$ has a convergent subsequence. Without loss of generality, we can assume that $t_{n_k} \to t_0 \in T(A)$. Then,

$$t_0 \in \limsup_{n\to\infty} T_n(x_n).$$

By the virtue of (ii), we have $t_0 \in T(x_0)$. Next, we show that, for all $y \in K(x_0)$,

$$f(t_0, x_0, y) \subset P(x_0).$$

As $\{K_n\}$ inner converges continuously to K and $x_{n_k} \to x_0$, for any $y \in K(x_0)$, there exists a sequence $y_{n_k} \in K_{n_k}(x_{n_k})$ such that $y_{n_k} \to y$ as $k \to \infty$. Applying (3.1), we have

$$f_{n_k}(t_{n_k}, x_{n_k}, y_{n_k}) \subset P(x_{n_k}), \forall k \in \mathbb{N}. \tag{3.2}$$

Since $\{f_{n_k}\}$ converges continuously to f at (t_0, x_0, y) and $(t_{n_k}, x_{n_k}, y_{n_k}) \to (t_0, x_0, y)$, we can conclude that

$$\limsup_{k\to\infty} f_{n_k}(t_{n_k}, x_{n_k}, y_{n_k}) \subset f(t_0, x_0, y) \subset \liminf_{k\to\infty} f_{n_k}(t_{n_k}, x_{n_k}, y_{n_k}).$$

Thus, for any fixed $w_0 \in f(t_0, x_0, y)$, we have $w_0 \in \liminf_{k\to\infty} f_{n_k}(t_{n_k}, x_{n_k}, y_{n_k})$, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \in f_{n_k}(t_{n_k}, x_{n_k}, y_{n_k})$, and $w_{n_k} \to w_0$ as $k \to \infty$. By (3.2), we get $w_{n_k} \in P(x_{n_k})$. By the virtue of the closedness of P, we have $(x_0, w_0) \in$ graph P, that is $w_0 \in P(x_0)$. Thus, we actually conclude that

$$f(t_0, x_0, y) \subset P(x_0),$$

which gives that $x_0 \in S(T, K, f)$. This complete the proof. □

The following examples show that none of the assumptions in Theorem (3.1) can be dropped.

Example 3.1. (Assumption (i) can not be dropped)
Let $X = Y = Z = \mathbb{R}, P(x) = [0, +\infty), K(x) = [-1, 1], K_n(x) = [-\frac{1}{n}, \frac{1}{n}], T(x), T_n(x) = [-1, 1]$. We define a mappings $f, f_n : E \times K \times K \to \mathbb{R}$ by

$$f(t, x, y) = y - x \text{ and } f_n(t, x, y) = (1 + \frac{1}{n})(y - x).$$

Thus, assumptions (ii)-(iv) holds. It follows from a direct computation $S(T, K, f) = \{-1\}$ and $S(T_n, K_n, f_n) = \{-\frac{1}{n}\}$. However, the result of Theorem 3.1 does not hold. In fact,

$$\{0\} = \limsup_{n\to\infty} S(T_n, K_n, f_n) \not\subseteq S(T, K, f) = \{-1\}.$$

The reason is that, $\{K_n\}$ does not converge in the sense of Painlevé-Kuratowski to K.

Example 3.2. (Assumption (ii) can not be dropped)
Let $X = Y = Z = \mathbb{R}, P(x) = [0, +\infty)$. Let $K(x), K_n(x) = [-1, 1]$, $T(x) = [0, 1], T_n(x) = [-1 - \frac{1}{n}, 1]$. Define a mappings $f, f_n : E \times K \times K \to \mathbb{R}$ by

$$f(t, x, y) = t + y - x \text{ and } f_n(t, x, y) = (t + \frac{1}{n}) + (y - x).$$

Thus, assumptions (i),(iii), and (iv) holds. It follows from a direct computation $S(T, K, f) = \{-1\}$ and $S(T_n, K_n, f_n) = \{-1, 1\}$. However, the result of Theorem 3.1 does not hold. In fact,

$$\{-1, 1\} = \limsup_{n\to\infty} S(T_n, K_n, f_n) \not\subseteq S(T, K, f) = \{-1\}.$$

The reason is that, $\{T_n\}$ does not converge continuously to T.

Example 3.3. (Assumption (iii) can not be dropped)
Let $X = Y = Z, P(x) = \mathbb{R}^2_+$. Let $K(x), K_n(x) = [0,1], T(x), T_n(x) = [1, 1+x^2]$. Define a mappings $f, f_n : E \times K \times K \to \mathbb{R}^2$ by

$$f(t,x,y) = (0, x-y) \text{ and } f_n(t,x,y) = \Big(\frac{1}{n}(y-x), 0\Big).$$

Thus, assumptions (i),(ii), and (iv) holds. It follows from a direct computation $S(T,K,f) = \{1\}$ and $S(T_n, K_n, f_n) = \{0\}$. However, the result of Theorem 3.1 does not hold. In fact,

$$\{0\} = \limsup_{n\to\infty} S(T_n, K_n, f_n) \nsubseteq S(T,K,f) = \{1\}.$$

The reason is that, $\{f_n\}$ does not converge continuously to f.

