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On New Fixed Point Results in E_b-Metric Spaces

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Abstract : In this paper, we introduce the concepts of an extended cyclic Banach contraction and an extended cyclic orbital $\mathcal{F}$-expanding contraction. Thereby, we prove pertinent fixed point theorems in an extended b-metric space (simply E_b-metric space). Moreover, we present the characterization of the Hardy and Rogers mapping theorem for (a pair of) non-self maps, which gives a positive answer to the question raised by C. Boateng Ampadu [12] (Fixed Point Theory, 19(2018), No. 2, 449-452 DOI: 10.24193/fpt-ro.2018.2.35).

Keywords : Extended b-metric space, an extended cyclic Banach contraction, extended cyclic orbital $\mathcal{F}$-expanding contraction, proximal cyclic Hardy and Rogers type mapping.
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1 Introduction and Preliminaries

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The Banach contraction principle is a one of the superior results in Nonlinear Analysis; and has always been at the forefront of creating and supplying outstanding generalizations for its researchers. Many authors generalized and utilized Banach contraction principle in their pertinent research. Thus we can easily conclude that, the largest part of the fixed point theory was occupied by various generalizations of Banach contraction principle.

Below are the some of the well known generalizations of Banach contraction principle.

Cyclic contraction by Kirk et al. [1] $\Leftrightarrow$ there exists $k \in (0,1)$ such that

$$d(\mathcal{T}x, \mathcal{T}y) \leq kd(x,y) \text{ for all } x \in A \text{ and } y \in B.$$

(Note that, a mapping $\mathcal{T} : A \cup B \to A \cup B$ is called cyclic if $\mathcal{T}(A) \subseteq B$ and $\mathcal{T}(B) \subseteq A$, where A, B are nonempty subsets of a metric space (X, d).)

Cyclic orbital contraction by Karpagam [2] $\Leftrightarrow d(\mathcal{T}^{2n}x, \mathcal{T}y) \leq \gamma d(\mathcal{T}^{2n-1}x, y)$; for all $x \in \mathcal{A}$, $\gamma \in (0,1)$; where $\mathcal{A}$ and $\mathcal{B}$ are non-empty closed subsets of X and $\mathcal{T} : A \cup B \to A \cup B$ is a cyclic map.

F-contraction by Wardowski [3] $\Leftrightarrow$ there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(\mathcal{T}x, \mathcal{T}y) > 0 \Rightarrow \tau + F(d(\mathcal{T}x, \mathcal{T}y)) \leq F(d(x,y)),$$

F-Expanding by Gornicki [4]$\Leftrightarrow$ there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(x,y) > 0 \Rightarrow F(d(\mathcal{T}x, \mathcal{T}y)) \geq F(d(x,y)) + \tau.$$

where $F : \mathbb{R}_0^+ \to \mathbb{R}$ is a mapping satisfying,
$(F1)$. F is strictly increasing, i.e for all $\alpha, \beta \in \mathbb{R}_0^+$ such that if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;
$(F2)$. For each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

$(F3)$. There exists $k \in (0,1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Let $\mathcal{F}$ denote the set of all functions satisfying the conditions $(F1)$-$(F3)$.

For more generalizations of Banach contraction principle, the reader can refer to [5, 6, 7, 8]. Recently, a new kind of generalized metric space was introduced by T. Kamran *et al.* [9].

Definition 1.1. *[9] Let X be a non-empty set and $s : X \times X \to [1, \infty)$. A function $E_b : X \times X \to [0, \infty)$ is called an E_b-metric if, for all $x, y, z \in X$, if it satisfies*

(i) $E_b(x,y) = 0$ *if and only if* $x = y$;

(ii) $E_b(x,y) = E_b(y,x)$;

(iii) $E_b(x,y) \leq s(x,y)[E_b(x,z) + E_b(z,y)]$.

The pair (X, E_b) *is called an* E_b*-metric space.*

It is clear that, if $s(x,y)$, in Definition 1.1, is a constant in $[1,\infty)$, the pair (X, E_b) coincides with a b-metric space.

