



Random Fixed Point of Random Hardy-Roger Almost Contraction for Solving Nonlinear Stochastic Integral Equations

Plern Saipara[†], Dhananjay Gopal[‡] and Wiyada Kumam^{‡, 1}

[†]Division of Mathematics, Department of Science, Faculty of Science and
Agricultural Technology, Rajamangala University of Technology Lanna Nan,
59/13 Fai Kaeo, Phu Phiang, Nan 55000 Thailand
e-mail : splernn@gmail.com

[‡]Department of Mathematics and Humanities,
National Institute of Technology, Surat, Gujarat 395007, India
e-mail : gopal.dhananjay@rediffmail.com

[‡]Program in Applied Statistics, Department of Mathematics and Computer
Science, Faculty of Science and Technology, Rajamangala University of
Technology Thanyaburi (RMUTT), Thanyaburi, Pathumthani 12110, Thailand
e-mail : wiyada.kum@rmutt.ac.th

Abstract : The objective of this article is to prove a new random fixed point theorem for a random Hardy-Roger almost contraction mapping. The main result in this article is the identification of some random fixed point theorems and the interrelated applications. This result is convenient to determine the existence of solution of a stochastic nonlinear integral equation in a Banach space.

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⁰Thanks! Rajamangala University of Technology Lanna (RMUTL)

¹Corresponding author email: wiyada.kum@rmutt.ac.th (WK)

1 Introduction

The random fixed point theorems for random operators on polish spaces were first studied by Spacek [2] and Hans [3, 4]. In addition, many mathematicians studied various topics of random fixed point on various mappings and various spaces, see in [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Recently, Saha and Ganguly [17] proved a random fixed point theorem in a separable Banach space for some class of random operators.

On the other hand, Banach's contraction [19] is the most important result in some fields of mathematical analysis, especially nonlinear analysis. It has been the source of metric fixed point theory and its importance rests in its various enforcement in many fields of mathematics. For a complete metric space, this result runs as follows (in [20, 21]).

Theorem 1.1. *(Banach's contraction principle) If (X, d) be a complete metric space and $T : X \rightarrow X$ be a self-mapping such that,*

$$d(Tx, Ty) \leq \alpha d(x, y), \quad (1.1)$$

for each $x, y \in X$ for some $\alpha \in [0, 1)$, then T has a unique fixed point.

Sometime, the mappings are not necessarily continuous property. In 2000, Ciric [22] dealt with a class of mappings which are imposed on a complete convex metric space and shown the existence of a unique fixed point on this space which is a double generalization of Gregus [23] as follows:

Theorem 1.2. *Let C be a closed convex subset of a complete convex metric space X and $T : C \rightarrow C$ be a mapping satisfying*

$$d(Tx, Ty) \leq ad(x, y) + b \max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)] \quad (1.2)$$

where $a \in (0, 1)$, $a + b = 1$, $c \leq \frac{4-a}{8-b} \forall x, y \in C$. Then T has a unique fixed point.

Also, Berinde [24] extended the Zamfirescu fixed point theorem to almost contractions, a class of contractive type mappings which shows new property with respect to the ones of the specific results incorporated as follows:

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ an almost contraction, that is, a mapping for which there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X. \quad (1.3)$$

Then

1. $F(T) = \{x \in X : Tx = x\} \neq \emptyset$;
2. For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^\infty$ given by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots$$

converges to some $x^* \in F(T)$;

3. The following estimate holds

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots, \quad i = 1, 2, 3, \dots$$

Later, in 2008, Berinde [25] stated and proved some results in a metric space as follows:

Theorem 1.4. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a Ciric almost contraction, that is, a mapping for which there exist a constant $\alpha \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \alpha M(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X, \tag{1.4}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then

1. (1) $F(T) = \{x \in X : Tx = x\} \neq \emptyset$;
2. (2) For any $x_0 = x \in X$, the Picard iteration $\{x_n\}_{n=0}^\infty$ given by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots$$

converges to some $x^* \in F(T)$;

3. (3) The following estimate holds

$$d(x_n, x^*) \leq \frac{\alpha^n}{(1-\alpha)^2} d(x, Tx), \quad n = 1, 2, \dots$$

Also recently, in the sense of random fixed points, Saha and Ganguly [17] proved a theorem of random fixed point in a separable Banach space for some class of contractive mappings as follows.

Theorem 1.5. *Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \rightarrow X$ be a continuous random operator such that for $\omega \in \Omega$, T satisfies*

$$\begin{aligned} & \|T(\omega, x_1) - T(\omega, x_2)\| & (1.5) \\ \leq & a(\omega) \max\{\|x_1 - x_2\|, c(\omega)(\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_1)\|)\} \\ & + b(\omega) \max\{\|x_1 - T(\omega, x_1)\|, \|x_2 - T(\omega, x_2)\|\} \end{aligned}$$

for all random variables $x_1, x_2 \in X$ where $a(\omega), b(\omega), c(\omega)$ are real-valued random variables such that $a(\omega) \in (0, 1)$, $a(\omega) + b(\omega) = 1$, $c(\omega) \leq \frac{4-a(\omega)}{8-a(\omega)}$ almost surely. Then there exist a unique random fixed point of T .

