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Constraint Qualifications for Uncertain Convex Optimization without Convexity of Constraint Data Uncertainty

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Abstract : Considering an uncertain convex optimization problem, within the present paper we study constraint qualifications as well as necessary and sufficient optimality conditions. Following the robust optimization approach, we present constraint qualifications for Lagrange multiplier characterizations of the robust constrained convex optimization with a robust convex feasible set described by locally Lipschitz constraints which are satisfied for all possible uncertainties within the prescribed uncertainty sets and establish relations among various known constraint qualifications. A new constraint qualification are described, and it is shown that this constraint qualification is the weakest constraint qualification for guaranteeing the Lagrange multiplier conditions to be necessary for optimality of the robust constrained convex optimization problem. Consequently, we present how the robust best approximation that is immunized against data uncertainty can be obtained by characterizing the best approximation to any x from the robust counterpart of the robust feasible set without convexity of constraint data uncertainty, improving the corresponding results in the literature.

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1 Introduction

Constraint qualifications are corner stones to solve the classical convex programming problems, which commonly referred to the problem of minimizing a convex function subject to convex inequality constraints, because they guarantee the existence of Lagrange multipliers for optimality. The best-known constraint qualifications are the interior point conditions, also known as the Slater-type constraint qualifications which commonly used to obtain necessary and sufficient Karush-Kuhn-Tucker conditions. Nevertheless, the Karush-Kuhn-Tucker conditions may fail under the Slater-type constraint qualification to characterize optimality of the following convex optimization:

$$\min_{x\in\mathbb{R}^n}\{f(x) : x \in C,\ g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}, \tag{P}$$

where C is a nonempty closed convex subset of the Euclidean space $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous convex function, the functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are differentiable functions, and the set $\{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$, is a nonempty convex subset of $\mathbb{R}^n$. For a simple example, as shown in [1, Example 3.4], it can be observed that the following set in $\mathbb{R}$, $\{x \in \mathbb{R} : x^3 \leq 0\} = -\mathbb{R}_+$ is convex and the constraint function x^3 is not convex but differentiable and the Slater constraint qualification is satisfied. Moreover, $\bar{x} := 0$ is a global minimizer of the problem (P) with $C := \mathbb{R}$ while the KarushKuhnTucker condition does not hold at $\bar{x}$. In fact, Slaters condition along with an additional condition on the constraints has been shown to ensure that the KarushKuhnTucker conditions are necessary and sufficient for optimality of the problem (P) in the case of $C = \mathbb{R}^n$ [2, 3, 4, 5], where apart from [2, 5] in other references inequality constraints are not assumed to be differentiable. Recent reviews of constraint qualifications and all relations among those constraint qualifications can be found in [6].

Recently, new links among various known constraint qualifications that guarantee necessary KarushKuhnTucker conditions for the problem (P) was discussed in [1], where each g_i, $i = 1, 2, \ldots, m$, was assumed to be continuously differentiable. Consequently, the authors presented Lagrange multiplier characterizations for the the best approximation to any x in $\mathbb{R}^n$ from the set $C \cap \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$, which commonly assume accurate values for the data or parameters in the constraints $g_i(x) \leq 0,\ i = 1, 2, \ldots, m$. Unfortunately, in reality, such precise information is rarely available because of forecasting errors or lack of complete information [7, 8]. In addition, this problem (P) in the face of constraint data uncertainty can be captured by the following optimization problem:

$$\min_{x\in\mathbb{R}^n}\{f(x) : x \in C,\ g_i(x, v_i) \leq 0,\ i = 1, 2, \dots, m\}, \tag{UP}$$

where v_i is the uncertain parameter which belongs to an uncertainty set $\mathcal{V}_i \subseteq \mathbb{R}^{q_i}$ and $g_i : \mathbb{R}^n \times \mathcal{V}_i \to \mathbb{R}$, $i = 1, 2, \dots, m$, are given functions.

The computationally powerful approach to dealing with the data uncertainty is to treat uncertainty as deterministic and is known as robust optimization [7, 8, 9, 10]. Following the robust optimization approach [7], one usually associates the so-called *robust (or worst-case) counterpart* of (UP)

$$\min_{x\in\mathbb{R}^n}\{f(x) : x \in C,\ g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ \ i = 1, 2, \dots, m\}, \tag{RP}$$

where the uncertain constraint are enforced for every possible value of the parameters within their prescribed uncertainty and the global minimizer of the problem (RP) is known as *robust optimal solution* of the problem (UP). In almost all existing literature on robust convex optimization [11, 12, 13, 14, 15, 16, 17, 18, 19] and other references therein, the convexity assumption on the functions $g_i(\cdot, v_i)$, $i = 1, 2, \dots, m$, for all $v_i \in \mathcal{V}_i$, is principle and restrictive. In fact, even if $g_i(\cdot, v_i)$, $i = 1, 2, \dots, m$, are not convex for all $v_i \in \mathcal{V}_i$, it may happen that the so-called *robust feasible set* $\{x \in \mathbb{R}^n :\ x \in C,\ g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, 2, \dots, m\}$ is convex (see Example 4.4).

In this article, we aim to investigate constraint qualifications for Lagrange multiplier characterizations of the robust constrained convex optimization (RP) by just imposing the robust feasible set to be convex while $g_i(\cdot, v_i)$, $i = 1, 2, \dots, m$, are not assumed to be convex functions for all $v_i \in \mathcal{V}_i$, and to establish relations among various known constraint qualifications for Lagrange multiplier characterizations of the problem (RP), such as an extended nonsmooth MangasarianFromovitz constraint qualification [20] and the strong conical hull intersection property [21]. We also present a new constraint qualification which completely characterizes the robust optimal solution of (UP) in the sense that the constraint qualification holds if and only if for each robust optimal solution of (UP), there exist Lagrange multipliers satisfying the KarushKuhnTucker conditions. As an application, we establish the robust constrained best approximation in terms of Lagrange multipliers under a new constraint qualification.

The layout of the paper is as follows. Section 2 collects definitions, notations and preliminary results that will be used later in the paper. Section 3 discuss the relations among various known constraint qualifications of robust convex optimization. Section 4 establishes the weakest constraint qualification, and then obtains some complete characterizations of robust forms of Karush-Kuhn-Tucker conditions to be necessary and sufficient for robust optimal solutions of an uncertain convex optimization problem (UP). Section 5 provides Lagrange multiplier characterizations for the robust best approximation from the convex set $C \cap \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, 2, \dots, m\}$.

2 Preliminaries

We begin this section by fixing certain notations, definitions and preliminary results that will be used throughout the paper. We denote by $\mathbb{R}^n$ the Euclidean space with dimension n whose norm is denoted by $\|\cdot\|$ and $\langle x, y\rangle$ denotes the usual inner product between two vectors x, y in $\mathbb{R}^n$, that is, $\langle x, y\rangle = x^T y$. Let $\mathbb{R}^n_+ := \{x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n\}$ be non-negative orthant of $\mathbb{R}^n$. Given a nonempty set $A \subseteq \mathbb{R}^n$, we denote the *interior* of the set A by int$A$. We recall that a set A is *convex* whenever $\lambda x + (1-\lambda)y \in A$ for all $\lambda \in [0,1]$, $x, y \in A$. A set A is said to be a *cone* if $\lambda A \subseteq A$ for all $\lambda \geq 0$. The *polar cone* of A is defined by $A^\circ := \{u \in \mathbb{R}^n : \langle \xi, x\rangle \leq 0,\ \forall x \in A\}$. The *normal cone* at x to a closed convex set A, denoted by $N_A(x)$, is defined by $N_A(x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x\rangle \leq 0,\ \forall y \in A\}$. Furthermore, the *indicator function* $\delta_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of A is defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{elsewhere.} \end{cases}$$

Considering now a function $f : \mathbb{R}^n \to \mathbb{R}$, the *epigraph* of f, epi$f$, is defined as epi$f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$. The function f is said to be *convex* if for all $\mu \in [0,1]$ and $x, y \in \mathbb{R}^n$, $f(\mu x + (1-\mu)y) \leq \mu f(x) + (1-\mu) f(y)$. In addition, the *Fenchel conjugate* of f, f^*, is defined as $f^*(\xi) := \sup_{x \in \mathbb{R}^n}\{\langle \xi, x\rangle - f(x)\}$. Remember the fact that a convex function need not be differentiable everywhere. However if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function then the one-sided or rather right-sided directional derivative always exists and is finite. The *right-sided directional derivative* of f at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is denoted by denoted by $f'(x, d)$, is defined as

$$f'(x, d) := \lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t}.$$

It is important to note that for every fixed x the function $f'(x, .)$ is a positively homogeneous convex function. The *subdifferential of a convex function* f at x is defined as

$$\partial f(x) := \{\xi \in \mathbb{R}^n : f(y) \geq f(x) + \langle \xi, y - x\rangle, \text{ for all } y \in \mathbb{R}^n\}.$$

Definition 2.1. A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally Lipshitz* at $x \in \mathbb{R}^n$, if there exist an open neighborhood U and a constant L and p such that, for all y and z in U, one has

$$|h(y) - h(z)| \leq L\|y - z\|.$$

If the function h is locally Lipshitz at every point $x \in \mathbb{R}^n$, one says that h is a locally Lipshitz function on $\mathbb{R}^n$.