Example 1.2. *[10] Let* $X = [0,1]$ *and* $s : X \times X \to [1,\infty)$, $s(x,y) = \frac{xy+1}{x+y}$. *Define* $E_b : X \times X \to [0,\infty)$ *by*

$$E_b(x,y) = \begin{cases} \frac{1}{xy}, & \text{for } x,y \in (0,1], \quad x \neq y \\ 0, & \text{for } x,y \in [0,1], \quad x = y \end{cases}$$

Then (X, E_b) *is an* E_b*-metric space.*

Definition 1.3. *[9] Let* (X, E_b) *be an* E_b*-metric space and* $\{x_n\}$ *be a sequence in* X*. Then*

(a) $\{x_n\}$ *converges to* x *if and only if for every* $\epsilon > 0$ *there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that* $E_b(x_n, x) < \epsilon$, *for all* $n \geq N$. *For this particular case, we write* $\lim_{n\to\infty} x_n = x$.

(b) $\{x_n\}$ *is called Cauchy if and only if for every* $\epsilon > 0$ *there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that* $E_b(x_m, x_n) < \epsilon$, *for all* $m, n \geq N$.

Definition 1.4. *[9] An* E_b*-metric space* (X, E_b) *is complete if and only if every Cauchy sequence in* X *is convergent.*

Note that, usually a b-metric is not a continuous functional. Analogously, E_b-metric is also not necessarily a continuous functional.

Motivated by the above facts, we introduce and establish the concepts of an extended cyclic Banach contraction and an extended cyclic orbital $\mathcal{F}$-expanding contraction. Thereby, we prove pertinent fixed point theorems in E_b-metric space. Moreover, we present the characterization of the Hardy and Rogers mapping theorem for (a pair of) non-self maps, which gives a positive answer to the question raised by C. B. Ampadu [12](Fixed Point Theory, 19(2018), No.2, 449-452, DOI: 10.24193/fpt-ro.2018.2.35).

2 Fixed point theorems in E_b-metric spaces

Now, we start this section by introducing the following definition.

Definition 2.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be non-empty subsets of a* E_b*-metric space* (X, E_b)*. A cyclic map* $\mathcal{T} : \mathcal{A}\cup\mathcal{B} \to \mathcal{A}\cup\mathcal{B}$ *is said to be an extended cyclic Banach contraction, if*

$$E_b(\mathcal{T}x, \mathcal{T}y) \leq kE_b(x,y); \quad \forall x \in \mathcal{A}, y \in \mathcal{B} \quad \text{and} \quad k \in [0,1) \tag{2.1}$$

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty closed subsets of a complete E_b-metric space (X, E_b) such that E_b is a continuous functional. Let $\mathcal{T}$ be an extended cyclic Banach contraction such that for each $x_0 \in \mathcal{A}$, $\lim_{n,m\to\infty} s(x_n, x_m) < \frac{1}{k}$, here $x_n = \mathcal{T}^n x_0; n = 1, 2, 3....$ Then $\mathcal{T}$ has unique fixed point in $\mathcal{A} \cap \mathcal{B}$.*

Proof. Let us suppose $x_0 = x \in \mathcal{A}$(fixed). Define the iterative sequence

$$x_0 = \mathcal{T}x_0 = x_1, x_2 = \mathcal{T}x_1 = \mathcal{T}(\mathcal{T}x_0) = \mathcal{T}^2(x_0), ..., x_n = \mathcal{T}^n(x_0).$$

Using the extended cyclic Banach contraction, we get,

$$\begin{aligned} E_b(\mathcal{T}x, \mathcal{T}^2x) &= E_b(\mathcal{T}x, \mathcal{T}(\mathcal{T}x)) \\ &\le kE_b(x, \mathcal{T}x) \end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned} E_b(\mathcal{T}^2x, \mathcal{T}^3x) &= E_b(\mathcal{T}(\mathcal{T}x), \mathcal{T}(\mathcal{T}^2x)) \\ &\le kE_b(\mathcal{T}x, \mathcal{T}^2x) \\ &\le k^2E_b(x, \mathcal{T}x) \end{aligned} \tag{2.3}$$