Very recently, Saipara et al. [18] defined the random Hardy-Roger contraction and proved some random fixed point theorems for Hardy-Roger self-random mappings in separable Banach spaces as follows:

Definition 1.6. Let $T : \Omega \times X \rightarrow X$ be a continuous random mapping. The random mapping T is called Hardy-Roger contraction if, for any $\omega \in \Omega$,

$$\begin{aligned} & \|T(\omega, x_1) - T(\omega, x_2)\| \\ \leq & \alpha_1(\omega)\|x_1 - x_2\| + \alpha_2(\omega)\|x_1 - T(\omega, x_1)\| & (1.6) \\ & + \alpha_3(\omega)\|x_2 - T(\omega, x_2)\| + \alpha_4(\omega)\|x_1 - T(\omega, x_2)\| \\ & + \alpha_5(\omega)\|x_2 - T(\omega, x_1)\| \end{aligned}$$

for all $x_1, x_2 \in X$ and $\alpha_i : \Omega \rightarrow \mathbb{R}_+ \cup \{0\}$ for $i = 1, 2, 3, 4, 5$ such that $\sum_{i=1}^5 \alpha_i(\omega) < 1$.

Theorem 1.7. Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \rightarrow X$ be a continuous random mapping satisfying Hardy-Roger contraction. Then there exists a unique random fixed point of T in X .

Motivated and inspired by Theorem 1.3, 1.4, 1.5, Definition 1.6 and Theorem 1.7, we proposed the definition of Hardy-Roger almost contraction as follows:

Definition 1.8. Let $T : \Omega \times X \rightarrow X$ be a continuous random mapping. The random mapping T is called Hardy-Roger almost contraction if, for any $\omega \in \Omega$, there exists a constant $\delta \in [0, 1)$ such that

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq \delta M(x_1, x_2) + L(\omega)\|x_2 - T(\omega, x_1)\| \tag{1.7}$$

where

$$\begin{aligned} M(x_1, x_2) = & \alpha_1(\omega)\|x_1 - x_2\| + \alpha_2(\omega)\|x_1 - T(\omega, x_1)\| \\ & + \alpha_3(\omega)\|x_2 - T(\omega, x_2)\| + \alpha_4(\omega)\|x_1 - T(\omega, x_2)\| \\ & + \alpha_5(\omega)\|x_2 - T(\omega, x_1)\| \end{aligned}$$

for all $x_1, x_2 \in X$ and $\alpha_i, L : \Omega \rightarrow \mathbb{R}_+ \cup \{0\}$ for $i = 1, 2, 3, 4, 5$ such that $\sum_{i=1}^5 \alpha_i(\omega) + L(\omega) < 1$.

The objective of this article is to prove a random fixed point theorem for a random Hardy-Roger almost contraction operator. The article is organized as follows. Sections 1 and 2 contains Introduction and Preliminaries, respectively. The main results are presented in section 3. The last section contains some application to a random nonlinear integral equations.

2 Preliminaries

Let (X, β_X) be a separable Banach space, where β_X is a σ -algebra of Borel subsets of X , (Ω, β, μ) be a complete probability measure space. More details can be seen the article of Joshi et.al. [26].

Definition 2.1. (1) The mapping $x : \Omega \rightarrow X$ is called an X -valued random variable if $x^{-1}(B) \in \beta$ for any $B \in \beta_X$.

(2) The mapping $x : \Omega \rightarrow X$ is called a finitely valued random variable if it is constant on any finite number of disjoint sets $A_i \in \beta$ and is equal to 0 over $\Omega \setminus (\bigcup_{i=1}^n A_i)$. The mapping x is said to be a simple random variable if it's finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

(3) The mapping $x : \Omega \rightarrow X$ is called a strong random variable if there is a simple random variables sequence $\{x_n(\omega)\}$ converges to $x(\omega)$ almost surely, that is, there is a set $A_0 \in \beta$ with $\mu(A_0) = 0$ so that $\lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)$ for any $\omega \in \Omega \setminus A_0$.

(4) The mapping $x : \Omega \rightarrow X$ is called a weak random variable if the function $x^*(x(\cdot))$ is a real valued random variable for any $x^* \in X^*$, where X^* denotes the first normed dual space of X .

In a separable Banach space X , the notions of strong and weak random variables coincide ([26]).

Theorem 2.2. ([26]) Assume that the mapping $x, y : \Omega \rightarrow X$ be strong random variables and α, β be constants. Then the following assumption hold:

- (1) $\alpha x + \beta y$ is a strong random variable.
- (2) If $f : \Omega \rightarrow \mathbb{R}$ is a real valued random variable, then fx is a strong random variable.
- (3) If x_n is a sequence of strong random variables converges strongly to x almost surely, then $x(\omega)$ is a strong random variable.

Let Y be another Banach space.

Definition 2.3. (1) The mapping $F : \Omega \times X \rightarrow Y$ is called a random mapping if $F(\omega, x) = Y(\omega)$ is a Y -valued random variable $\forall x \in X$.

(2) The mapping $F : \Omega \times X \rightarrow Y$ is called a continuous random mapping if $\mu(\{\omega \in \Omega : F(\omega, x) \text{ is a continuous function of } x\}) = 1$.

(3) The mapping $F : \Omega \times X \rightarrow Y$ is called a demi-continuous at the $x \in X$ if $\|x_n - x\| \rightarrow 0$ implies $F(\omega, x_n) \rightarrow F(\omega, x)$ almost surely.