Definition 2.2. *[22]* Let $h : \mathbb{R}^n \to \mathbb{R}$ be locally Lipshitz at a given point $x \in \mathbb{R}^n$. The *Clarke generalized directional derivative* of h at x in the direction $d \in \mathbb{R}^n$, denoted $h^o(x, d)$, is defined as

$$h^o(x, d) := \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{h(y + td) - h(y)}{t}.$$

Definition 2.3. *[22]* Let $h : \mathbb{R}^n \to \mathbb{R}$ be locally Lipshitz at a given point $x \in \mathbb{R}^n$. The *Clarke generalized subdifferential* of h at x, denoted by $\partial^o h(x)$, is defined as

$$\partial^o h(x) := \{\xi \in \mathbb{R}^n : h^o(x,d) \geq \langle \xi, d\rangle \text{ for all } d \in \mathbb{R}^n\}.$$

From the definition of the Clarke generalized subdifferential, it follows that

$$h^o(x,d) = \max_{\xi \in \partial^o h(x)} \langle \xi, d\rangle, \ \forall d \in \mathbb{R}^n.$$

Definition 2.4. Let $h : \mathbb{R}^n \to \mathbb{R}$ be locally Lipshitz at a given point $x \in \mathbb{R}^n$. The function h is said to be *regular* (in the sense of Clarke) at $x \in \mathbb{R}^n$ if $h'(x,\cdot)$ and $h^o(x,\cdot)$ both exist and coincide.

Assumptions [20, p. 2041] Let $\mathcal{V}$ be a compact subset of $\mathbb{R}^q$. Suppose $g : \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$, is a function satisfying the following conditions:

(A1) $g(x,v)$ is upper semicontinuous in (x,v);

(A2) g is locally Lipschitz in the first argument uniformly in the second argument, i.e. for all $x \in \mathbb{R}^n$, there exist neighborhood U of x and a constant $L > 0$ such that for all y and z in U, and $v \in \mathcal{V}$, one has

$$|g(y,v) - g(z,v)| \leq L\|y - z\|;$$

(A3) g is regular with respect to x;

(A4) The generalized gradient $\partial^o_x g(x,v)$ with respect to the first component is upper semicontinuous in (x,v).

Remark 2.5. *Note that, if one of the following conditions holds, then the conditions (A2), (A3), and (A4) hold:*

(i) *The function g is convex in x and continuous in v.*

(ii) *The derivative $\nabla_x g(x,v)$ with respect to x exists and is continuous in (x,v).*

The following lemmas which will be useful in our later analysis.

Lemma 2.6 (Danskin theorem in nonsmooth setting)**.** *[23, Theorem 2](see also [24, Theorem 2.1]) Let $\mathcal{V}$ be a nonempty compact subset of $\mathbb{R}^q$ and let $g : \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$ be such that the conditions (A1)-(A4) are fulfilled. Let $\psi(x) := \sup_{v\in\mathcal{V}} g(x,v)$. Denote $\mathcal{V}(x) := \{v \in \mathcal{V} : g(x,v) = \psi(x)\}$. Then the function ψ is locally Lipschitz, directionally differentiable, regular for each $x \in \mathbb{R}^n$ and*

$$\begin{aligned}\psi^o(x,d) &= \max\{g^o_x(x,v,d) : v \in \mathcal{V}(x)\} \\ &= \max\{\langle \xi, d\rangle : \xi \in \partial^o_x g(x,v), v \in \mathcal{V}(x)\}, \ \forall d \in \mathbb{R}^n.\end{aligned}$$

Lemma 2.7. *[25] Let $\mathcal{V}$ be a nonempty compact convex subset of $\mathbb{R}^q$ and let $g : \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$ be such that the conditions (A1)-(A4) are fulfilled. In addition, suppose that $g(x,\cdot)$ is concave on $\mathcal{V}$, for each $x \in U$. Then the following statements hold:*

(i) *The set* $\mathcal{V}(x)$ *is convex and compact.*

(ii) *The set*

$$\partial_x^o g(x, \mathcal{V}(x)) := \{\xi \in \mathbb{R}^n : \exists v \in \mathcal{V}(x) \text{ s.t. } \xi \in \partial_x^o g(x, v)\}$$

is convex and compact.

(iii) $\partial^o \psi(x) = \partial_x^o g(x, \mathcal{V}(x))$, *where* ψ *is defined in Lemma 2.6.*

We conclude this section by recalling the notion of the strong conical hull intersection property (the strong CHIP, in brief).

Definition 2.8 (Strong CHIP [27, 26])**.** Let C_1 and C_2 be closed convex sets in $\mathbb{R}^n$ and let $x \in C_1 \cap C_2$. Then, the pair $\{C_1, C_2\}$ is said to have the *strong CHIP* at x if

$$(C_1 \cap C_2 - x)^\circ = (C_1 - x)^\circ + (C_2 - x)^\circ.$$

The pair $\{C_1, C_2\}$ is said to have the strong CHIP if it has the strong CHIP at each $x \in C_1 \cap C_2$.

3 Robust type constraint qualifications

In this section, we examine the robust feasible set

$$\Omega := \{x \in \mathbb{R}^n : \ x \in C, \ g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, 2, \dots, m\}, \tag{3.1}$$

where C is a closed convex subset of $\mathbb{R}^n$, $\mathcal{V}_i \subseteq \mathbb{R}^{q_i}$, $i = 1, \dots, m$, are the specified nonempty convex and compact uncertainty sets,

$$K := \{x \in \mathbb{R}^n : \ g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, 2, \dots, m\}, \tag{3.2}$$

each $g_i : \mathbb{R}^n \times \mathcal{V}_i \to \mathbb{R}$, $i = 1, 2, \dots, m$, is a function satisfying the given assumptions (A1)-(A4), and $g_i(x, \cdot)$ is a concave function on $\mathcal{V}_i$ for each $x \in \mathbb{R}^n$. Throughout this paper, we alway suppose that $C \cap K \neq \emptyset$. For each $i = 1, 2, \dots, m$, define a function ψ_i by

$$\psi_i(x) := \max_{v_i \in \mathcal{V}_i} g_i(x, v_i), x \in \mathbb{R}^n.$$

It follows from Lemma 2.6 that, for each $i = 1, 2, \dots, m$, ψ_i is a locally Lipschitz function on $\mathbb{R}^n$, and so is a continuous function. Consequently, the set K is closed and for each $x \in \mathbb{R}^n$ the set, by Lemma 2.7,

$$\mathcal{V}_i(x) := \{v_i \in \mathcal{V}_i : g_i(x, v_i) = \psi_i(x)\}$$

is a nonempty compact subset of $\mathbb{R}^{q_i}$.

Corresponding to any $\bar{x} \in K$, for notational simplicity, we denote $I := \{1, 2, \dots, m\}$ and decompose it into two index sets $I = I_1(\bar{x}) \cup I_2(\bar{x})$, where $I_1(\bar{x}) := \{i \in I : \exists v_i \in \mathcal{V}_i \text{ s.t. } g_i(\bar{x}, v_i) = 0\}$ is the active index set at $\bar{x} \in K$ and $I_2 := I \backslash I_1(\bar{x})$. Let $\mathcal{V}_i^0 := \{v_i \in \mathcal{V}_i : g_i(\bar{x}, v_i) = 0\}$ for $i \in I_1(\bar{x})$.

Definition 3.1.