Then by successively applying extended cyclic Banach contraction condition, we get,

$$E_b(\mathcal{T}^nx, \mathcal{T}^{n+1}x) \le k^nE_b(x, \mathcal{T}x); \tag{2.4}$$

By triangle inequality and (4), for $m > n$ we have,

$$\begin{aligned} E_b(\mathcal{T}^nx, \mathcal{T}^mx) &= E_b(x_n, x_m) \\ &\le s(x_n, x_m)k^nE_b(x_0, x_1) + s(x_n, x_m)s(x_{n+1}, x_m)k^{n+1}E_b(x_0, x_1) \\ &\quad + s(x_n, x_m)s(x_{n+1}, x_m)s(x_{n+2}, x_m)...s(x_{m-2}, x_m)s(x_{m-1}, x_m)k^{m-1}E_b(x_0, x_1) \\ &\le E_b(x_0, x_1)[s(x_1, x_m)s(x_2, x_m)\dots s(x_{n-1}, x_m)s(x_n, x_m)k^n \\ &\quad + s(x_1, x_m)s(x_2, x_m)\dots s(x_n, x_m)s(x_{n+1}, x_m)k^{n+1} \\ &\quad \vdots \\ &\quad + s(x_1, x_m)s(x_2, x_m)\dots s(x_{m-2}, x_m)s(x_{m-1}, x_m)k^{m-1}]. \end{aligned}$$

Since $\lim_{n,m\to\infty} s(x_{n+1}, x_m)k < 1$, so that the series $\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} s(x_i, x_m)$ converges by ratio test for each $m \in \mathbb{N}$.
Let,

$$\mathcal{S} = \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} s(x_i, x_m)$$

$$\mathcal{S}_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j} s(x_i, x_m)$$

Thus for $m > n$ above inequality implies $E_b(\mathcal{T}^n x, \mathcal{T}^m x) \leq E_b(x_0, x_1)[\mathcal{S}_{m-1} - \mathcal{S}_{n-1}]$. Letting $n \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, we see that $\{\mathcal{T}^n x\}$ converges to some $\upsilon \in X$.

We note that $\{\mathcal{T}^{2n} x\}$ is a sequence in $\mathcal{A}$ and $\{\mathcal{T}^{2n-1} x\}$ is a sequence in $\mathcal{B}$ such a way that both sequences tend to the same limit υ.

Since $\mathcal{A}$ and $\mathcal{B}$ are closed, we have $\upsilon \in \mathcal{A} \cap \mathcal{B}$, and then $\mathcal{A} \cap \mathcal{B} \neq \emptyset$.
Now we will prove that υ is a fixed point of $\mathcal{T}$.
Consider,

$$E_b(\mathcal{T}^n x, \mathcal{T}\upsilon) \leq s(x,\upsilon)[E_b(\mathcal{T}^n x, \mathcal{T}^{n+1} x) + E_b(\mathcal{T}^{n+1} x, \mathcal{T}\upsilon)]. \tag{2.5}$$

Since T is continuous, $\lim_{n\to\infty} E_b(\mathcal{T}^{n+1} x, \mathcal{T}\upsilon) = 0$ and $E_b(\mathcal{T}^n x, \mathcal{T}^{n+1} x) \leq k^n E_b(x, \mathcal{T}x)$. Letting $n \to \infty$, (5) yields $E_b(\upsilon, \mathcal{T}\upsilon) = 0$, since $0 \leq k < 1$. Thus $\upsilon = \mathcal{T}\upsilon$.