Theorem 2.4. ([26]) *Let the mapping $F : \Omega \times X \rightarrow Y$ be a demi-continuous random mapping where a Banach space Y is separable. Then, for any X -valued random variable x , the function $F(\omega, x(\omega))$ is a Y -valued random variable.*

Following Joshi et.al. [26], we recall some necessary Definitions and results:

Definition 2.5. (1) $F(\omega, x(\omega)) = x(\omega)$ is said to be a random fixed point equation, where F is a random mapping.

(2) For each the mapping $x : \Omega \rightarrow X$ which satisfies the random fixed point equation almost surely is called a wide sense solution of the fixed point equation.

(3) For each X -valued random variable $x(\omega)$ which satisfies $\mu\{\omega : F(\omega, x(\omega)) = x(\omega)\} = 1$ is called a random fixed point of $F : \Omega \rightarrow X$.

3 Main Results

Now, we prove the random fixed point theorem for a random Hardy-Roger almost contraction as follows:

Theorem 3.1. *Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \rightarrow X$ be a continuous random mapping satisfying Hardy-Roger almost contraction in Definition 1.8. Then there exists a unique random fixed point of T in X .*

Proof. Let

$$A = \{\omega \in \Omega : T(\omega, x) \text{ is a continuous function of } x\},$$

$$B = \left\{ \omega \in \Omega : \sum_{i=1}^5 (\alpha_i(\omega) + L(\omega)) < 1 \right\}$$

and

$$C_{x_1, x_2} = \left\{ \omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| \leq \delta M(x_1, x_2) + L(\omega)\|x_2 - T(\omega, x_1)\| \right\}.$$

Let S be a countable dense set which $S \subseteq X$. Now, we show that

$$\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) = \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B).$$

Now, for all $s_1, s_2 \in S$, we have

$$\begin{aligned} & \|T(\omega, s_1) - T(\omega, s_2)\| \\ & \leq \delta M(s_1, s_2) + L(\omega)\|s_2 - T(\omega, s_1)\| \\ & = \delta\left(\alpha_1(\omega)\|s_1 - s_2\| + \alpha_2(\omega)\|s_1 - T(\omega, s_1)\| \right. \\ & \quad \left. + \alpha_3(\omega)\|s_2 - T(\omega, s_2)\| + \alpha_4(\omega)\|s_1 - T(\omega, s_2)\| \right. \\ & \quad \left. + \alpha_5(\omega)\|s_2 - T(\omega, s_1)\|\right) + L(\omega)\|s_2 - T(\omega, s_1)\|. \end{aligned} \tag{3.1}$$

Since S is dense subset of X , for any $\delta_i(x_i) > 0$, there exist $s_1, s_2 \in S$ such that $\|x_i - s_i\| < \delta_i(x_i)$ for each $i = 1, 2$. Note that, for any $x_1, x_2 \in X$,

$$\|s_1 - s_2\| \leq \|s_1 - x_1\| + \|x_1 - x_2\| + \|x_2 - s_2\|, \tag{3.2}$$

$$\begin{aligned} \|s_1 - T(\omega, s_1)\| & \leq \|s_1 - x_1\| + \|x_1 - T(\omega, x_1)\| \\ & \quad + \|T(\omega, x_1) - T(\omega, s_1)\|, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \|s_2 - T(\omega, s_2)\| & \leq \|s_2 - x_2\| + \|x_2 - T(\omega, x_2)\| \\ & \quad + \|T(\omega, x_2) - T(\omega, s_2)\|, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \|s_1 - T(\omega, s_2)\| & \leq \|s_1 - x_1\| + \|x_1 - T(\omega, x_2)\| \\ & \quad + \|T(\omega, x_2) - T(\omega, s_2)\| \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|s_2 - T(\omega, s_1)\| & \leq \|s_2 - x_2\| + \|x_2 - T(\omega, x_1)\| \\ & \quad + \|T(\omega, x_1) - T(\omega, s_1)\|. \end{aligned} \tag{3.6}$$

Suppose that

$$\begin{aligned} & \|T(\omega, s_1) - T(\omega, s_2)\| \\ & \leq \delta\alpha_1(\omega)\|s_1 - s_2\| + \delta\alpha_2(\omega)\|s_1 - T(\omega, s_1)\| \\ & \quad + \delta\alpha_3(\omega)\|s_2 - T(\omega, s_2)\| + \delta\alpha_4(\omega)\|s_1 - T(\omega, s_2)\| \\ & \quad + \delta\alpha_5(\omega)\|s_2 - T(\omega, s_1)\| + L(\omega)\|s_2 - T(\omega, s_1)\| \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned} & \|T(\omega, x_1) - T(\omega, x_2)\| \\ & \leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_1) - T(\omega, s_2)\| \\ & \quad + \|T(\omega, s_2) - T(\omega, x_2)\|, \end{aligned} \tag{3.8}$$

substituting (3.7) in (3.8), we have

$$\begin{aligned}
& \|T(\omega, x_1) - T(\omega, x_2)\| \\
\leq & \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\| \\
& + \delta\alpha_1(\omega)\|s_1 - s_2\| + \delta\alpha_2(\omega)\|s_1 - T(\omega, s_1)\| \\
& + \delta\alpha_3(\omega)\|s_2 - T(\omega, s_2)\| + \delta\alpha_4(\omega)\|s_1 - T(\omega, s_2)\| \\
& + (\delta\alpha_5(\omega) + L(\omega))\|s_2 - T(\omega, s_1)\|.
\end{aligned} \tag{3.9}$$