Theorem 3.2. *For* (GSVQEP)*, assume the following conditions are satisfied:*

(i) K *is continuous on* A*;*

(ii) f *is isc on* $E \times A \times A$*;*

(iii) T *is osc on* A*;*

(iv) A *and* $K(A)$ *are compact sets;*

(v) P *is closed.*

Then, $S(T,K,f)$ *is compact.*

Proof. First, we prove that $S(T,K,f)$ is a closed set. Take any sequence $\{x_n\} \in S(T,K,f)$ with $x_n \to x_0$. Then, for any $y \in K(x_n)$, there exists a sequence $t_n \in T(x_n)$ such that

$$f(t_n, x_n, y) \subset P(x_n). \tag{3.3}$$

As $x_n \in K(x_n)$ and $x_n \to x_0$, one has $x_0 \in \liminf_{n\to\infty} K(x_n)$. Hence, $x_0 \in \limsup_{n\to\infty} K(x_n)$. By the outer semicontinuity of K at x_0, we have

$$\limsup_{n\to\infty} K(x_n) \subset K(x_0).$$

Hence, $x_0 \in K(x_0)$. Further, since $t_n \in T(x_n) \subset T(A)$ and $T(A)$ is compact, then $\{t_n\}$ has a convergent subsequence which converges in $T(A)$. Without loss of generality, we may assume that

$$t_n \to t_0 \in T(A).$$

Then, $t_0 \in \limsup_{n\to\infty} T(x_n)$. By (iii), we have $t_0 \in T(x_0)$. Finally, we shall prove that, for any $y \in K(x_0)$,

$$f(t_0, x_0, y) \subset P(x).$$

For any $y \in K(x_0)$, since K is isc at x_0, one has $y \in \liminf_{n\to\infty} K(x_n)$. Then, there is a sequence $\{y_n\} \subset K(x_n)$ such that $y_n \to y$. Let $z_0 \in f(t_0, x_0, y)$ be arbitrary. As f is isc on $E \times A \times A$ and $(t_n, x_n, y_n) \to (t_0, x_0, y)$, one has

$$f(t_0, x_0, y) \subset \liminf_{n\to\infty} f(t_n, x_n, y_n).$$

Hence, $z_0 \in \liminf_{n\to\infty} f(t_n, x_n, y_n)$. Then, there exists a sequence $\{z_n\} \subset f(t_n, x_n, y_n)$ such that

$$z_n \to z_0 \text{ as } n \to \infty.$$

Applying (3.3), we have $\{z_n\} \subset P(x_n)$. By the closedness of P, $(x_0, z_0) \in \text{graph}P$, that is, $z_0 \in P(x_0)$. Thus, we get $x_0 \in S(T, K, f)$. So, $S(T, K, f)$ is a closed set. As $S(T, K, f) \subset K(A)$ and $K(A)$ be a compact set, we obtain that $S(T, K, f)$ is a compact set. □

Similarly, we have the following result.

Theorem 3.3. *For any n, suppose that*

(i) $\{K_n\}$ *is continuous on* A*;*

(ii) $\{f_n\}$ *is isc on* $E \times A \times A$*;*

(iii) $\{T_n\}$ *is osc on* A*;*

(iv) A *and* $K_n(A)$ *are compact sets;*

(v) P *is closed.*

Then, $S(T_n, K_n, f_n)$ *is a compact set.*

4 Painlevé-Kuratowski lower convergence of the solution sets.

In this section, we mainly discuss the Painlevé-Kuratowski lower convergence of the sequence sets $S(T_n, K_n, f_n)$. First of all, we give the concept of introduce a sequence of gap functions based on the nonlinear scalarization for (GSVQEP) and establish a key assumption (H_g) imposed on the sequence of gap functions. Set

$$\tilde{K} := \{x \in A | x \in K(x)\} \text{ and } \tilde{K_n} := \{x \in A | x \in K_n(x)\}.$$

For the set-valued mapping with compact values $f : E \times A \times A \to 2^Z$ and $f_n : E \times A \times A \to 2^Z$, we introduce respectively two mappings $\varphi : E \times \tilde{K} \times \tilde{K} \to \mathbb{R}$ and $\varphi_n : E \times \tilde{K_n} \times \tilde{K_n} \to \mathbb{R}$ as the following :

$$\varphi(t, x, y) = \max_{v \in -f(t,x,y)} \xi_e(x, v);$$

and

$$\varphi_n(t_n, x_n, y) = \max_{v_n \in -f_n(t_n,x_n,y)} \xi_e(x_n, v_n),$$

where $\xi_e : X \times Z \to \mathbb{R}$ is the nonlinear scalarization function.