Hence υ is a fixed point of $\mathcal{T}$. Finally, to obtain the uniqueness of a fixed point, let $\upsilon^* \in X$ be another fixed point of $\mathcal{T}$ such that $\mathcal{T}\upsilon^* = \upsilon^*$.
Then we have,

$$E_b(\upsilon, \upsilon^*) = E_b(\mathcal{T}\upsilon, \mathcal{T}\upsilon^*) \leq k E_b(\upsilon, \upsilon^*)$$

which yields $E_b(\upsilon, \upsilon^*) = 0$. Thus, $\upsilon = \upsilon^*$.
Thus υ is a unique fixed point of $\mathcal{T}$. This completes the proof of the theorem. □

If we take $s(x, y) = 1$ a constant function, then above theorem reduces to a metric space. If $s(x, y) = s; \quad s \geq 1$ then we get above theorem for a b-metric space.

Example 2.3. *Let $X = \mathbb{R}$. Define $E_b(x, y) : X \times X \to \mathbb{R}_0^+$ and $s : X \times X \to [1, \infty)$ as $E_b(x, y) = (x - y)^2$ and $s(x, y) = x + y + 1$. Then E_b is a complete E_b-metric on X. Let $A = [-1, 0],\ B = [0, 1]$ Define $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ by $\mathcal{T}x = -\frac{x}{2}$. Further $\mathcal{T}(\mathcal{A}) \subset \mathcal{B}$ and $\mathcal{T}(\mathcal{B}) \subset \mathcal{A}$. Thus $\mathcal{T}$ is a cyclic map.*
Consider

$$\begin{aligned} E_b(\mathcal{T}x, \mathcal{T}y) &= E_b(\frac{-x}{2}, \frac{-y}{2}) \\ &= (\frac{y-x}{2})^2 \\ &\leq \frac{1}{2}(y-x)^2 \\ &\leq k E_b(x, y); \quad \text{for all } k, \frac{1}{2} \leq k < 1. \end{aligned}$$

Thus all the conditions of above theorem are satisfied and 0 is a unique fixed point.

Definition 2.4. *[10] Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty sub sets of an E_b-metric space (X, E_b). Let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclic map such that for some $x \in \mathcal{A}$ there exists $\tau > 0$ such that $\forall x, y \in X$ satisfying $E_b(\mathcal{T}x, \mathcal{T}y) > 0$, the following holds:*

$$\tau + F(E_b(\mathcal{T}^{2n} x, \mathcal{T}y)) \leq F(E_b(\mathcal{T}^{2n-1} x, y)), \tag{2.6}$$

where $n \in \mathbb{N}, y \in \mathcal{A}$ such that for each $x_0 \in X$, $\lim_{n,m\to\infty} s(x_n, x_m) < 1$, here $x_n = \mathcal{T}^n x_0, \quad n = 1, 2, 3....$

Then $\mathcal{T}$ is called an *extended cyclic orbital $\mathcal{F}$-contraction.*

Theorem 2.5. *[10] Let (X, E_b) be a complete E_b-metric space such that E_b is continuous functional and let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be an extended cyclic orbital $\mathcal{F}$-contraction. Then $\mathcal{A} \cap \mathcal{B}$ is non-empty and $\mathcal{T}$ has a unique fixed point in $\mathcal{A} \cap \mathcal{B}$.*

Definition 2.6. *[11] Let (X, E_b) be an E_b-metric space. A mapping $\mathcal{T} : X \to X$ is said to be an extended expanding if*

$$\forall \;\; x, y \in X, \quad E_b(\mathcal{T}x, \mathcal{T}y) \geq \kappa E_b(x, y); \quad \textit{where} \quad \kappa > 1.$$

Definition 2.7. *[10] Let (X, E_b) be E_b-metric space. Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets of E_b-metric space and $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclic map such that for some $x \in \mathcal{A}$ there exists a $k_x > 1$ such that,*

$$E_b(\mathcal{T}^{2n}x, \mathcal{T}y) \geq k_x(E_b(\mathcal{T}^{2n-1}x, y); \quad n \in \mathbb{N}, y \in \mathcal{A}. \tag{2.7}$$

Then $\mathcal{T}$ is called an extended expanding cyclic orbital contraction.