Thus, from (3.2), (3.3), (3.4), (3.5), (3.6), (3.9), it follows that

$$\begin{aligned}
& \|T(\omega, x_1(\omega)) - T(\omega, x_2(\omega))\| \\
\leq & \delta\alpha_1(\omega)\|x_1(\omega) - x_2(\omega)\| + \delta\alpha_2(\omega)\|x_1(\omega) - T(\omega, x_1(\omega))\| \\
& + \delta\alpha_3(\omega)\|x_2 - T(\omega, x_2)\| + \delta\alpha_4(\omega)\|x_1 - T(\omega, x_2)\| \\
& + (\delta\alpha_5(\omega) + L(\omega))\|x_2 - T(\omega, x_1)\| + \frac{\varepsilon}{4} \\
& + \delta\alpha_1(\omega)\|s_1 - x_2\| + \delta\alpha_1(\omega)\|x_2 - s_2\| \\
& + \delta\alpha_2(\omega)\|s_1 - x_1\| + \delta\alpha_2(\omega)\|T(\omega, x_1) - T(\omega, s_1)\| \\
& + \delta\alpha_3(\omega)\|s_2 - x_2\| + \delta\alpha_3(\omega)\|T(\omega, x_2) - T(\omega, s_2)\| \\
& + \delta\alpha_4(\omega)\|s_1 - x_2\| + \delta\alpha_4(\omega)\|T(\omega, x_2) - T(\omega, s_2)\| \\
& + (\delta\alpha_5(\omega) + L(\omega))\|s_2 - x_2\| + (\delta\alpha_5(\omega) + L(\omega))\|T(\omega, x_1) - T(\omega, s_1)\|.
\end{aligned} \tag{3.10}$$

For any $\omega \in \Omega$, since $T(\omega, x)$ is a continuous function of $x(\omega)$, for any $\varepsilon > 0$, there exists $\delta_i(x_i) > 0$ ($i = 1, 2$) such that

$$\|T(\omega, x_1) - T(\omega, s_1)\| < \frac{\varepsilon}{8} \tag{3.11}$$

whenever $\|x_1 - s_1\| < \delta_1(x_1)$ and

$$\|T(\omega, x_2) - T(\omega, s_2)\| < \frac{\varepsilon}{8} \tag{3.12}$$

whenever $\|x_2 - s_2\| < \delta_1(x_2)$. Now, choosing

$$\delta_1 = \min \left\{ \delta_1(x_1), \frac{\varepsilon}{8} \right\} \tag{3.13}$$

and

$$\delta_2 = \min \left\{ \delta_2(x_2), \frac{\varepsilon}{8} \right\}, \tag{3.14}$$

by (3.10), we have

$$\begin{aligned}
 & \|T(\omega, x_1) - T(\omega, x_2)\| \\
 \leq & \delta\alpha_1(\omega)\|x_1 - x_2\| + \delta\alpha_2(\omega)\|x_1 - T(\omega, x_1)\| \\
 & + \delta\alpha_3(\omega)\|x_2 - T(\omega, x_2)\| + \delta\alpha_4(\omega)\|x_1 - T(\omega, x_2)\| \\
 & + (\delta\alpha_5(\omega) + L(\omega))\|x_2 - T(\omega, x_1)\| + \frac{\varepsilon}{4} \\
 & + \delta\alpha_1(\omega)\frac{\varepsilon}{8} + \delta\alpha_1(\omega)\frac{\varepsilon}{8} + \delta\alpha_2(\omega)\frac{\varepsilon}{8} + \delta\alpha_2(\omega)\frac{\varepsilon}{8} \\
 & + \delta\alpha_3(\omega)\frac{\varepsilon}{8} + \delta\alpha_3(\omega)\frac{\varepsilon}{8} + \delta\alpha_4(\omega)\frac{\varepsilon}{8} + \delta\alpha_4(\omega)\frac{\varepsilon}{8} \\
 & + (\delta\alpha_5(\omega) + L(\omega))\frac{\varepsilon}{8} + (\delta\alpha_5(\omega) + L(\omega))\frac{\varepsilon}{8} \\
 = & \delta\alpha_1(\omega)\|x_1 - x_2\| + \delta\alpha_2(\omega)\|x_1 - T(\omega, x_1)\| \\
 & + \delta\alpha_3(\omega)\|x_2 - T(\omega, x_2)\| + \delta\alpha_4(\omega)\|x_1 - T(\omega, x_2)\| \\
 & + (\delta\alpha_5(\omega) + L(\omega))\|x_2 - T(\omega, x_1)\| \\
 & + (2 + 2\delta \sum_{i=1}^5 (\alpha_i(\omega) + L(\omega)))\frac{\varepsilon}{8}.
 \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\begin{aligned}
 & \|T(\omega, x_1) - T(\omega, x_2)\| \\
 \leq & \delta\alpha_1(\omega)\|x_1 - x_2\| + \delta\alpha_2(\omega)\|x_1 - T(\omega, x_1)\| \\
 & + \delta\alpha_3(\omega)\|x_2 - T(\omega, x_2)\| + \delta\alpha_4(\omega)\|x_1 - T(\omega, x_2)\| \\
 & + (\delta\alpha_5(\omega) + L(\omega))\|x_2 - T(\omega, x_1)\| \\
 = & \delta\left(\alpha_1(\omega)\|x_1 - x_2\| + \alpha_2(\omega)\|x_1 - T(\omega, x_1)\| \right. \\
 & \left. + \alpha_3(\omega)\|x_2 - T(\omega, x_2)\| + \alpha_4(\omega)\|x_1 - T(\omega, x_2)\| \right. \\
 & \left. + \alpha_5(\omega)\|x_2 - T(\omega, x_1)\| \right) + L(\omega)\|x_2 - T(\omega, x_1)\| \\
 = & \delta M(x_1, x_2) + L(\omega)\|x_2 - T(\omega, x_1)\|.
 \end{aligned} \tag{3.15}$$