Lemma 4.1. *1. For (GSVQEP), suppose that for every fixed $x \in \tilde{K}$, $-f(\cdot, x, \cdot)$ is lower semicontinuous with compact values on $E \times A$ and K has closed values on $\tilde{K}$. Then, for any fixed $x \in \tilde{K}$,* $\max_{y \in K(x)} \varphi(\cdot, x, y)$ *is lower semicontinuous on E.*

2. For (GSVQEP)$_n, n \in \mathbb{N}$, suppose that, for every fixed $x \in \tilde{K_n}$, $-f_n(\cdot, x, \cdot)$ is lower semicontinuous with compact values on $E \times A$ and K_n has closed values on $\tilde{K_n}$. Then, for any fixed $x \in \tilde{K_n}$, $\max_{y \in K_n(x)} \varphi_n(\cdot, x, y)$ *is lower semicontinuous on E.*

Proof. (i) Since, for every fixed $x \in \tilde{K}$, $-f(\cdot, x, \cdot)$ is a lower semicontinuous set-valued mapping on $E \times \tilde{K}$, it follows from [3] that $\varphi(\cdot, x, \cdot)$ is lower semicontinuous on $E \times \tilde{K}$, for any fixed $x \in \tilde{K}$. Thus, we have that $\max_{y \in K(x)} \varphi(\cdot, x, y)$ is lower semicontinuous on E.

Similarly, we can show that (ii) holds. □

For (GSVQEP) and (GSVQEP)$_n, n \in \mathbb{N}$, suppose that $K(x), T(x), K_n(x)$ and $T_n(x)$ are compact for all $x \in A$. Then, from Lemma 4.1, we define the following two real-valued functions $g : \tilde{K} \to \mathbb{R}$ and $g_n : \tilde{K_n} \to \mathbb{R}$ by

$$g(x) = -\min_{t \in T(x)} \max_{y \in K(x)} \varphi(t, x, y), \ \forall x \in \tilde{K}, \tag{4.1}$$

and

$$g_n(x) = -\min_{t \in T_n(x)} \max_{y \in K_n(x)} \varphi_n(t, x, y), \ \forall x \in \tilde{K_n}. \tag{4.2}$$

Lemma 4.2. *1. Suppose that, for every fixed $x \in \tilde{K}$, $-f(\cdot, \cdot, x)$ is a lower semi-continuous set-valued mapping with compact values on $E \times \tilde{K}$, K and T have closed values on $\tilde{K}$. If $-f(t, x, x) \cap -\partial P(x) \neq \emptyset$, $\forall x \in \tilde{K}$ and $t \in T(x)$, then g defined by* (4.1) *is a gap function for* (GSVQEP).

2. Suppose that, for every fixed $x \in \tilde{K_n}$, $-f_n(\cdot, \cdot, x)$ is a lower semi-continuous set-valued mapping with compact values on $E \times \tilde{K_n}$, K_n and T_n have closed values on $\tilde{K_n}$. If $-f_n(t, x, x) \cap -\partial P(x) \neq \emptyset$, $\forall x \in \tilde{K_n}$ and $t \in T_n(x)$, then g_n defined by (4.1) *is a gap function for* (GSVQEP)$_n$.

Proof. (i) (a) We first show that, for all $x \in \tilde{K}$, $g(x) \leq 0$. Since $-f(t, x, x) \cap -\partial P(x) \neq \emptyset$, $\forall x \in \tilde{K}$ and $t \in T(x)$, there exists $w_{xt} \in -f(t, x, y)$ such that $w_{xt} \in -\partial P(x)$. It follows from Proposition 2.6 (v) that

$$\xi_e(x, w_{xt}) = 0.$$

Then, for all $x \in \tilde{K}$ and $t \in T(x)$,

$$\max \xi_e(x, -f(t, x, x)) \geq 0.$$

Naturally, for all $x \in \tilde{K}$,

$$g(x) = -\min_{t \in T(x)} \max_{y \in K(x)} \max \xi_e(x, -f(t, x, y)) \leq 0. \tag{4.3}$$