(i) **(An extended nonsmooth MangasarianFromovitz constraint qualification [20])**. The set K is said to satisfy an *extended nonsmooth MangasarianFromovitz constraint qualification* (ENMFCQ, briefly) at $\bar{x} \in K$ for (RP) (with respect to the given representation) if there exists $d \in \mathbb{R}^n$ such that for each $i \in I_1(\bar{x})$, it holds

$$g^o_{ix}(\bar{x}, v_i, d) < 0, \ \forall v_i \in \mathcal{V}^0_i.$$

(ii) **(Robust nondegeneracy condition)**. One says that K satisfies the *robust nondegeneracy condition* at $\bar{x} \in K$ if for each $i \in I_1(\bar{x})$, it holds

$$0 \notin \partial^o_x g_i(\bar{x}, v_i), \ \forall v_i \in \mathcal{V}^0_i.$$

If the robust nondegeneracy condition holds at every point $x \in K$, one says that K satisfies the robust nondegeneracy condition.

(iii) **(Robust Slater constraint qualification [11])**. The set $\Omega := C \cap K$ is said to satisfy the *robust Slater constraint qualification* (RSCQ for short) if there exists $x_0 \in C$ such that for each $i \in I$, it holds

$$g_i(x_0, v_i) < 0, \ \forall v_i \in \mathcal{V}_i.$$

In [3], the following characterization of a convex set in terms of the Clarke directional derivative in the absent of data uncertainty were proved.

Lemma 3.2. *[3, Proposition 2.2] Let $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, be locally Lipshitz and regular in the sense of Clarke. Let $\widetilde{K} := \{x \in \mathbb{R}^n : h_i(x) \leq 0, \ i = 1, 2, \ldots, m\}$ be nonempty. If $\widetilde{K}$ is convex, then, for each $i = 1, 2, \ldots, m$,*

$$h^o_i(x, y - x) \leq 0, \ \forall x, y \in \widetilde{K} \text{ with } h_i(x) = 0. \tag{3.3}$$

Moreover, if there exists $x_0 \in \mathbb{R}^n$ such that $h_i(x_0) < 0$ for all $i = 1, 2, \ldots, m$ (Slater's constraint qualification), and $0 \notin \partial^o h_i(x)$ whenever $x \in \widetilde{K}$ and $h_i(x) = 0$ (nondegeneracy condition), then, (3.3) *implies that $\widetilde{K}$ is convex.*

Lemma 3.3 (Characterizing convexity of robust feasible set)**.** *Let K be defined as in* (3.2) *and $C = \mathbb{R}^n$. If K is convex, then, for each $\bar{x} \in K$,*

$$g^o_{ix}(\bar{x}, v_i, x - \bar{x}) \leq 0, \ \forall x \in K, \ \forall i \in I_1(\bar{x}), \ \forall v_i \in \mathcal{V}^0_i, \tag{3.4}$$

equivalently,

$$\psi^o_i(\bar{x}, x - \bar{x}) \leq 0, \ \forall x \in K, \ \forall i \in I_1(\bar{x}).$$

Furthermore, if (RSCQ) holds and the robust nondegeneracy condition is satisfied, then (3.4) *implies that K is convex.*

Proof. The conclusion will follow from Lemma 3.2 if we show that ψ_i, $i \in I$, are regular in the sense of Clarke, for any $\bar{x} \in K$ the nondegeneracy condition is satisfied, and the Slater's constraint qualification holds. The first and the second requirements will follow from Lemma 2.6 and Lemma 2.7 that for each $\bar{x} \in K$ and $i \in I_1(\bar{x})$, one has $\mathcal{V}_i^0 = \mathcal{V}_i(\bar{x})$,

$$\psi_i'(\bar{x}, d) = \psi_i^o(\bar{x}, d) = \max\{g_{ix}^o(\bar{x}, v_i, d) : v_i \in \mathcal{V}_i(\bar{x})\}, \ \forall d \in \mathbb{R}^n, \tag{3.5}$$

and

$$0 \in \bigcap_{v_i \in \mathcal{V}_i^0} \mathbb{R}^n \backslash \Big(\partial_x^o g_i(\bar{x}, v_i)\Big) = \mathbb{R}^n \backslash \Big(\bigcup_{v_i \in \mathcal{V}_i(\bar{x})} \partial_x^o g_i(\bar{x}, v_i)\Big) = \mathbb{R}^n \backslash \partial^o \psi_i(\bar{x}).$$

Finally, the robust Slater constraint qualification (RSCQ) leads us to the following strict inequality

$$\psi_i(x_0) = \max\{g_i(x_0, v_i) : v_i \in \mathcal{V}_i\} < 0, \ \forall i \in I \text{ for some } x_0 \in \mathbb{R}^n,$$

which means that the system $x \in \mathbb{R}^n$, $\psi_i(x) \leq 0$ $(i \in I)$ satisfies the Slater's constraint qualification. Taking into account Lemma 3.2, the proof is complete. □

Remark 3.4. *In Lemma 3.3 without the validity of (RSCQ) and robust nondegeneracy condition, we easily obtain that if K is convex, then*

$$K \subseteq \{x \in \mathbb{R}^n : g_{ix}^o(\bar{x}, v_i, x - \bar{x}) \leq 0, \ \forall \bar{x} \in K, \ \forall i \in I_1(\bar{x}), \ \forall v_i \in \mathcal{V}_i^0\},$$

and consequently, for every $\bar{x} \in K$ one has

$$\partial_x^o g_i(\bar{x}, v_i) \subseteq N_K(\bar{x}) \textit{ whenever } i \in I_1(\bar{x}) \textit{ and } v_i \in \mathcal{V}_i^0.$$

Now we turn our attention to the comparison of robust type constraint qualifications.

Theorem 3.5. *Let K be defined as in* (3.2), *C a closed convex subset of $\mathbb{R}^n$, and $\bar{x} \in C \cap K$. Consider the following assertions:*

(a) *there exists $x \in C$ such that, for each $i \in I_1(\bar{x})$,*

$$g_{ix}^o(\bar{x}, v_i, x - \bar{x}) < 0, \ v_i \in \mathcal{V}_i^0.$$

(b) *Robust nondegeneracy condition holds at $\bar{x}$.*

(c) *For each $i \in I_1(\bar{x})$, $v_i \in \mathcal{V}_i^0$, and $\xi_i \in \partial_x^o g_i(\bar{x}, v_i)$, one has*

$$\langle \xi_i, x - \bar{x} \rangle \neq 0 \textit{ for some } x \in K.$$

Then,

(i) *If (a) is satisfied, then (RSCQ) holds. Furthermore, the robust nondegeneracy condition is satisfied at* $\bar{x}$*, that is (b) is true.*

(ii) *If* K *is convex and (RSCQ) holds, then (a), (b) and (c) are equivalent.*

Proof. (i) Suppose that (a) holds, that is there exists $x \in C$ such that, for each $v_i \in \mathcal{V}_i^0$,

$$g_{ix}^o(\bar{x}, v_i, x - \bar{x}) < 0, \; i \in I_1(\bar{x}).$$

It follows easily from Lemma 2.6 that, for each $i \in I_1(\bar{x})$, $\mathcal{V}_i(\bar{x}) = \mathcal{V}_i^0$ and

$$\psi_i^o(\bar{x}, x - \bar{x}) < 0, \; \forall i \in I_1(\bar{x}). \tag{3.6}$$

This together with the regularity of ψ_i, $i \in I_1(\bar{x})$, at $\bar{x}$ implies that, for each $i \in I_1(\bar{x})$,

$$-\psi_i'(\bar{x}, x - \bar{x}) > 0.$$

So, $x \neq \bar{x}$ and for some $\delta_i > 0$, it holds

$$\frac{\psi_i(\bar{x} + t_i(x - \bar{x})) - \psi_i(\bar{x})}{t_i} - \psi_i'(\bar{x}, x - \bar{x}) < -\psi_i'(\bar{x}, x - \bar{x}), \; \forall t_i \in (0, \delta_i).$$

On the one hand, the continuity of ψ_i, $i \in I_2(\bar{x})$ implies that there exists $r_i > 0$ such that $\psi_i(u) < 0$ for all $i \in I_2(\bar{x})$, and $u \in \mathbb{B}_{r_i}(\bar{x}) := \{w \in \mathbb{R}^n : \|w - \bar{x}\| < r_i\}$. Denote

$$\delta := \min\left\{\min_{i \in I_1(\bar{x})} \delta_i, \min_{i \in I_2(\bar{x})} \frac{r_i}{\|x - \bar{x}\|}, 1\right\}.$$

Thus, by taking $t_0 \in (0, \delta)$, one has

$$\psi_i(\bar{x} + t_0(x - \bar{x})) < 0, \; \forall i \in I.$$

Put $x_0 := \bar{x} + t_0(x - \bar{x}) \in C$. Then $\psi_i(x_0) < 0$, $\forall i \in I$, which actually means that there exists $x_0 \in C$ such that $g_i(x_0, v_i) < 0$ for all $v_i \in \mathcal{V}_i$, $i \in I$, that is (RSCQ) is satisfied. Furthermore, clearly, (a) implies (b).