Thus we have $\omega \in \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B)$, which implies that

$$\bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B) \subset \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B).$$

Also, we have

$$\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) \subset \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B).$$

Therefore, we have

$$\bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B) = \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B).$$

Let $N' = \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B)$. Then $\mu(N') = 1$. Next, we prove that $\forall \omega \in N'$, $T(\omega, x)$ is a deterministic continuous operators satisfying the mapping referred in [25].

Let $x : \Omega \rightarrow X$ be a random variable defined for some $x^* \in X$ by

$$x(\omega) = \begin{cases} x_\omega, & \omega \in N' \\ x^*, & \omega \notin N'. \end{cases}$$

Next, we show that $x(\omega)$ is the random variable. We construct a sequence of random variable $x_n(\omega)$ as follows. Let $x_0(\omega)$ be an arbitrary random variable and $x_1(\omega) = T(\omega, x_0(\omega))$. Thus $x_1(\omega)$ is a random variable. Next, we get $x_{n+1}(\omega) = T(\omega, x_n(\omega))$, by repeated generating, it gives that $\{x_n(\omega)\}_{n=1,2,\dots}$ is a random variables sequence converge to $x(\omega)$. So, $x(\omega)$ is a random variable.

Finally, we show that $x(\omega)$ is a unique. Let $y : \Omega \rightarrow X$ be another random fixed point. We want to prove that $x(\omega) = y(\omega)$ almost surely. Let $M = \{\omega \in N' : x(\omega) = y(\omega)\}$. To prove $\mu(M) = 0$. Suppose $\mu(M) > 0$, thus $\mu(M \cap N') > 0$ implies $M \cap N' \neq \emptyset$, for all $\omega \in M \cap N'$. Let $\omega \in M \cap N'$, thus $x(\omega) \neq y(\omega)$. But $x(\omega)$ and $y(\omega)$ are fixed point of $T(\omega, \cdot) : X \rightarrow X$, thus $x(\omega) = y(\omega)$. So $\mu(M) = 0$ which is contradiction. Thus, $x(\omega)$ is a unique. Therefore, $x(\omega)$ is a unique random fixed point of T . This completes the proof. \square

From Theorem 3.1, if $L(\omega) = 0$, then we obtain the following Corollary for Hardy-Roger contraction:

Corollary 3.2. *Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \rightarrow X$ be a continuous random mapping satisfying Hardy-Roger contraction. Then there exists a unique random fixed point of T in X .*

From Theorem 3.1, if $\alpha_4(\omega) = \alpha_5(\omega) = L(\omega) = 0$, then we obtain the following Corollary for Reich's contraction:

Corollary 3.3. *Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \rightarrow X$ be a continuous random mapping satisfying the following condition: for any $\omega \in \Omega$,*

$$\begin{aligned} & \|T(\omega, x_1) - T(\omega, x_2)\| \\ & \leq \alpha_1(\omega)\|x_1 - x_2\| + \alpha_2(\omega)\|x_1 - T(\omega, x_1)\| \\ & \quad + \alpha_3(\omega)\|x_2 - T(\omega, x_2)\| \end{aligned}$$

for all $x_1, x_2 \in X$ and $\alpha_i : \Omega \rightarrow \mathbb{R}_+ \cup \{0\}$ for $i = 1, 2, 3$ such that $\sum_{i=1}^3 \alpha_i(\omega) < 1$. Then there exists a unique random fixed point of T in X .

From Theorem 3.1, if $\alpha_1(\omega) = \alpha_4(\omega) = \alpha_5(\omega) = L(\omega) = 0$, then we obtain the following Corollary for Kannan's contraction:

Corollary 3.4. *Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \rightarrow X$ be a continuous random mapping satisfying the following condition: for any $\omega \in \Omega$,*

$$\begin{aligned} & \|T(\omega, x_1(\omega)) - T(\omega, x_2)\| \\ \leq & \alpha_2(\omega)\|x_1 - T(\omega, x_1)\| + \alpha_3(\omega)\|x_2 - T(\omega, x_2)\| \end{aligned}$$

for all $x_1, x_2 \in X$ and $\alpha_i : \Omega \rightarrow \mathbb{R}_+ \cup \{0\}$ for $i = 2, 3$ such that $\alpha_2(\omega) + \alpha_3(\omega) < 1$. Then there exists a unique random fixed point of T in X .