(b) Next, we want to prove that $g(\bar{x}) = 0$ if and only if $\bar{x}$ is a solution of (GSVQEP). Firstly, we suppose that there is $\bar{x} \in \tilde{K}$ such that $g(\bar{x}) = 0$. Applying Lemma 4.1, there exists a point $\bar{t} \in T(\bar{x})$ such that

$$g(\bar{x}) = -\max_{y \in K(\bar{x})} \max \xi_e(x, -f(\bar{t}, \bar{x}, y)) = 0.$$

which gives that,

$$\max \xi_e(x, -f(\bar{t}, \bar{x}, y)) \leq 0, \, \forall y \in K(\bar{x}).$$

By the virtue of Proposition 2.6 (ii), we have

$$-f(\bar{t}, \bar{x}, y) \subset -P(\bar{x}), \text{ for all } y \in K(\bar{x});$$

that is, for all $y \in K(\bar{x})$,

$$f(\bar{t}, \bar{x}, y) \subset P(\bar{x}).$$

Then, we can conclude that $\bar{x}$ is a solution of (GSVQEP).

Conversely, suppose that $\bar{x}$ is a solution of (GSVQEP). Then, there is $\bar{t} \in T(\bar{x})$ such that

$$f(\bar{t}, \bar{x}, y) \subset P(\bar{x}), \, \forall y \in K(\bar{x}),$$

i.e.,

$$-f(\bar{t}, \bar{x}, y) \subset -P(\bar{x}), \, \forall y \in K(\bar{x}).$$

Using the Proposition 2.6 (ii), we get that

$$\xi_e(x, v) \leq 0, \, \forall v \in -f(\bar{t}, \bar{x}, y) \text{ and } y \in K(\bar{x}).$$

It follows that

$$g(\bar{x}) \geq 0. \tag{4.4}$$

From (4.3) and (4.4), we get $g(\bar{x}) = 0$. Hence, the mapping g be a gap function for (GSVQEP).

Similarly, we can show that (ii) holds. □

Proposition 4.3. *Assume for the problem (GSVQEP) that*

1. *For every fixed $x \in \tilde{K}$, $-f(\cdot, \cdot, x)$ is a lower semi-continuous set-valued mapping with compact values on $E \times \tilde{K}$;*
2. *K is continuous with compact values in A;*
3. *T is continuous with compact values in A;*
4. *P is upper semicontinuous in A.*

Then, g is continuous in $\tilde{K}$.

Proof. (a) First, for any $r \in \mathbb{R}$, we prove that the level set $L := \{x \in \tilde{K} : g(x) \leq r\}$ is closed. To this end, suppose that $\{x_n\} \subset L$ satisfying $x_n \to x_0$ as $n \to \infty$. It follows that, for each $n \in \mathbb{N}$,

$$g(x_n) = -\min_{t \in T(x_n)} \max_{y \in K(x_n)} \varphi(t, x_n, y) \leq r,$$

which gives that

$$\max_{t \in T(x_n)} (-\max_{y \in K(x_n)} \varphi(t, x_n, y)) \leq r,$$

For any $t_0 \in T(x_0)$, the lower semicontimuity of T implies that there exists a sequence $\{t_n\}$ with $t_n \in T(x_n)$ such that $t_n \to t_0$ and so, we have

$$(-\max_{y \in K(x_n)} \varphi(t_n, x_n, y)) \leq r,$$

By the compactness of $K(x_n)$, there exists a sequence $\{y_n\}$ with $y_n \in K(x_n)$ such that

$$-\varphi(t_n, x_n, y_n)) \leq r, \text{ for all } n \in \mathbb{N}. \tag{4.5}$$

Since K is upper semicontinuous with compact values, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$ for some $y_0 \in K(x_0)$. From (4.5), it is clear that

$$-\varphi(t_{n_k}, x_{n_k}, y_{n_k})) \leq r, \text{ for all } k \in \mathbb{N}. \tag{4.6}$$

On taking the limit (as $k \to \infty$) in the last inequality, we obtain

$$-\varphi(t_0, x_0, y_0) \leq r. \tag{4.7}$$

It follows from (4.7) that

$$-\max_{y \in K(x_0)} \{\varphi(t_0, x_0, y)\} \leq r.$$

Since $t_0 \in T(x_0)$ is arbitrary, it follows from the last inequality that

$$g(x_0) = -\min_{t_0 \in T(x_0)} \max_{y \in K(x_0)} \varphi(t_0, x_0, y) \leq r.$$

This proves that, for $r \in \mathbb{R}$, the level set $\{x \in \tilde{K} : g(x) \leq r\}$ is closed. Hence, g is lower semicontinuous in $\tilde{K}$.

(b) Using the same argument as in the proof of (a), we can prove that for $r \in \mathbb{R}$, the level set $\{x \in \tilde{K} : g(x) \geq r\}$ is closed. Hence, g is upper semicontinuous in $\tilde{K}$.

□

Similarly, we have the following result.