(ii) Next, suppose that K is convex and (RSCQ) is satisfied. We shall prove that (a), (b) and (c) are all equivalent. It is sufficient to prove the following cases.

[(b)⇒(a)]. Suppose that (b) holds. Assume by contradiction that for any $x \in C$ there exists $i_0 \in I_1(\bar{x})$ such that

$$\psi_{i_0}^o(\bar{x}, x - \bar{x}) \geq 0. \tag{3.7}$$

On the other hand, by the assumption, there exists $x_0 \in C$ such that $g_i(x_0, v_i) < 0$ for all $v_i \in \mathcal{V}_i$, $i \in I$. That is, $\psi_i(x_0) < 0$ for all $i \in I$. Then, since ψ_i, $i \in I$ is continuous at x_0, there exists $r > 0$ such that $\psi_i(x_0 + ru) < 0$ for all $i \in I$, and $u \in \mathbb{B} := \{w \in \mathbb{R}^n : \|w\| < 1\}$. That is, $x_0 + ru \in K$ for all $u \in \mathbb{B}$. So, since $\bar{x} \in K$ and K is convex, we conclude from Lemma 3.3 that

$$\psi_i^o(\bar{x}, x_0 + ru - \bar{x}) \leq 0, \; \forall u \in \mathbb{B}, \; \forall i \in I_1(\bar{x}). \tag{3.8}$$

In particular, taking $u = 0 \in \mathbb{B}$, it leads to

$$\psi_i^o(\bar{x}, x_0 - \bar{x}) \leq 0, \ \forall i \in I_1(\bar{x}). \tag{3.9}$$

This together with the fact that $x_0 \in C$ and (3.7) in turn gives us the following equality $\psi_{i_0}^o(\bar{x}, x_0 - \bar{x}) = 0$. Thus, by Lemma 2.6 and the assertion (b), we can find $\bar{v}_{i_0} \in \mathcal{V}_{i_0}(\bar{x}) = \mathcal{V}_{i_0}^0$ and $\xi_{i_0} \in \partial_x^o g_{i_0}(\bar{x}, \bar{v}_{i_0})\backslash\{0\}$ such that

$$\langle \xi_{i_0}, x_0 - \bar{x} \rangle = 0.$$

It then follows from (3.8) that for every $u \in \mathbb{B}$,

$$r\langle \xi_{i_0}, u \rangle = \langle \xi_{i_0}, ru \rangle + \langle \xi_{i_0}, x_0 - \bar{x} \rangle = \langle \xi_{i_0}, x_0 + ru - \bar{x} \rangle \leq \psi_{i_0}^o(\bar{x}, x_0 + ru - \bar{x}) \leq 0. \tag{3.10}$$

By taking $u := \frac{1}{\|\xi_{i_0}\|}\xi_{i_0}$ in (3.10), we get $0 \geq \langle \xi_{i_0}, \frac{1}{\|\xi_{i_0}\|}\xi_{i_0} \rangle = \frac{1}{\|\xi_{i_0}\|}\|\xi_{i_0}\|^2 = \|\xi_{i_0}\|$, which implies that $\xi_{i_0} = 0$. This contradicts the fact that $\xi_{i_0} \neq 0$, and completes the proof the implication $(b) \Rightarrow (a)$.

[(b)⇒(c)]. The proof is done by contradiction. Assume assertion (b) holds true and there exist $i_0 \in I_1(\bar{x})$, $v_{i_0} \in \mathcal{V}_{i_0}^0$ and $\xi_{i_0} \in \partial_x^o g_{i_0}(\bar{x}, v_{i_0})$ such that

$$\langle \xi_{i_0}, x - \bar{x} \rangle = 0, \ \forall x \in K. \tag{3.11}$$

As $0 \notin \partial_x^o g_{i_0}(\bar{x}, v_{i_0})$, we have $\xi_{i_0} \neq 0$. Since, Ω satisfies (RSCQ), there exists $x_0 \in C$ such that $\psi_i(x_0) < 0$ for all $i \in I$. Then, by continuity of ψ_i, $i \in I$, there exists $r > 0$ such that $\psi_i(x_0 + ru) < 0$ for all $i \in I$, and all $u \in \mathbb{B}$. This implies that $x_0 + ru \in K$ for all $u \in \mathbb{B}$. In view of (3.11), one has

$$\langle \xi_{i_0}, x_0 + ru - \bar{x} \rangle = 0, \ \forall u \in \mathbb{B}. \tag{3.12}$$

Taking $u = 0$ in relation (3.12), we conclude that $\langle \xi_{i_0}, x_0 - \bar{x} \rangle = 0$. This together with (3.12) yields $\langle \xi_{i_0}, u \rangle = 0, \ \forall u \in \mathbb{B}$. It follows that $0 = \langle \xi_{i_0}, \frac{1}{\|\xi_{i_0}\|}\xi_{i_0} \rangle = \|\xi_{i_0}\|$, that is, $\xi_{i_0} = 0$. This contradicts the fact that $\xi_{i_0} \neq 0$. So, (c) holds.

Clearly, (c) implies (b) without the validity of (RSCQ) and the convexity of K. □

Remark 3.6. *Note that in Theorem 3.5, if each function $g_i(\cdot, v_i)$, $i \in I$, is convex for all $v_i \in \mathcal{V}_i$ and (RSCQ) holds, then the robust nondegeneracy condition is satisfied at $\bar{x}$. Indeed, if there exist $i \in I_1(\bar{x})$ and $\bar{v}_i \in \mathcal{V}_i^0$ such that $0 \in \partial_x^0 g_i(\bar{x}, \bar{v}_i) = \partial_x g_i(\bar{x}, \bar{v}_i)$, then, by the convexity of the function $g_i(\cdot, \bar{v}_i)$, one has $0 = g_i(\bar{x}, \bar{v}_i) \leq g_i(x, \bar{v}_i)$ for all $x \in \mathbb{R}^n$. So, the robust Slater constraint qualification (RSCQ) does not hold, which is a contradiction.*

Also, even in the case where $g_i(\cdot, v_i)$, $i \in I$, are convex for each $v_i \in \mathcal{V}_i$, it may happen that $0 \notin \partial_x^o g_i(\bar{x}, v_i)$ whenever $\bar{x} \in K$, $i \in I_1(\bar{x})$ and $v_i \in \mathcal{V}_i^0$, while there is no $x_0 \in \mathbb{R}^n$ such that $g_i(x_0, v_i) < 0$ for all $v_i \in \mathcal{V}_i$ and all $i \in I$. For example, let $x := (x_1, x_2) \in \mathbb{R}^2$, $C := \mathbb{R}^2$, $\mathcal{V}_1 := [0, 1]$, $\mathcal{V}_2 := [-1, 0]$, $g_1(x, v_1) := x_1 + v_1 x_2$, $g_2(x, v_2) := -x_1 + v_2$ and $\bar{x} := (0, 0)$. Then $I_1(\bar{x}) = \{1, 2\}$, $\mathcal{V}_1^0 = [0, 1]$, $\mathcal{V}_2^0 = \{0\}$, $\partial_x^o g_1(\bar{x}, v_1) = \{(1, v_1)\}$ and $\partial_x^o g_2(\bar{x}, v_2) = \{(-1, 0)\}$. We see *that* $(0,0) \notin \partial_x^o g_1(\bar{x}, v_1)$ *for all* $v_1 \in \mathcal{V}_1^0$ *and* $(0,0) \notin \partial_x^o g_2(\bar{x}, v_2)$ *for all* $v_2 \in \mathcal{V}_2^0$. *On the other hand, assume if possible that there exists* $\hat{x} := (\hat{x}_1 \hat{x}_2) \in \mathbb{R}^2$ *such that* $\hat{x}_1 + v_1\hat{x}_2 < 0$ *for all* $v_1 \in \mathcal{V}_1$ *and* $-\hat{x}_1 + v_2 < 0$ *for all* $v_2 \in \mathcal{V}_2$. *It follows that* $\hat{x}_1 < 0$ *and* $-\hat{x}_1 < 0$, *a contradiction.*

Remark 3.7. *From the proof of Theorem 3.5, we see that the implications (a)⇒(b) and the equivalence (b)⇔(c) do not require the convexity of* K. *However, it is worth noting that the the implication (b)⇒(a) is not valid without the convexity of* K. *This fact will demonstrate in Example 3.10.*

Corollary 3.8. *If* K *is a convex set given by* (3.2), *and (RSCQ) holds, then the following assertions are equivalent:*

(i) K *satisfies (ENMFCQ) at* $\bar{x}$.