From Theorem 3.1, if $\alpha_1(\omega) = \alpha_2(\omega) = \alpha_3(\omega) = 0$, then we obtain the following Corollary for Chatterjea’s contraction:

Corollary 3.5. *Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \rightarrow X$ be a continuous random mapping satisfying the following condition: for all $\omega \in \Omega$,*

$$\begin{aligned} & \|T(\omega, x_1) - T(\omega, x_2)\| \\ \leq & \alpha_4(\omega)\|x_1 - T(\omega, x_2)\| + \alpha_5(\omega)\|x_2 - T(\omega, x_1)\| \end{aligned}$$

for all $x_1, x_2 \in X$ and $\alpha_i : \Omega \rightarrow \mathbb{R}_+ \cup \{0\}$ for $i = 4, 5$ such that $\alpha_4(\omega) + \alpha_5(\omega) < 1$. Then there exists a unique random fixed point of T in X .

Remark 3.6. *The random fixed point theorems for Hardy-Roger almost contraction reduced to the random fixed point theorems for Hardy-Roger contraction and Ćirić’s contraction.*

4 Application to random nonlinear operator equations

Now, we show the existence and uniqueness of a solution of a nonlinear stochastic integral equation of the Hammerstein type ([15]) by using Theorem 3.1:

$$x(t; \omega) = h(t; \omega) + \int_S k(t; s; \omega) f(s; x(s; \omega)) d\mu(s), \tag{4.1}$$

where

- (a) S is a locally compact metric space with metric d defined on $S \times S$ and μ_0 is a complete σ -finite measure defined on the collection of Borel subsets of S ;
- (b) $\omega \in \Omega$ where ω is the supporting set of the probability measure space (Ω, β, μ) ;

- (c) $x(t; \omega)$ is the unknown vector-valued random variable for each $t \in S$;
- (d) $h(t; \omega)$ is the stochastic free term defined for $t \in S$;
- (e) $k(t, s; \omega)$ is the stochastic kernel defined for t and s in S ;
- (f) $f(t, x)$ is a vector-valued function of $t \in S$ and x .

Note that the integral in the equation (4.1) is interpreted as a Bochner integral ([27]).

Further, we assume that the union of a countable family $\{C_n\}$ of compact sets with $C_{n+1} \subset C_n$ is defined as S such that, for each other compact set in S , there exists C_i which contains it ([28]).

We define $C = C(S, L_2(\Omega, \beta, \mu))$ as all continuous functions space from S into the space $L_2(\Omega, \beta, \mu)$ with the topology of uniform convergence on compact sets of S , that is, $x(t; \omega)$ is a vector-valued random variable for each fixed $t \in S$ such that

$$\|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}^2 = \int_{\Omega} |x(t; \omega)|^2 d\mu(\omega) < \infty.$$

Noted that $C(S, L_2(\Omega, \beta, \mu))$ is a space of locally convex ([27]) whose topology is defined by the countable family of semi-norms given by

$$\|x(t; \omega)\|_n = \sup_{t \in C_n} \|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}$$

for each $n \geq 1$. Furthermore, since $L_2(\Omega, \beta, \mu)$ is complete, $C(S, L_2(\Omega, \beta, \mu))$ is complete relative to this topology.

Next, we define $BC = BC(S, L_2(\Omega, \beta, \mu))$ as a Banach space of all bounded continuous functions from S into $L_2(\Omega, \beta, \mu)$ with the norm

$$\|x(t; \omega)\|_{BC} = \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}.$$

The space $BC \subset C$ is a space of all second order vector-valued stochastic processes defined on S which are bounded and continuous in mean-square.

Now, we consider the functions $h(t; \omega)$ and $f(t, x(t; \omega))$ to be in the $C(S, L_2(\Omega, \beta, \mu))$ space with respect to the stochastic kernel and assume that, for each pair (t, s) , $k(t, s; \omega) \in L_{\infty}(\Omega, \beta, \mu)$ and the norm denoted by

$$\| \|k(t, s; \omega)\| \| = \|k(t, s; \omega)\|_{L_{\infty}(\Omega, \beta, \mu)} = \mu - \text{ess sup}_{\omega \in \Omega} |k(t, s; \omega)|.$$

Also, we suppose that $k(t, s; \omega) \in L_{\infty}(\Omega, \beta, \mu)$ is such that

$$\| \|k(t, s; \omega)\| \| = \|x(s; \omega)\|_{L_2(\Omega, \beta, \mu)}$$

is μ -integrable with respect to s for each $t \in S$ and $x(s; \omega) \in C(S, L_2(\Omega, \beta, \mu))$ and there exists a real-valued function G μ -a.e. on S such that $G(S)\|x(s; \omega)\|_{L_2(\Omega, \beta, \mu)}$ is μ -integrable and, for each pair $(t, s) \in S \times S$,

$$\| \|k(t, u; \omega) - k(s, u; \omega)\| \| \cdot \|x(u; \omega)\|_{L_2(\Omega, \beta, \mu)} \leq G(u)\|x(u; \omega)\|_{L_2(\Omega, \beta, \mu)} \quad \mu - \text{a.e.}$$

Forward, assume that, for almost all $s \in S$, $k(t, s; \omega)$ is continuous in t from S into $L_\infty(\Omega, \beta, \mu)$.