Proposition 4.4. *Assume for the problem (GSVQEP)$_n$ that, for each $n \in \mathbb{N}$,*

1. *For every fixed $x \in \tilde{K}$, $-f_n(\cdot, \cdot, x)$ is a lower semi-continuous set-valued mapping with compact values on $E \times \tilde{K}_n$;*

2. K_n is continuous with compact values in A;

3. T_n is continuous with compact values in A;

4. P is upper semicontinuous in A.

Then, for each $n \in \mathbb{N}$, g_n is continuous in $\tilde{K_n}$.

Lemma 4.5. *Assume for the problem (GSVQEP)$_n$ that, for each $n \in \mathbb{N}$,*

1. *For every fixed $x \in \tilde{K}$, $-f_n(\cdot,\cdot,x)$ is a lower semi-continuous set-valued mapping with compact values on $E \times \tilde{K_n}$;*
2. *K_n is continuous with compact values in A;*
3. *T_n is continuous with compact values in A;*
4. *P is upper semicontinuous in A.*

Then, for any $\delta > 0$, $x_0 \in \tilde{K}$ and a sequence $\{x_n\} \subset \tilde{K_n}$ satisfying $x_n \to x_0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{N} \in \mathbb{N}$ such that

$$g_{n_j}(x_{n_j}) \geq g(x_0) - \delta, \text{ for all } j \geq \bar{N}.$$

Next, we establish that the condition (H_g) is a sufficient and necessary condition for the lower Painleavé-Kuratowski convergence of the solution sets for generalized strong vector quasi-equilibrium problem with domination structure (GSVQEP). In view of hypothesis (H_g) of [5, 10, 12, 15, 27], we introduce the following key assumption:

(H_g) : For any open neighborhood U of the origin in X, there exists $\alpha > 0$ and an $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$ with $x_n \in \tilde{K_n} \backslash (S(T_n, K_n, f_n) + U)$, one has $g_n(x_n) \leq -\alpha$.

Remark 4.6. *Geometrically, the hypothesis (H_g) means that, given a small open neighborhood U of 0_X, we can find a small positive number α and a large enough positive number n_0 such that for all $n \geq n_0$, if a feasible point x_n is not in the set $S(T_n, K_n, f_n) + U$, then a "gap" by an amount of at least $-\alpha$ will be yielded.*

We now need the following result to establish the Painlevé-Kuratowski lower convergence of the sequence sets $S(T_n, K_n, f_n)$.

Lemma 4.7. *Suppose that all the conditions in Theorem 3.2 are satisfied. Then, $S(T, K, f) \subset \liminf_{n\to\infty} S(T_n, K_n, f_n)$ if and only if for each open neighborhood U of the origin in X, there is $N > 0$ such that*

$$S(T, K, f) \subset S(T_n, K_n, f_n) + U,$$

for all $n \geq N$.

Proof. Suppose that $S(T,K,f) \subset \liminf_{n\to\infty} S(T_n,K_n,f_n)$ and there is an open neighborhood U of the origin in X such that, for each $n > 0$, there exists $N_n \geq n$ such that

$$S(T,K,f) \not\subseteq S(T_{N_n},K_{N_n},f_{N_n}) + U.$$

Without loss of generality, we assume that $\{N_n\}$ is strictly increasing. Hence, there exists a sequence $\{x_n\}$ such that, for each $n \in \mathbb{N}$,

$$x_n \in S(T,K,f)\backslash(S(T_{N_n},K_{N_n},f_{N_n}) + U).$$

Hence, for each $x \in S(T_{N_n},K_{N_n},f_{N_n})$, we have

$$x_n - x \notin U. \tag{4.8}$$

Since $S(T,K,f)$ is compact, without loss of generality, we can assume $x_n \to x_0 \in S(T,K,f)$. Since $S(T,K,f) \subset \liminf_{n\to\infty} S(T_n,K_n,f_n)$, there is a sequence $z_n \in S(T_n,K_n,f_n)$ such that $z_n \to x_0$ as $n \to \infty$. For each $n \in \mathbb{N}$, denote $w_n := x_n - z_n$. Applying (4.8), we see that $w_{N_n} := x_{N_n} - z_{N_n} \in U^c$, the complement of U. Since U^c is closed and $w_{N_n} \to 0$, we have $0 \in U^c$. This is a contradiction to the given definition of U.