Theorem 2.8. *[10] Let (X, E_b) be a complete E_b-metric space such that E_b is a continuous functional. Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets of an E_b-metric space (X, E_b) and $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be an extended cyclic orbital contraction. Suppose that for each $x_0 \in \mathcal{A}$, $\lim_{n,m\to\infty} s(x_n, x_m) < \dfrac{1}{k_{x_0}}$, here $x_n = \mathcal{T}^n x_0$; $\;\; n = 1, 2, 3...$ Then $\mathcal{A} \cap \mathcal{B}$ is non-empty and $\mathcal{T}$ has a unique fixed point.*

Theorem 2.9. *Let (X, E_b) be a complete E_b-metric space such that E_b is a continuous functional. Let $\mathcal{T}$ be a surjective and an extended expanding cyclic orbital contraction. Then $\mathcal{T}$ is bijective and has a unique fixed point in $\mathcal{A} \cap \mathcal{B}$.*

Proof. Suppose there exists an x, (say x_0), $\in \mathcal{A}$ satisfying (7). Define an iterative sequence $\{x_n\}$ starting by x_0, as follows:

$$\mathcal{T}x_0 = x_1, x_2 = \mathcal{T}x_1 = \mathcal{T}(\mathcal{T}x_0) = \mathcal{T}^2(x_0).....x_n = \mathcal{T}^n(x_0)....$$

If $x = \mathcal{T}x$, then x is a fixed point of $\mathcal{T}$. Hence the proof is completed.
Thus, let us suppose that $x \neq \mathcal{T}x$.

$$E_b(\mathcal{T}^2x, \mathcal{T}x) \geq k_x E_b(\mathcal{T}x, x) > 0$$
$$\Rightarrow E_b(\mathcal{T}^2x, \mathcal{T}x) > 0$$
$$\Rightarrow \mathcal{T}^2x \neq \mathcal{T}x.$$

Hence $\mathcal{T}$ is bijective. Since $\mathcal{T}$ is bijective, it has an inverse on its range. It can be noted that $\mathcal{T}^{-1}$ is an extended expanding cyclic orbital contraction.

Since $\frac{1}{k_x} < 1$, by using Theorem 2.8, we can easily prove that $\mathcal{T}^{-1}$ has a unique fixed point in $\mathcal{A} \cap \mathcal{B}$.

□

Theorem 2.10. *Let (X, E_b) be a complete E_b-metric space such that E_b is continuous functional. If $\mathcal{T} : X \to X$ is surjective then there exists a mapping $\mathcal{T}^* : X \to X$ such that $\mathcal{T} \circ \mathcal{T}^*$ is the identity map on X.*

The proof is omitted as it is easy to prove.

Definition 2.11. *Let (X, E_b) be a complete E_b-metric space such that E_b is continuous functional. Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets of an E_b-metric space (X, E_b). A cyclic map $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is called an extended cyclic orbital $\mathcal{F}$-expanding contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for some $x \in \mathcal{A}$,*

$$E_b(x, y) > 0; \quad F(E_b(\mathcal{T}^{2n}x, \mathcal{T}y)) \geq F(E_b(\mathcal{T}^{2n-1}x, y)) + \tau \tag{2.8}$$

$n \in \mathbb{N}, y \in \mathcal{A}$ such that for each $x_0 \in X$, $\lim_{n,m\to\infty} s(x_n, x_m) < 1$. Here $x_n = \mathcal{T}^n x_0$; $n = 1, 2, 3...$

Theorem 2.12. *Let (X, E_b) be a complete E_b-metric space such that E_b is continuous functional and let $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets of an E_b-metric space (X, E_b). Suppose $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is an extended cyclic orbital $\mathcal{F}$-expanding contraction and surjective. Then $\mathcal{A} \cap \mathcal{B}$ is non-empty and $\mathcal{T}$ has a unique fixed point in $\mathcal{A} \cap \mathcal{B}$.*