(ii) *The condition (a) in Theorem 3.5 is satisfied at* $\bar{x}$.

Proof. It is clear that the implication (ii)⇒(i) holds. Conversely, if K satisfies (ENMFCQ) at $\bar{x}$, then, by the definition of Clarke generalized subdifferential, $0 \notin \partial_x^o g_i(\bar{x}, v_i)$ for all $v_i \in \mathcal{V}_i^0$ and $i \in I_1(\bar{x})$. Thus, the robust nondegeneracy condition is satisfied at $\bar{x}$, and so the condition (a) will follow from Theorem 3.5 (the equivalence (a)⇔(b)). □

Corollary 3.9. *If* $C = \mathbb{R}^n$, K *is a convex set given by* (3.2), *and* $\bar{x} \in K$, *then the following assertions are equivalent:*

(i) K *satisfies (ENMFCQ) at* $\bar{x}$.

(ii) *The condition (RSCQ) holds and the robust nondegeneracy condition is satisfied at* $\bar{x}$.

Proof. We see, in the case of $C = \mathbb{R}^n$, that (ENMFCQ) is equivalent to (a) in Theorem 3.5, and so the result follows from Theorem 3.5 (the equivalence (a)⇔(b)). □

Example 3.10. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $K := \{x \in \mathbb{R}^2 : g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, 2\}$, where $g_1(x, v_1) := x_1^3 - v_1 x_2 + v_1 - 1$, $g_2(x, v_2) := -x_1^2 + x_2 + v_2$, $\mathcal{V}_1 := [0, 1]$ and $\mathcal{V}_2 := [-1, 0]$. We see that $g_1((-1, 0), v_1) = -2 < 0$ for all $v_1 \in \mathcal{V}_1$ and $g_2((-1, 0), v_2) = -1 + v_2 < 0$ for all $v_2 \in \mathcal{V}_2$, that is, robust Slater constraint qualification (RSCQ) holds. Also, for $\bar{x} := (0, 0)$, one has $I_1(\bar{x}) = \{1, 2\}$, $\mathcal{V}_1^0 = \{1\}$ $\mathcal{V}_2^0 = \{0\}$, $(0, 0) \notin \partial_x^o g_1(\bar{x}, 1) = \{(0, -1)\}$ and $(0, 0) \notin \partial_x^o g_2(\bar{x}, 0) = \{(0, 1)\}$, which implies that the robust nondegeneracy holds at $\bar{x}$. It should be observed that K is not convex, and moreover (ENMFCQ) is invalid at $\bar{x}$. In fact, there is no $\hat{x} := (\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $0 > g_{1x}^o(\bar{x}, 1, \hat{x}) = -\hat{x}_2$ and $0 > g_{2x}^o(\bar{x}, 0, \hat{x}) = \hat{x}_2$. So, in the absence of the convexity of K, the validity of both the robust Slater constraint qualification (RSCQ) and the robust nondegeneracy condition at $\bar{x}$ does not guarantee the validity of (ENMFCQ) at $\bar{x} \in K$, and consequently, the condition (a) as well.

4 Weakest CQ: multiplier characterization

Along the lines of [1, 13], in this section, we present a new robust type constraint qualification characterizing the robust optimal solution of the problem (UP) where each g_i, $i \in I$, is satisfied the condition (A1)-(A4) and additionally $g_i(x,\cdot)$ is a concave function on $\mathcal{V}_i$ for all $x \in \mathbb{R}^n$, and K is a convex set given by (3.2).

Definition 4.1 (G-RS Strong CHIP)**.** The pair $\{C, K\}$ is said to have the *generalized robust sharpened strong conical hull intersection property* (the G-RS Strong CHIP) at $x \in C \cap K$ if

$$(C \cap K - x)^\circ = \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in \Pi_{i\in I}\mathcal{V}_i}} \left\{ \sum_{i \in I} \lambda_i \partial^o_x g_i(x, v_i) : \lambda_i g_i(x, v_i) = 0,\ i \in I \right\} + (C - x)^\circ, \tag{4.1}$$

where $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $v := (v_1, v_2, \ldots, v_m)$. The pair $\{C, K\}$ is said to have the G-RS strong CHIP if it has the G-RS strong CHIP at every $x \in C \cap K$.

Remark 4.2. *In the special case when all $\mathcal{V}_i$, $i \in I$, are singletons and all the functions g_i, $i \in I$, are continuously differentiable, the G-RS strong CHIP becomes the generalized sharpened strong conical hull intersection property (the G-S strong CHIP), which was introduced in [1, Definition 3.1].*

Remark 4.3. *In the case of $g_i(\cdot, v_i)$, $i \in I$, are convex for each $v_i \in \mathcal{V}_i$, the G-RS Strong CHIP becomes the following robust type subdifferential constraint qualification,*

$$\partial \delta_{C \cap K}(x) = \partial \delta_C(x) + \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in \Pi_{i\in I}\mathcal{V}_i}} \left\{ \sum_{i \in I} \lambda_i \partial_x g_i(x, v_i) : \lambda_i g_i(x, v_i) = 0,\ i \in I \right\},$$

which was introduced in [18, Definition 3.3].

We point out that the above definition of the G-RS strong CHIP generalized the following notion the so-called *robust sharpened strong conical hull intersection property*, which states an analogous manner as in [21] that

$$(C \cap K - x)^\circ = \tilde{M}(x) + (C - x)^\circ, \tag{4.2}$$

where

$$\tilde{M}(x) := \left\{ u \in \mathbb{R}^n : (u, \langle u, x \rangle) \in \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in \Pi_{i\in I}\mathcal{V}_i}} \operatorname{epi} \left(\sum_{i \in I} \lambda_i g_i(\cdot, v_i) \right)^* \right\}$$

and $g_i(\cdot, v_i)$, $i \in I$, are convex for all $v_i \in \mathcal{V}_i$. In the case where $g_i(\cdot, v_i)$, $i \in I$, are convex for each $v_i \in \mathcal{V}_i$, with the similar methods in [13, Theorem 3.1, p. 289] together with [28, Proposition 2.1.], we can easily obtain that

$$\bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in \Pi_{i \in I} \mathcal{V}_i}} \left\{ \sum_{i \in I} \lambda_i \partial^o_x g_i(x, v_i) : \lambda_i g_i(x, v_i) = 0,\ i \in I \right\} = \tilde{M}(x).$$

However, it is worth to mention that the above equality fails to be true if $g_i(\cdot, v_i)$, $i \in I$, are not convex for all $v_i \in \mathcal{V}_i$, even in the case of K is convex, as the next example illustrates.

Example 4.4. Let $\mathcal{V} := [0, 1]$. For any $v \in \mathcal{V}$, we define $g(x, v) := x - vx^3$ for all $x \in \mathbb{R}$. Then $K := \{x \in \mathbb{R} : g(x, v) \leq 0,\ \forall v \in \mathcal{V}\} = [-1, 0]$, which is convex, while $g(\cdot, v)$ are not convex for each $v \in \mathcal{V}$. We see that, for each $\lambda \in \mathbb{R}_+$, $v \in \mathcal{V}$, and each $\xi \in \mathbb{R}$,

$$(\lambda g(\cdot, v))^*(\xi) = \begin{cases} 0, & \text{if } \lambda = 0,\ \xi = 0, \\ +\infty, & \text{elsewhere.} \end{cases}$$

This implies that $\bigcup_{\substack{\lambda \in \mathbb{R}_+ \\ v \in \mathcal{V}}} \operatorname{epi}(\lambda g(\cdot, v))^* = \{0\} \times \mathbb{R}_+$, and thus, $\tilde{M}(0) = \{0\}$. On the other hand, $\bigcup_{\substack{\lambda \in \mathbb{R}_+ \\ v \in \mathcal{V}}} \{\lambda \partial^o_x g(0, v) : \lambda g(0, v) = 0\} = \mathbb{R}_+$. So, $\bigcup_{\substack{\lambda \in \mathbb{R}_+ \\ v \in \mathcal{V}}} \{\lambda \partial^o_x g(0, v) : \lambda g(0, v) = 0\} \neq \tilde{M}(0)$.