Conversely, suppose that for each open neighborhood U of the origin in X, there is $N > 0$ such that

$$S(T,K,f) \subset (S(T_n,K_n,f_n) + U),$$

for all $n \geq N$. We will prove that, $S(T,K,f) \subset \liminf_{n\to\infty} S(T_n,K_n,f_n)$. Let $x_0 \in S(T,K,f)$ be arbitrary. For each $k \in \mathbb{N}$, $U_k := B(0,\frac{1}{k})$ is an open neighborhood of the origin in X, and hence there exists $N_k > 0$ such that

$$x_0 \in S(T_n,K_n,f_n) + U_k,\ \forall n \geq N_k.$$

Thus, for any $n \geq N_k$, there exists a sequence $\{x_n^k\}_{k=1}^{\infty}$ in $S(T_n,K_n,f_n)$ such that

$$x_0 - x_n^k \in U_k.$$

By the compactness of $S(T_n,K_n,f_n)$ in Theorem (3.2) and $x_n^k \in S(T_n,K_n,f_n)$, there exists a subsequence of $\{x_n^k\}$ which converges in $S(T_n,K_n,f_n)$. Without loss of generality, we may assume that

$$x_n^k \to x_n \in S(T_n,K_n,f_n) \text{ as } k \to \infty.$$

We have

$$\begin{aligned} d(x_n,x_0) &\leq d(x_n,x_n^k) + d(x_n^k,x_0) \\ &\leq d(x_n,x_n^k) + \frac{1}{k} \to 0 \text{ as } n,k \to \infty. \end{aligned}$$

Hence, $x_n \to x_0$, i.e.,

$$S(T,K,f) \subset \liminf_{n\to\infty} S(T_n,K_n,f_n).$$

This complete the proof. □

Motivated by [15], we also study the characterizating (H_g).

Lemma 4.8. *Suppose that all the conditions in Proposition 4.3 are satisfied. For any open neighborhood U of the origin in X, we let*

$$\Theta_U^n := \inf_{x \in \tilde{K}_n \setminus (S(T_n, K_n, f_n) + U)} -g_n(x).$$

Then, (H_g) holds if and only if for any open neighborhood U of the origin in X, one has

$$\liminf_{n\to\infty} \Theta_U^n > 0.$$

Proof. If (H_g) holds, then for any open neighborhood U of the origin in X, there exists $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ and $x \in \tilde{K}_n \setminus (S(T_n, K_n, f_n) + U)$, one has $g_n(x) \leq -\alpha$, that is, $-g_n(x) \geq \alpha > 0$. It follows that

$$\liminf_{n\to\infty} \Theta_U^n \geq \alpha > 0.$$

Conversely, for any open neighborhood U of the origin in X,

$$\pi := \liminf_{n\to\infty} \Theta_U^n > 0.$$

Then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $\alpha := \frac{1}{2}\pi$, we have

$$\Theta_U^n = \inf_{x \in \tilde{K}_n \setminus (S(T_n, K_n, f_n) + U)} -g_n(x) \geq \alpha > 0.$$

It then follows that, for each $n \geq n_0$ and $x_n \in \tilde{K}_n \setminus (S(T_n, K_n, f_n) + U)$, one has

$$-g_n(x_n) \geq \alpha > 0.$$

Hence, (H_g) is obtained.

□

Now, we are in a position to state and prove the lower Painlevé-Kuratowski convergence of the solution sets $S(T_n, K_n, f_n)$ in the following theorem.

Theorem 4.9. *Suppose that all the conditions in Theorem 3.2 and Theorem 3.3 are satisfied. Suppose that assumption (H_g) holds and assume that K and g are lsc in A and $\tilde{K}(A)$, respectively. Then, (H_g) holds if and only if*

$$S(T, K, f) \subset \liminf_{n\to\infty} S(T_n, K_n, f_n). \tag{4.9}$$

Proof. We first prove the sufficient condition. Suppose on the contrary that (H_g) holds but

$$S(T, K, f) \not\subseteq \liminf_{n\to\infty} S(T_n, K_n, f_n).$$

Then due to Lemma 4.7, there is an open neighborhood U_0 of the origin in X such that for each $n > 0$, there exists $N_n \geq n$ such that we get

$$S(T,K,f) \nsubseteq S(T_{N_n}, K_{N_n}, f_{N_n}) + U_0.$$

Thus, there is a sequence $\{x_{N_n}\}$ in $S(T,K,f)$ such that

$$x_{N_n} \notin S(T_{N_n}, K_{N_n}, f_{N_n}) + U_0, \text{ for all } n \in \mathbb{N}. \tag{4.10}$$