Proof. Suppose there exists an $x \in \mathcal{A}$ (say x_0) satisfying (8). From Theorem 2.10, there exists a mapping $\mathcal{T}^* : X \to X$ such that $\mathcal{T} \circ \mathcal{T}^*$ is the identity mapping on X. If $x_0 = \mathcal{T}x_0$, there is nothing to prove. Thus, let us suppose that $x_0 \neq \mathcal{T}x_0$ (say (y)) and let $\eta = \mathcal{T}^*x$ and $\gamma = \mathcal{T}^*y$. Obviously, $\eta \neq \gamma$ (since $\mathcal{T}$ is injective). From the definition of an extended cyclic orbital $\mathcal{F}$-expanding contraction, let us define for $n \in \mathbb{N}$

$$\mathcal{T}^{2n}x = \mathcal{T}^{*2n-1}x = Ix.$$

Since

$$\begin{aligned}
\mathcal{T}^{2n}\eta &= \mathcal{T}^{2n}(\mathcal{T}^*x) \\
&= \mathcal{T}^{2n-1}(\mathcal{T} \circ \mathcal{T}^*x) \\
&= \mathcal{T}^{2n-1}(x) \\
&= \mathcal{T}^{*2n-1}x
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^{2n-1}\eta &= \mathcal{T}^{2n-1}(\mathcal{T}^*x) \\
&= \mathcal{T}^*(\mathcal{T}^{2n-1}x) \\
&= \mathcal{T}^*(\mathcal{T}^{*2n-1}x) \\
&= \mathcal{T}^{*2n}x
\end{aligned}$$

From (8),

$$F(E_b(\mathcal{T}^{*2n-1}x, y)) \geq F(E_b(\mathcal{T}^{*2n}x, \mathcal{T}^*y)) + \tau$$

Thus $\mathcal{T}^*$ satisfies an extended cyclic orbital $\mathcal{F}$-expanding contraction. Thus from the hypothesis, we can prove that $\mathcal{T}^*$ has a unique fixed point, say $\delta \in X$.

Now we will prove that δ is the fixed point of $\mathcal{T}$.
Consider $\mathcal{T}\delta = \mathcal{T}(\mathcal{T}^*\delta) = \mathcal{T} \circ \mathcal{T}^*\delta = \delta$. Thus δ is also a fixed point of $\mathcal{T}$.
To prove uniqueness, let us suppose that $\mathcal{T}$ has two fixed points say δ_1 and δ_2 $(\delta_1 \neq \delta_2)$.
Thus, $\mathcal{T}\delta_1 = \delta_1$ and $\mathcal{T}\delta_2 = \delta_2$.

$$\begin{aligned} F(E_b(\delta_1, \delta_2)) &= F(E_b(\delta_1, \mathcal{T}\delta_2)) \\ &= F(E_b(\mathcal{T}^{2n}\delta_1, \mathcal{T}\delta_2)) \\ &\geq F(E_b(\mathcal{T}^{2n-1}\delta_1, \delta_2)) + \tau \\ &= F(E_b(\delta_1, \delta_2)) + \tau \end{aligned}$$

Hence $0 = F(E_b(\delta_1, \delta_2)) - F(E_b(\delta_1, \delta_2)) \geq \tau$, which is a contradiction. Hence $\delta_1 = \delta_2$. □

3 Characterization of the Hardy and Rogers mapping theorem for (a pair of) non-self maps in a complete metric space

Let A and B be nonempty subsets of a metric space X. We use the following notations subsequently.

(a) $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$

(b) $A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}$

(c) $B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}$

Definition 3.1. *Let $S : A \mapsto B$ and $T : B \mapsto A$ be non-self mappings. The pair (S, T) will be said to form a proximal cyclic Hardy and Rogers type mapping if there exists a nonnegative number $k < \frac{1}{5}$ such that $d(u, Sx) = d(A, B)$ and $d(v, Ty) = d(A, B)$ implies that*

$$d(u, v) \leq k[d(x, u) + d(y, v) + d(x, v) + d(y, u) + d(x, y)] + (1 - 5k)d(A, B)$$

for all $x, u \in A$ and $y, v \in B$.