Now, we define the random integral operator T on $C(S, L_2(\Omega, \beta, \mu))$ by

$$(Tx)(t; \omega) = \int_S k(t, s; \omega)x(s; \omega)d\mu(s), \tag{4.2}$$

where the integral is a Bochner integral. From the conditions on $k(t, s; \omega)$, it follows that, for each $t \in S$, $(Tx)(t; \omega) \in L_2(\Omega, \beta, \mu)$ and $(Tx)(t; \omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem, that is, $(Tx)(t; \omega) \in C(S, L_2(\Omega, \beta, \mu))$.

Lemma 4.1. ([15]) *The linear operator T defined by equation (4.2) is continuous from $C(S, L_2(\Omega, \beta, \mu))$ into itself.*

Proof. See [15]. □

Definition 4.2. ([29], [30]) *Let B and D be Banach spaces. The pair (B, D) is said to be admissible with respect to a linear operator T if $T(B) \subset D$.*

Lemma 4.3. ([15]) *If T is a continuous linear operator from $C(S, L_2(\Omega, \beta, \mu))$ into itself and $B, D \subset C(S, L_2(\Omega, \beta, \mu))$ are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to T , then T is continuous from B into D .*

By a *random solution* of the equation (4.1), we mean a function

$$x(t; \omega) \in C(S, L_2(\Omega, \beta, \mu))$$

which satisfies the equation (4.1) $\mu - a.e.$

Now, by using Theorem 3.1, we prove the following:

Theorem 4.4. *If the stochastic integral equation (4.1) is subject to the following conditions:*

- (1) *B and D are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to the integral operator defined by (4.2);*
- (2) *$x(t; \omega) \mapsto f(t, x(t; \omega))$ is an operator from the set $Q(\rho) = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$ into the space B satisfying*

$$\begin{aligned} & \|f(t, x_1(t, \omega)) - f(t, x_2(t, \omega))\|_B \\ \leq & \delta\alpha_1(\omega)\|x_1(t, \omega) - x_2(t, \omega)\| + \delta\alpha_2(\omega)\|x_1(t, \omega) - f(t, x_1(t, \omega))\| \\ & + \delta\alpha_3(\omega)\|x_2(t, \omega) - f(t, x_2(t, \omega))\| + \delta\alpha_4(\omega)\|x_1(t, \omega) - f(t, x_2(t, \omega))\| \\ & + \delta\alpha_5(\omega)\|x_2(t, \omega) - f(t, x_1(t, \omega))\| + L(\omega)\|x_2(t, \omega) - f(t, x_1(t, \omega))\| \end{aligned} \tag{4.3}$$

for all $x_1(t, \omega), x_2(t, \omega) \in Q(\rho)$ and $\alpha_i, L : \Omega \rightarrow \mathbb{R}_+ \cup \{0\}$ for $i = 1, 2, 3, 4, 5$ such that $\sum_{i=1}^5 \alpha_i(\omega) + L(\omega) < 1$ almost surely;

$$(3) \quad h(t; \omega) \in D,$$

then there exists a unique random solution of (4.1) in $Q(\rho)$ provided

$$\begin{aligned} & \|h(t, \omega)\|_D + l(\omega) \|f(t, 0)\|_B \left(\frac{1 + \delta\alpha_3(\omega) + \delta\alpha_4(\omega)}{1 - \delta\alpha_2(\omega) + \delta\alpha_5(\omega) - L(\omega)} \right) \\ & \leq \rho \left(1 - \frac{l(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega) - L(\omega)} \right), \end{aligned}$$

where the norm of $T(\omega)$ denoted by $l(\omega)$.

Proof. Let a mapping $\mathcal{U}(\omega) : Q(\rho) \rightarrow D$ defined by

$$(\mathcal{U}x)(t, \omega) = h(t, \omega) + \int_S k(t, s, \omega) f(s, x(s, \omega)) d\mu_0(s).$$

Then we have

$$\begin{aligned} \|(\mathcal{U}x)(t, \omega)\|_D & \leq \|h(t, \omega)\|_D + l(\omega) \|f(t, x(t, \omega))\|_B \\ & \leq \|h(t, \omega)\|_D + l(\omega) \|f(t, 0)\|_B + l(\omega) \|f(t, x(t, \omega)) - f(t, 0)\|_B. \end{aligned}$$

Thus it follows from (4.3) that

$$\begin{aligned} & \|f(t, x(t, \omega)) - f(t, 0)\|_B \\ & \leq \delta\alpha_1(\omega) \|x(t, \omega)\|_D + \delta\alpha_2(\omega) \|x(t, \omega) - f(t, x(t, \omega))\|_D \\ & \quad + \delta\alpha_3(\omega) \|f(t, 0)\|_D + \delta\alpha_4(\omega) \|x(t, \omega) - f(t, 0)\|_D \\ & \quad + \delta\alpha_5(\omega) \|f(t, x(t, \omega))\|_D + L(\omega) \|f(t, x(t, \omega))\|_D \\ & \leq \delta\alpha_1(\omega) \|x(t, \omega)\|_D + \delta\alpha_2(\omega) \|x(t, \omega)\|_D \\ & \quad + \delta\alpha_2(\omega) \|f(t, x(t, \omega)) - f(t, 0)\|_B + \delta\alpha_2(\omega) \|f(t, 0)\|_D \\ & \quad + \delta\alpha_3(\omega) \|f(t, 0)\|_D + \delta\alpha_4(\omega) \|x(t, \omega)\|_D + \alpha_4(\omega) \|f(t, 0)\|_D \\ & \quad + \delta\alpha_5(\omega) \|f(t, x(t, \omega)) - f(t, 0)\|_B + \delta\alpha_5(\omega) \|f(t, 0)\|_D \\ & \quad + L(\omega) \|f(t, x(t, \omega)) - f(t, 0)\|_B + L(\omega) \|f(t, 0)\|_D \end{aligned}$$