By Theorem 3.2, $S(T,K,f)$ is a compact set. Hence, we can assume without loss of generality that, as $n \to \infty$,

$$x_{N_n} \to x_0 \in S(T,K,f). \tag{4.11}$$

For U_0, in virtue of Lemma 2.6, there exists a balanced open neighborhood V_0 of 0_X such that $V_0 + V_0 + V_0 \subset U_0$. Furthermore, it is clear that for any given $\varepsilon > 0$, $(x_0 + \varepsilon V_0) \cap K(x_0) \neq \emptyset$. Since K is lsc at x_0, there exists some k_0 such that

$$(x_0 + \varepsilon V_0) \cap K(x_{N_n}) \neq \emptyset, \forall n \geq k_0.$$

For a given $\varepsilon \in (0,1]$ and $n \geq k_0$, suppose that $y_{N_n} \in (x_0 + \varepsilon V_0) \cap K(x_{N_n})$. We claim that

$$y_{N_n} \notin S(T_{N_n}, K_{N_n}, f_{N_n}) + V_0. \tag{4.12}$$

Otherwise, there exists $z_{N_n} \in S(T_{N_n}, K_{N_n}, f_{N_n})$ such that $y_{N_n} - z_{N_n} \in V_0$. By (4.11), without loss of generality, we may assume that $x_{N_n} - x_0 \in V_0$, whenever n is sufficiently large. Consequently, we get

$$\begin{aligned} x_{N_n} - z_{N_n} &= (x_{N_n} - x_0) + (x_0 - y_{N_n}) + (y_{N_n} - z_{N_n}) \\ &\in V_0 + (-\varepsilon V_0) + V_0 \\ &\subset V_0 + V_0 + V_0 \\ &\subset U_0. \end{aligned}$$

Hence, for each $n \geq k_0$, $x_{N_n} \in S(T_{N_n}, K_{N_n}, f_{N_n}) + U_0$, which contradicts to (4.10). Thus, (4.12) is proved. By hypothesis (H_g), there exists a constant $\alpha > 0$ and $n_0 \in \mathbb{N}$ with $n_0 \geq k_0$ such that, for each $n \geq n_0$ with $x_n \in \tilde{K_n} \backslash (S(T_n, K_n, f_n) + U)$, one has $g_n(x_n) \leq -\alpha$. In particular, it follows from (4.12) that

$$g_{N_n}(y_{N_n}) \leq -\alpha, \text{ for } n \text{ large enough.}$$

Applying Lemma 4.5, for any $\delta > 0$, there exists a subsequence $\{y_{N_{n_j}}\}$ of $\{y_{N_n}\}$ and $\bar{N} \in \mathbb{N}$ such that

$$g_{N_{n_j}}(y_{N_{n_j}}) \geq g(x_0) - \delta, \text{ for all } j \geq \bar{N}.$$

We choose an δ with $\delta < \alpha$, and hence

$$g(x_0) \leq g_{N_{n_j}}(y_{N_{n_j}}) + \delta \leq -\alpha + \delta < 0, \text{ for all } j \geq \bar{N}.$$

Therefore,

$$-\min_{t\in T(x_0)}\max_{y\in K(x_0)}\max\xi_e(x_0,-f(t,x_0,y))<0.$$

So, for any $t\in T(x_0)$, there exists $y_0\in K(x_0)$ such that

$$\max\xi_e(x_0,-f(t,x_0,y_0))>0.$$

In other words, there exists $v_0\in -f(t,x_0,y_0)$ such that $\xi_e(x_0,v_0)>0$. From Proposition 2.6(iv), one has $v_0\notin -P(x_0)$. This also implies that $-f(t,x_0,y_0)\nsubseteq -P(x_0)$, i.e.,

$$f(t,x_0,y_0)\nsubseteq P(x_0).$$

This is a contradiction to (4.11). Thus,

$$S(T,K,f)\subset\liminf_{n\to\infty}S(T_n,K_n,f_n).$$

Next, we show the necessary condition. Suppose to the contrary that $S(T,K,f)\subset\liminf_{n\to\infty}S(T_n,K_n,f_n)$, but (H_g) does not hold. Then there exists an open neighborhood U of the origin in X such that

$$\liminf_{n\to\infty}\Theta_U^n=0,$$

where

$$\Theta_U^n:=\inf_{x\in\tilde{K}_n\backslash(S(T_n,K_n,f_n)+U)}-g_n(x).$$

It then follows that there exists a sequence $\{\Theta_U^{n_j}\}$ of $\{\Theta_U^n\}$ such that $\lim_{j\to\infty}\Theta_U^{n_j}=\liminf_{n\to\infty}\Theta_U^n=0$. For each $j\in\mathbb{N}$, since $\tilde{K}_{n_j}\backslash(S(T_{n_j},K_{n_j},f_{n_j})+U)$ is a compact set and g_{n_j} is continuous (from Proposition 4.3), there exists $x_{n_j}\in\tilde{K}_{n_j}\backslash(S(T_{n_j},K_{n_j},f_{n_j})+U)$ satisfying $\Theta_U^{n_j}=-g_{n_j}(x_{n_j})$. Consequently,

$$\lim_{j\to\infty}g_{n_j}(x_{n_j})=0. \tag{4.13}$$

Since $x_{n_j}\in K_{n_j}(x_{n_j})\subseteq A$ and A is compact, we may assume that $x_{n_j}\to x_0$ as $j\to\infty$. Consequently, $x_0\in\limsup_{n\to\infty}K_n(x_n)$. As $\{K_n\}$ outer converges continuously to K, we have $\limsup_{n\to\infty}K_n(x_n)\subset K(x_0)$. Hence, $x_0\in K(x_0)$.