Remark 3.2. *In the above definition, if $A = B$ and $S = T$ then T is a Hardy and Rogers type mapping.*

Definition 3.3. *A mapping $S : A \mapsto B$ will be called a proximal Hardy and Rogers type mapping of the first kind if there exists a nonnegative number $k < \frac{1}{5}$ such that $d(u_1, Sx_1) = d(A, B)$ and $d(u_2, Sx_2) = d(A, B)$ implies that*

$$d(u_1, u_2) \leq k[d(x_1, x_2) + d(x_1, u_1) + d(x_2, u_2) + d(x_2, u_1) + d(x_1, u_2)]$$

for all $u_1, u_2, x_1, x_2 \in A$.

Remark 3.4. *In the above definition, if $A = B$, then S is a self map, and thus a Hardy and Rogers type mapping.*

Definition 3.5. *A mapping $S : A \mapsto B$ will be called a proximal Hardy and Rogers type mapping of the second kind if there exists a nonnegative number $k < \frac{1}{5}$ such that $d(u_1, Sx_1) = d(A, B)$ and $d(u_2, Sx_2) = d(A, B)$ implies that*

$$d(Su_1, Su_2) \leq k[d(Sx_1, Sx_2)+d(Sx_1, Su_1)+d(Sx_2, Su_2)+d(Sx_2, Su_1)+d(Sx_1, Su_2)]$$

for all $u_1, u_2, x_1, x_2 \in A$.

Remark 3.6. *A self mapping that is a proximal Hardy and Rogers type mapping of the second kind is a Hardy and Rogers type mapping.*

Definition 3.7. *Given a mapping $S : A \mapsto B$ and an isometry $g : A \mapsto A$, the mapping S is said to preserve isometric distance with respect to g if*

$$d(Sgx_1, Sgx_2) = d(Sx_1, Sx_2)$$

for all $x_1, x_2 \in A$.

Definition 3.8. *An element $x \in A$ is said to be a best proximity point of the mapping $S : A \mapsto B$ if it satisfies the condition $d(x, Sx) = d(A, B)$.*

Remark 3.9. *If the underlying map in the previous definition is a self-mapping, then the best proximity point reduces to a fixed point.*

Theorem 3.10. *Let A and B be non-void closed subsets of a complete metric space such that A_0 and B_0 are non-void. Let $S : A \mapsto B$, $T : B \mapsto A$, and $g : A \cup B \mapsto A \cup B$ satisfy the following conditions*

(a) *S and T are proximal Hardy and Rogers type mappings of the first kind;*

(b) *$S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$;*

(c) *The pair (S, T) forms a proximal cyclic Hardy and Rogers type mapping;*

(d) *g is an isometry;*

(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

Then there exists a unique element $x \in A$ and a unique element $y \in B$ satisfying the conditions that

$$d(gx, Sx) = d(A, B)$$

$$d(gy, Ty) = d(A, B)$$

$$d(x, y) = d(A, B)$$

Proof. Let x_0 be an element in A_0. In view of the fact that $S(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, it follows that there exists an element x_1 in A_0 such that $d(gx_1, Sx_0) = d(A, B)$. Again since $S(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, it follows that there exist an element x_2 in A_0 such that $d(gx_2, Sx_1) = d(A, B)$. Continuing, one has $d(gx_{n+1}, Sx_n) = d(A, B)$ for all $n \geq 0$, since $S(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$. Since g is an isometry and S is a proximal Hardy and Rogers type mapping of the first kind, we have,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(gx_n, gx_{n+1}) \\ &\leq k[d(x_n, x_{n+1}) + d(x_n, x_{n-1}) + d(x_{n+1}, x_n) + d(x_{n+1}, x_{n-1}) + d(x_n, x_n)] \\ &= k[2d(x_n, x_{n+1}) + d(x_n, x_{n-1}) + d(x_{n+1}, x_{n-1})] \\ &\leq k[2d(x_n, x_{n+1}) + d(x_n, x_{n-1}) + d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\ &= 3kd(x_n, x_{n+1}) + 2kd(x_{n-1}, x_n) \end{aligned}$$