In the sequel, we turn our attention to define for simplicity,

$$M(x) := \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in \Pi_{i \in I} \mathcal{V}_i}} \left\{ \sum_{i \in I} \lambda_i \partial^o_x g_i(x, v_i) : \lambda_i g_i(x, v_i) = 0,\ i \in I \right\}, \tag{4.3}$$

where $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_m)$, and $v := (v_1, v_2, \ldots, v_m)$.

From now on, for each continuous convex function $f : \mathbb{R}^n \to \mathbb{R}$, denote by (RP_f) the following optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x),$$
$$\text{subject to } x \in C \cap K, \tag{4.4}$$

where the robust feasible set K is convex, given by (3.2), and C is closed convex subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$.

In what follows, we now show that the G-RS strong CHIP of $\{C, K\}$ is necessary and sufficient for the Lagrange multiplier characterization for the case where K is convex while $g_i(\cdot, v_i)$, $i \in I$, are not necessarily convex for all $v_i \in \mathcal{V}_i$.

Theorem 4.5 (Weakest CQ for necessary optimality conditions)**.** *Let $\bar{x} \in C \cap K$. Then the following assertions are equivalent:*

(i) $\{C, K\}$ *has the G-RS strong CHIP at $\bar{x}$;*

(ii) *For each continuous convex function $f:\mathbb{R}^n\to\mathbb{R}$ attaining its global minimizer over $C\cap K$ at $\bar{x}$, one has*

$$0\in\partial f(\bar{x})+M(\bar{x})+(C-\bar{x})^\circ. \tag{4.5}$$

Proof. [(i)⇒(ii)]. Suppose that (i) holds. Let f be any continuous convex function such that $\bar{x}\in C\cap K$ is a global minimizer of (RP_f). Using the Fermat rule and the Moreau-Rockafellar theorem [29, Theorem 23.8, p. 223], we get

$$0\in\partial f(\bar{x})+(C\cap K-\bar{x})^\circ.$$

So, in view of (4.1), it follows that (4.5) holds.

[(ii)⇒(i)]. Suppose that (ii) holds. Let $u\in(C\cap K-\bar{x})^\circ$ be arbitrary. Then, by the definition of the polar cone, $\langle u,x-\bar{x}\rangle\leq 0$ for all $x\in C\cap K$. So, noting that $\bar{x}\in C\cap K$, we see that $f(x):=-\langle u,x\rangle,\ x\in\mathbb{R}^n$, is a continuous convex function attaining its global minimizer over $C\cap K$ at $\bar{x}$. By (4.5), it gets

$$0\in\{-u\}+M(\bar{x})+(C-\bar{x})^\circ,$$

whence $u\in M(\bar{x})+(C-\bar{x})^\circ$. This shows that

$$(C\cap K-\bar{x})^\circ\subseteq M(\bar{x})+(C-\bar{x})^\circ. \tag{4.6}$$

To show the reverse inclusion, let us take any $u\in M(\bar{x})+(C-\bar{x})^\circ$. Then there exist $\bar{\lambda}:=(\bar{\lambda}_1,\bar{\lambda}_2,\ldots,\bar{\lambda}_m)\in\mathbb{R}^m_+$, $\bar{v}_i\in\mathcal{V}_i$, $l_i\in\partial^o_x g_i(\bar{x},\bar{v}_i)$, $i\in I$, and $x^*\in(C-\bar{x})^\circ$ such that $\bar{\lambda}_i g_i(\bar{x},\bar{v}_i)=0$ for all $i\in I$ and $u=\sum_{i\in I}\bar{\lambda}_i l_i+x^*$. If $\bar{\lambda}_i=0$ for all $i\in I$, then $\langle u,x-\bar{x}\rangle=\langle x^*,x-\bar{x}\rangle\leq 0$ for every $x\in C\cap K$, and so $u\in(C\cap K-\bar{x})^\circ$. Otherwise, suppose that $\widetilde{I}:=\{i\in I:\bar{\lambda}_i>0\}\neq\emptyset$. Let $x\in C\cap K$ be arbitrary. As $\bar{\lambda}_i g_i(\bar{x},\bar{v}_i)=0$ for all $i\in I$, we have $\widetilde{I}\subseteq I_1(\bar{x})$ and so, $\bar{v}_i\in\mathcal{V}_i^0$ for all $i\in\widetilde{I}$. Thus, by Lemma 3.3,

$$\langle l_i,x-\bar{x}\rangle\leq g^o_{ix}(\bar{x},\bar{v}_i,x-\bar{x})\leq 0,\ \forall i\in\widetilde{I}.$$

It follows that

$$\langle u,x-\bar{x}\rangle=\left\langle\sum_{i\in I}\bar{\lambda}_i l_i+x^*,x-\bar{x}\right\rangle=\left\langle\sum_{i\in\widetilde{I}}\bar{\lambda}_i l_i,x-\bar{x}\right\rangle+\langle x^*,x-\bar{x}\rangle\leq 0.$$

This gives us that, for each $x\in C\cap K$, $\langle u,x-\bar{x}\rangle\leq 0$. So, $u\in(C\cap K-\bar{x})^\circ$, and consequently,

$$M(\bar{x})+(C-\bar{x})^\circ\subseteq(C\cap K-\bar{x})^\circ. \tag{4.7}$$

Therefore, (4.6) and (4.7) give to conclusion that the G-RS strong CHIP holds at $\bar{x}$. □

Definition 4.6 (KKT Condition)**.** Let K be as in (3.2), for the problem (RP_f), let $\bar{x}\in C\cap K$. One says that KKT condition holds at $\bar{x}$ whenever

$$\begin{aligned}0\in&\partial f(\bar{x})+\sum_{i\in I}\lambda_i\partial^o_{ix}g_i(\bar{x},v_i)+(C-\bar{x})^\circ,\\&\text{for some }\lambda_i\geq 0,\ v_i\in\mathcal{V}_i\text{ with }\lambda_i g_i(\bar{x},v_i)=0,\ i\in I.\end{aligned} \tag{4.8}$$

The next theorem presents conditions insuring the G-S strong CHIP to be valid.

Theorem 4.7 (Conditions for the G-RS strong CHIP)**.** *Let K be as in* (3.2)*, and $\bar{x} \in C \cap K$. Suppose that K is a convex set, the condition (RSCQ) holds, and the robust nondegeneracy condition is satisfied at $\bar{x}$. Then, $\{C, K\}$ has the G-RS strong CHIP at $\bar{x}$.*

Consequently, $\bar{x} \in C \cap K$ is a global minimizer of the problem (RP_f), if and only if KKT condition holds at $\bar{x}$ for (RP_f), where the problem (RP_f) is defined by (4.4).

Proof. In the hypothesis that the robust Slater constraint qualification (RSCQ) holds, by topological reformulation, it holds $C \cap \text{int}K \neq \emptyset$. Hence, by the Moreau-Rockafellar theorem [29, Theorem 23.8, p. 223] (see also [30, Proposition 2.3.]), we have

$$(C \cap K - \bar{x})^\circ = (C - \bar{x})^\circ + N_K(\bar{x}).$$

So, by Remark 3.4, it is sufficient to show that

$$N_K(\bar{x}) \subseteq M(\bar{x}). \tag{4.9}$$

To see this, let $u \in N_K(\bar{x})$ be arbitrary. Then $\langle u, x - \bar{x}\rangle \leq 0$ for all $x \in K$. So, we see that $f(x) := -\langle u, x\rangle$, $x \in \mathbb{R}^n$, is a convex function attaining its global minimizer over K at $\bar{x}$. In addition, from the proof of Lemma 3.3, we see by assumption that $0 \notin \partial^o \psi_i(\bar{x})$ whenever $\psi_i(\bar{x}) = 0$, and the system $x \in \mathbb{R}^n$, $\psi_i(x) \leq 0$ $(i \in I)$ satisfies the Slater's condition. So, [3, Theorem 2.4] gives us the multipliers $\bar{\lambda}_i \geq 0$, $i \in I$, satisfying

$$0 \in \{-u\} + \sum_{i \in I} \bar{\lambda}_i \partial^o \psi_i(\bar{x}) \text{ and } \bar{\lambda}_i \psi_i(\bar{x}) = 0, \ \forall i \in I.$$

By taking $\partial^o \psi_i(\bar{x}) = \{\xi_i \in \mathbb{R}^n : \exists v_i \in \mathcal{V}_i(\bar{x}) \text{ s.t. } \xi_i \in \partial_x^o g_i(\bar{x}, v_i)\}$ for each $i \in I$ into consideration, there exist $\bar{v}_i \in \mathcal{V}_i(\bar{x})$, $i \in I$, such that

$$u \in \sum_{i \in I} \bar{\lambda}_i \partial_x^o g_i(\bar{x}, \bar{v}_i).$$

Moreover, for each $i \in I$, we get $\bar{\lambda} g_i(\bar{x}, \bar{v}_i) = \bar{\lambda}_i \psi_i(\bar{x}) = 0$. It follows that $u \in M(\bar{x})$, and hence, (4.9) has been justified. Therefore, the G-RS strong CHIP holds at $\bar{x}$.