and so

$$\begin{aligned} & (1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega) - L(\omega)) \|f(t, x(t, \omega)) - f(t, 0)\|_B \\ & \leq (\delta\alpha_1(\omega) + \delta\alpha_2(\omega) + \delta\alpha_4(\omega)) \rho \\ & \quad + (\delta\alpha_2(\omega) + \delta\alpha_3(\omega) + \delta\alpha_4(\omega) + \delta\alpha_5(\omega) + L(\omega)) \|f(t, 0)\|_D. \end{aligned}$$

Hence we have

$$\begin{aligned} \|f(t, x(t, \omega)) - f(t, 0)\|_B & \leq \left(\frac{\delta\alpha_1(\omega) + \delta\alpha_2(\omega) + \delta\alpha_4(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega) - L(\omega)} \right) \rho \quad (4.4) \\ & \quad + \left(\frac{\delta\alpha_2(\omega) + \delta\alpha_3(\omega) + \delta\alpha_4(\omega) + \delta\alpha_5(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega) - L(\omega)} \right) \|f(t, 0)\|_D. \end{aligned}$$

Therefore, by (4.4), we have

$$\begin{aligned}
 \|(\mathcal{U}x)(t, \omega)\|_D &\leq \|h(t, \omega)\|_D + l(\omega)\|f(t, 0)\|_B + \left(\frac{\delta\alpha_1(\omega) + \delta\alpha_2(\omega) + \delta\alpha_4(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega)}\right)l(\omega)\rho \\
 &\quad + \left(\frac{\delta\alpha_2(\omega) + \delta\alpha_3(\omega) + \delta\alpha_4(\omega) + \delta\alpha_5(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega)}\right)l(\omega)\|f(t, 0)\|_B \\
 &\leq \|h(t, \omega)\|_D + \left(\frac{\delta\alpha_1(\omega) + \delta\alpha_2(\omega) + \delta\alpha_4(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega) - L(\omega)}\right)l(\omega)\rho \\
 &\quad + \left(1 + \frac{\delta\alpha_2(\omega) + \delta\alpha_3(\omega) + \delta\alpha_4(\omega) + \delta\alpha_5(\omega) + L(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega) - L(\omega)}\right)l(\omega)\|f(t, 0)\|_B \\
 &\leq \|h(t, \omega)\|_D + \left(\frac{\delta\alpha_1(\omega) + \delta\alpha_2(\omega) + \delta\alpha_4(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega) - L(\omega)}\right)l(\omega)\rho \\
 &\quad + \left(\frac{1 + \delta\alpha_3(\omega) + \delta\alpha_4(\omega)}{1 - \delta\alpha_2(\omega) - \delta\alpha_5(\omega) - L(\omega)}\right)l(\omega)\|f(t, 0)\|_B \tag{4.5} \\
 &< \rho
 \end{aligned}$$

and so, by (4.5), $(\mathcal{U}x)(t, \omega) \in Q(\rho)$. Thus, for any $x_1(t, \omega), x_2(t, \omega) \in Q(\rho)$ and, by the condition (2), we have

$$\begin{aligned}
 &\|(\mathcal{U}x_1)(t, \omega) - (\mathcal{U}x_2)(t, \omega)\|_D \\
 &= \left\| \int_S k(t, s, \omega)[f(s, x_1(s, \omega)) - f(s, x_2(s, \omega))]d\mu_0(s) \right\|_D \\
 &\leq l(\omega)\|f(s, x_1(s, \omega)) - f(s, x_2(s, \omega))\|_B \\
 &\leq \delta\alpha_1(\omega)\|x_1(t, \omega) - x_2(t, \omega)\|_D + \delta\alpha_2(\omega)\|x_1(t, \omega) - (\mathcal{U}x_1)(t, \omega)\|_D \\
 &\quad + \delta\alpha_3(\omega)\|x_2(t, \omega) - (\mathcal{U}x_2)(t, \omega)\|_D + \delta\alpha_4(\omega)\|x_1(t, \omega) - (\mathcal{U}x_2)(t, \omega)\|_D \\
 &\quad + \delta\alpha_5(\omega)\|x_2(t, \omega) - (\mathcal{U}x_1)(t, \omega)\|_D + L(\omega)\|x_2(t, \omega) - (\mathcal{U}x_1)(t, \omega)\|_D \\
 &= \delta M(x_1(t, \omega), x_2(t, \omega)) + L(\omega)\|x_2(t, \omega) - (\mathcal{U}x_1)(t, \omega)\|_D.
 \end{aligned}$$

Consequently, $\mathcal{U}(\omega)$ is a random contractive mapping on $Q(\rho)$. Hence, by Theorem 3.1, there exists a random fixed point of $\mathcal{U}(\omega)$, which is the random solution of the equation (4.1). This completes the proof. □

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