Put $\alpha := \frac{2k}{1-3k} < 1$, then from the above it follows that $d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n)$. Consequently, the sequence $\{x_n\}$ is Cauchy and converges to some element $x \in A$ by completeness of the space. Similarly, since $T(B_0)$ is contained in A_0 and B_0 is contained in $g(B_0)$, it follows that there is a sequence $\{y_n\}$ of elements in B_0 such that $d(gy_{n+1}, Ty_n) = d(A, B)$. Since g is an isometry and T is a proximal Hardy and Rogers type mapping of the first kind, we have, $d(y_n, y_{n+1}) \leq \alpha d(y_{n-1}, y_n)$ with $\alpha := \frac{2k}{1-3k} < 1$. Consequently, the sequence $\{y_n\}$ is Cauchy and converges to some element $y \in B$ by completeness of the space. Since the pair (S, T) forms a proximal cyclic Hardy and Rogers type mapping and g is an isometry, we deduce the following

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(gx_{n+1}, gy_{n+1}) \\ &\leq \alpha d(x_n, y_n) + (1-\alpha)d(A, B) \end{aligned}$$

where $\alpha := \frac{2k}{1-3k} < 1$. Now going in the limit of the inequality immediately above, we deduce that $d(x, y) = d(A, B)$. It follows that x is a member of A_0 and y is a member of B_0. Since $S(A_0)$ is contained in B_0 and $T(B_0)$ is contained in A_0, there is an element $u \in A$ and an element $v \in B$ such that $d(u, Sx) = d(A, B)$ and $d(v, Ty) = d(A, B)$. Since S is a proximal Hardy and Rogers type mapping of the first kind, then it follows that

$$d(u, gx_{n+1}) \leq \alpha d(x, x_n)$$

where $\alpha := \frac{2k}{1-3k} < 1$. Thus in the limit of the inequality immediately above, we get $u = gx$ and so $d(gx, Sx) = d(A, B)$. Similarly, it can be shown that $v = gy$, and so $d(gy, Ty) = d(A, B)$. Finally, we show uniqueness. We suppose that $d(gx^*, Sx^*) = d(A, B)$ for $x^* \in A$ and $d(gy^*, Sy^*) = d(A, B)$ for $y^* \in B$. Since g is an isometry, S and T are proximal Hardy and Rogers type mappings of the first kind, it follows that with $\alpha := \frac{2k}{1-3k} < 1$, one has

$$\begin{aligned} d(x, x^*) &= d(gx, gx^*) \\ &\leq \alpha d(x, x^*) \end{aligned}$$

$$\begin{aligned} d(y, y^*) &= d(gy, gy^*) \\ &\leq \alpha d(y, y^*) \end{aligned}$$

From the above two inequalities we conclude that $x = x^*$ and $y = y^*$, and the proof is finished. □

If g is the identity in the above theorem, then we get the following.

Corollary 3.11. *Let A and B be non-void closed subsets of a complete metric space such that A_0 and B_0 are non-void. Let $S : A \mapsto B$ and $T : B \mapsto A$ satisfy the following conditions*

(a) *S and T are proximal Hardy and Rogers type mappings of the first kind*

(b) *$S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$*

(c) *The pair (S, T) forms a proximal cyclic Hardy and Rogers type mapping.*

Then there exists a unique element $x \in A$ and a unique element $y \in B$ satisfying the conditions that

$$d(x, Sx) = d(A, B)$$

$$d(y, Ty) = d(A, B)$$

$$d(x, y) = d(A, B)$$

Remark 3.12. *If $S = T$ in the above Corollary or $S = T$ and g is the identity in Theorem 3.10, then the open problem contained in [12] is solved.*

Remark 3.13. *If, in addition to the conditions in the previous remark, we take $A = B$, then we get Theorem 1(a) contained in [13].*

4 Open Problem

How do we characterize the $(\delta, 1 - 3\delta)$-weak contraction mapping theorem contained in [14] for a non-self map?

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