Finally, if $\bar{x}$ is a global minimizer of the problem (RP_f), then by Theorem 4.5 (the implication (i)⇒(ii)), there exist $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}_+^m$ and $v := (v_1, v_2, \ldots, v_m) \in \Pi_{i \in I} \mathcal{V}_i$ with $\lambda_i g_i(\bar{x}, v_i) = 0$ for all $i \in I$ such that $0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial_x^o g_i(\bar{x}, v_i) + (C - \bar{x})^\circ$. That is, KKT condition holds at $\bar{x}$.

Conversely, assume that KKT condition holds at $\bar{x}$. Then, it comes that there exist $\bar{\lambda} := (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m) \in \mathbb{R}_+^m$ and $\bar{v} := (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m) \in \Pi_{i \in I} \mathcal{V}_i$ with $\lambda_i g_i(\bar{x}, \bar{v}_i) = 0$ for all $i \in I$ such that $0 \in \partial f(\bar{x}) + \sum_{i \in I} \bar{\lambda}_i \partial_x^o g_i(\bar{x}, \bar{v}_i) + (C - \bar{x})^\circ$. On the other hand, we have

$$\sum_{i \in I} \bar{\lambda}_i \partial_x^o g_i(\bar{x}, \bar{v}_i) + (C - \bar{x})^\circ \subseteq M(\bar{x}) + (C - \bar{x})^\circ = (C \cap K - \bar{x})^\circ.$$

Therefore, by using Moreau-Rockafellar's theorem [29, Theorem 23.8, p. 223], we get

$$0 \in \partial f(\bar{x}) + (C \cap K - \bar{x})^\circ = \partial f(\bar{x}) + \partial \delta_{C\cap K}(\bar{x}) = \partial (f + \delta_{C\cap K})(\bar{x}).$$

The latter, due to the convexity, implies that $\bar{x}$ is a global minimizer of the problem (RP_f). □

Next let us provide an example illustrating Theorem 4.7.

Example 4.8. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $v_1 := (v_{1,1}, v_{1,2})$, $v_2 := (v_{2,1}, v_{2,2})$, $\mathcal{V}_1 := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $\mathcal{V}_2 := [2,3] \times [0,1]$, $K := \{x \in \mathbb{R}^2 : g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1,2\}$, where $g_1(x, v_1) := v_{1,1}x_1 + v_{1,2}x_2 - x_1^3 - 2$ and $g_2(x, v_2) := v_{2,1}x_1 + v_{2,2}x_2$ for all $x := (x_1, x_2) \in \mathbb{R}^2$. Let $C := \{x \in \mathbb{R}^2 : x_2 \leq -x_1\}$. Then, it can be verify that $K = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} - x_1^3 - 2 \leq 0,\ \max\{2x_1 + x_2, x_1\} \leq 0\}$, and so C and K are closed convex subset of $\mathbb{R}^2$ such that $C \cap K \neq \emptyset$. Moreover, $g_1((-\frac{1}{2}, -\frac{1}{2}), v_1) = \frac{1}{2}(-v_{1,1} - v_{1,2}) - \frac{15}{8} \leq \frac{1}{2}(|v_{1,1}| + |v_{1,2}|) - \frac{15}{8} \leq \frac{1}{2}(2\sqrt{x_1^2 + x_2^2}) - \frac{15}{8} \leq -\frac{7}{8} < 0$ for all $v_1 \in \mathcal{V}_1$ and $g_2((-\frac{1}{2}, -\frac{1}{2}), v_2) = -\frac{1}{2}(v_{2,1} + v_{2,2}) < 0$ for all $v_2 \in \mathcal{V}_2$. That is, (RSCQ) holds. Also, for $\bar{x} := (\bar{x}_1, \bar{x}_2) = (0,0)$, we get $I_1(\bar{x}) = \{2\}$, $\mathcal{V}_2^0 = \mathcal{V}_2$ and $(0,0) \notin \partial_x^o g_2(\bar{x}, v_2) = \{(v_{2,1}, v_{2,2})\}$ for all $v_2 \in \mathcal{V}_2^0$. Hence, the robust nondegeneracy condition holds at $\bar{x}$. So, by Theorem 4.7, $\{C, K\}$ has the G-RS strong CHIP at $\bar{x}$.

In the case where $C := \mathbb{R}^n$, the following result follows from Theorem 4.7.

Corollary 4.9. *Consider the problem* (P_f) *with* K *given by* (3.2)*, and* $C := \mathbb{R}^n$*. Let the condition (RSCQ) hold and let the robust nondegeneracy condition be satisfied at* $\bar{x} \in K$*. Suppose that* K *is a convex set and* f *is a convex function. Then,* $\bar{x}$ *is a global minimizer of the problem* (RP_f) *if and only if it is a KKT point.*

Proof. This is an immediate consequence of Theorem 4.7. □

It is worth noting that Corollary 4.9 is not valid if the robust Slater constraint qualification (RSCQ) is removed. The following example illustrates this fact.

Example 4.10. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $C := \mathbb{R}^2$, $f(x) := x_1 + x_2$, $\mathcal{V}_1 := [0,1]$, $\mathcal{V}_2 := [-1, 0]$, $g_1(x, v_1) := x_1 - v_1 x_2^3$, $g_2(x, v_2) := -x_1 + v_2$ for all $x \in \mathbb{R}^2$, $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$. We see that $K := \{x \in \mathbb{R}^2 : g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1,2\} = \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \geq 0\}$, which is convex. Moreover, $\bar{x} := (0,0) \in K$ is a global minimizer of the problem (RP_f) at which the robust nondegeneracy condition is satisfied. However, $\bar{x}$ is not a KKT point. The reason is that the robust Slater constraint qualification (RSCQ) does not hold. Otherwise, if there exists $\hat{x} := (\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $\hat{x}_1 - v_1\hat{x}_2^3 < 0$ for all $v_1 \in \mathcal{V}_1$ and $\hat{x}_1 + v_2 < 0$ for all $v_2 \in \mathcal{V}_2$. It follows that $\hat{x}_1 < 0$ and $-\hat{x}_1 < 0$, a contradiction.

Also, the following example shows that the robust nondegeneracy condition is essential for the validity of Corollary 4.9.

Example 4.11. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $C := \mathbb{R}^2$, $f(x) := -x_1 - x_2$, $v_1 := (v_{1,1}, v_{1,2})$, $v_2 := (v_{2,1}, v_{2,2})$, $v_3 := (v_{3,1}, v_{3,2})$, $\mathcal{V}_1 := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $\mathcal{V}_2 := [0,1] \times [1,2]$, $\mathcal{V}_3 := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $K := \{x \in \mathbb{R}^2 : g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1,2,3\}$, where $g_1(x, v_1) := v_{1,1}x_1 + v_{1,2}x_2 - x_1^3 - 2$, $g_2(x, v_2) := -v_{2,1}x_1^3 + v_{2,2}\max\{-x_2, -x_2^2\}$ and $g_3(x, v_3) := v_{3,1}x_1 + v_{3,2}x_2$ for all $x \in \mathbb{R}^2$. Then, it can be verify that $K = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} \leq 1, \ \max\{-x_1, 0\} - x_2 \leq 0\}$, and so K is closed convex subset of $\mathbb{R}^2$. For $\bar{x} := (\bar{x}_1, \bar{x}_2) = (0,0)$, we have $\partial f(\bar{x}) = \{(-1,-1)\}$, $I_1(\bar{x}) = \{2,3\}$, $\mathcal{V}_2^0 = \mathcal{V}_2$, $\mathcal{V}_3^0 = \mathcal{V}_3$, $\partial_x^0 g_2(\bar{x}, v_2) = \{0\} \times [-v_{2,2}, 0]$ and $\partial_x^0 g_3(\bar{x}, v_3) = \{(v_{3,1}, v_{3,2})\}$. Selecting $\bar{v}_1 := (1,0)$, $\bar{v}_2 := (1,1)$, $\bar{v}_3 := (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\bar{\lambda}_1 := 0$, $\bar{\lambda}_2 := 1$ and $\bar{\lambda}_3 := \sqrt{2}$, we obtain $\bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$ for all $i = 1,2,3$ and

$$(0,0) \in \{(-1,-1)\} + (\{0\} \times [-1,0]) + \{(1,1)\} = \partial f(\bar{x}) + \sum_{i=1}^{3} \bar{\lambda}_i \partial_x^o g_i(\bar{x}, \bar{v}_i).$$

Thus, $\bar{x}$ is a KKT point. Moreover, it can be check that the robust Slater constraint qualification is satisfied. However, $\bar{x}$ is not a global minimizer. In fact, by taking $\widetilde{x} := (1,0) \in K$, one has $f(\widetilde{x}) = -1 < 0 = f(\bar{x})$.