By the compactness of $\tilde{K}_n\backslash(S(T_n,K_n,f_n)+U)$, we may assume that

$$x_n'\to x_0\in\tilde{K}_n\backslash(S(T_n,K_n,f_n)+U).$$

Since $x_n'\to x_0$ and g is continuous function , one implies that

$$g(x_n')\to g(x_0)\text{ as }n\to\infty. \tag{4.14}$$

By the uniqueness of the limits, we get that $g(x_0)=0$. Since g be a gap function for (GSVQEP), hence $x_0\in S(T,K,f)$.

For any fixed $t_0\in S(T,K,f)$, by our assumption, we have

$$t_0\in\liminf_{n\to\infty}S(T_n,K_n,f_n).$$

This implies that, there exists a sequence $\{t_n\} \subset S(T_n, K_n, f_n)$ such that $t_n \to t_0$. As $x'_n \in \tilde{K}_n \backslash (S(T_n, K_n, f_n) + U)$. it follows that

$$x'_n - t_n \notin U, \forall n.$$

Let $n \to \infty$, we get that

$$x_0 - t_0 \notin U.$$

This is a contradiction. Hence, (H_g) is valid. This complete the proof. □

Theorem 4.10. *Suppose that the conditions of Proposition 4.3 are satisfied and K_n is u.s.c with closed values in X. Then, $S(T, K, f) \subset \liminf_{n\to\infty} S(T_n, K_n, f_n)$ if and only if (H_g) holds.*

Proof. From Theorem (4.9), we only need to prove the necessity. Suppose to the contrary that $S(T, K, f) \subset \liminf_{n\to\infty} S(T_n, K_n, f_n)$, but (H_g) does not hold. Then there exists an open neighborhood U of the origin in X such that

$$\liminf_{n\to\infty} \Theta_U(x_n) = 0.$$

where

$$\Theta_U(x_n) := \inf_{x_n \in \tilde{K}_n \backslash (S(T_n, K_n, f_n) + U)} -g(x_n).$$

Then, there exists $N \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \Theta_U(x_n) = \lim_{n\to\infty} \Big(\inf_{x_n \in \tilde{K}_n \backslash S(T_n, K_n, f_n) + U} g(x_n) \Big) = 0,$$

for all $n \geq N$. Since $\tilde{K}_n \backslash (S(T_n, K_n, f_n) + U)$ is a compact set and g is continuous (from Proposition 4.3), there exists $x'_n \in \tilde{K}_n \backslash (S(T_n, K_n, f_n) + U)$ satisfying $\Theta_U(x'_n) = g(x'_n)$. This implies that

$$\lim_{n\to\infty} g(x'_n) = 0. \tag{4.15}$$

By the compactness of $\tilde{K}_n \backslash (S(T_n, K_n, f_n) + U)$, we may assume that

$$x'_n \to x_0 \in \tilde{K}_n \backslash (S(T_n, K_n, f_n) + U).$$

Since $x'_n \to x_0$ and g is continuous function , one implies that

$$g(x'_n) \to g(x_0) \text{ as } n \to \infty. \tag{4.16}$$

By the uniqueness of the limits, we get that $g(x_0) = 0$. Since g be a gap function for (GSVQEP), hence $x_0 \in S(T, K, f)$.

For any fixed $t_0 \in S(T, K, f)$, by our assumption, we have $t_0 \in \liminf_{n\to\infty} S(T_n, K_n, f_n)$. This implies that, there exists a sequence $\{t_n\} \subset S(T_n, K_n, f_n)$ such that $t_n \to t_0$. As $x'_n \in \tilde{K}_n \backslash (S(T_n, K_n, f_n) + U)$. it follows that

$$x'_n - t_n \notin U, \forall n.$$

Let $n \to \infty$, we get that

$$x_0 - t_0 \notin U.$$

This is a contradiction. Hence, (H_g) is valid. □

From Theorems 3.1 and 4.9, we can get the following result.

Corollary 4.11. *Suppose that all the assumptions of Theorems 3.1 and 4.9 are satisfies, then $S(T_n, K_n, f_n)$ converges to $S(T, K, f)$ in the sense of Painlevé-Kuratowski if and only if (H_g) holds.*

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