Proposition 4.12. *If $\{C, K\}$ has the G-RS strong CHIP at $\bar{x} \in C \cap K$, then it has the strong CHIP at $\bar{x}$.*

Proof. By Remark 3.4, the conclusion will easily follow from the fact that

$$(C - \bar{x})^\circ + M(\bar{x}) \subseteq (C - \bar{x})^\circ + N_K(\bar{x}) = (C - \bar{x})^\circ + (K - \bar{x})^\circ \subseteq (C \cap K - \bar{x})^\circ.$$

□

We will show that if (ENMFCQ) is satisfied at $\bar{x} \in C \cap K$, then the G-RS strong CHIP of $\{C, K\}$ at $\bar{x}$ is equivalent to the strong CHIP of $\{C, K\}$ at $\bar{x}$.

Proposition 4.13. *If (ENMFCQ) is satisfied at $\bar{x} \in C \cap K$, then $\{C, K\}$ has the G-RS strong CHIP at $\bar{x}$ if and only if $\{C, K\}$ has the strong CHIP at $\bar{x}$.*

Proof. As before, if $\{C, K\}$ has the G-RS strong CHIP at $\bar{x}$ then, by Proposition 4.12, $\{C, K\}$ has the strong CHIP at $\bar{x}$. The converse implication will follow from the fact, by Corollary 3.9 and Theorem 4.7, that $\{\mathbb{R}^n, K\}$ has the G-RS strong CHIP at $\bar{x}$, i.e., $(K - \bar{x})^\circ = M(\bar{x})$. □

The following example illustrates that without the (ENMFCQ) the strong CHIP does not necessarily imply the G-RS strong CHIP.

Example 4.14. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $C := \mathbb{R}_+ \times \{0\}$, $K := \{x \in \mathbb{R}^2 : g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1,2\}$, where $g_1(x, v_1) := x_1^3 - v_1 x_2 + v_1 - 1$, $g_2(x, v_2) := -v_2 x_1 + x_2^2 + v_2$, $\mathcal{V}_1 := [0,1]$ and $\mathcal{V}_2 := [-1,0]$. Then $K = -\mathbb{R}_+ \times \{0\}$, which is convex. Let $\bar{x} := (0,0) \in C \cap K = \{(0,0)\}$. A direct calculation shows that $(C - \bar{x})^\circ = \{x \in \mathbb{R}^2 : x_1 \leq 0\}$, $(K - \bar{x})^\circ = \{x \in \mathbb{R}^2 : x_1 \geq 0\}$, $M(\bar{x}) = \{0\} \times -\mathbb{R}_+$. It is now easy to verify that $\{C, K\}$ has the strong CHIP at $\bar{x}$; however, $\{C, K\}$

does not have the G-RS strong CHIP at $\bar{x}$ as $\mathbb{R}^2 = (C \cap K - \bar{x})^\circ \neq (C - \bar{x})^\circ + M(\bar{x}) = \{x \in \mathbb{R}^2 : x_1 \leq 0\}$. Note that $I_1(\bar{x}) = \{1, 2\}$, $\mathcal{V}_1^0 = \{1\}$, $\mathcal{V}_2^0 = \{0\}$, but there is no $\hat{x} := (\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $0 > g_{1x}^o(\bar{x}, 1, \hat{x}) = -\hat{x}_2$ and $0 > g_{2x}^o(\bar{x}, 0, \hat{x}) = 0$. So, (ENMFCQ) is invalid at $\bar{x}$.

Now we are ready to state the final result of this section by the global conditions which ensure the strong CHIP and the G-RS strong CHIP of $\{C, K\}$.

Proposition 4.15. *Let K be convex, given by* (3.2)*, and C be a closed convex subset of $\mathbb{R}^n$. Let $\bar{x} \in C \cap K$. Assume that the condition (a) in Theorem 3.5 holds at $\bar{x}$. Then, the following assertions holds:*

(i) *$\{C, K\}$ has the strong CHIP at $\bar{x}$.*

(ii) *$\{C, K\}$ has the G-RS strong CHIP at $\bar{x}$.*

(iii) *For each convex function $f : \mathbb{R}^n \to \mathbb{R}$, $\bar{x}$ is a global minimizer of the problem (RP_f) if and only if KKT condition holds at $\bar{x}$, where (RP_f) is defined by* (4.4)*.*

Proof. Suppose that the condition (a) in Theorem 3.5 holds. By Theorem 3.5, the robust Slater constraint qualification is fulfilled. In particular, $C \cap \text{int}K \neq \emptyset$. Hence, by the Moreau-Rockafellar theorem [29, Theorem 23.8, p. 223] (see also [30, Proposition 2.3.]), we have

$$(C \cap K - \bar{x})^\circ = (C - \bar{x})^\circ + (K - \bar{x})^\circ,$$

that is, $\{C, K\}$ has the strong CHIP at $\bar{x}$.

At the same time, since the condition (a) in Theorem 3.5 is satisfied at $\bar{x}$, then (ENMFCQ) is satisfied at $\bar{x}$. From this, by Proposition 4.13, $\{C, K\}$ has the G-RS strong CHIP at $\bar{x}$. Thus, (i) and (ii) hold. Finally, since (ii) holds, (iii) follows from Theorem 4.7.

□

5 Robust best approximation without convexity of constraint data uncertainty

In this section, we give a dual characterization of the robust best approximation $\bar{x}$ to x from the convex set $C \cap K$ in terms of the best approximation to a perturbation $(x - \sum_{i \in I} \lambda_i l_i)$ of x from the set C for some multipliers $\lambda_i \geq 0$, $v_i \in \mathcal{V}_i$, $l_i \in \partial_x^o g_i(\bar{x}, v_i)$ with $\lambda_i g_i(\bar{x}, v_i) = 0$ for all $i \in I$ whenever $\{C, K\}$ has the G-RS strong CHIP at some point $\bar{x} \in C \cap K$.

Here, for a nonempty subset A of $\mathbb{R}^n$, the *distance from a given point* $x \in \mathbb{R}^n$ to A is defined by $d(x, A) := \inf_{a \in A} \|x - a\|$, We say that a point $u \in A$ is a *best approximation* of $x \in \mathbb{R}^n$ if $\|x - u\| = d(x, A)$. In the following, the set of all best approximations of x in A is denoted by $P_A(x)$, that is,

$$P_A(x) := \{u \in A : \|x - u\| = d(x, A)\}.$$

Recall that if A is closed and convex, then for each $x \in \mathbb{R}^n$ there exists a unique best approximation $u_0 \in A$ and we write $u_0 = P_A(x)$ instead of $u_0 \in P_A(x)$.

In order to establish next theorem, we need the following characterization of best approximation, which is well known.

Lemma 5.1. *[27] Let A be a closed convex subset of $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $u \in A$. Then, $u = P_D(x)$ if and only if $x - u \in (A - u)^\circ$.*

We have the following result, which improves the corresponding known results in [31] for robust best approximation where the constraint data uncertainty $g_i(\cdot, v_i)$, $i = 1, 2, \dots, m$ are assumed to be convex for each $v_i \in \mathcal{V}_i$.

Theorem 5.2. *Let K be convex, given by (3.2), and C be a nonempty closed convex subset of $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $\bar{x} \in \Omega := C \cap K$. Assume that $\{C, K\}$ has the G-RS strong CHIP at $\bar{x}$. Then the following assertions are equivalent:*

(i) $\bar{x} = P_\Omega(x)$;

(ii) *There exist $\lambda_i \geq 0$, $v_i \in \mathcal{V}_i$, $l_i \in \partial_x^o g_i(\bar{x}, v_i)$ with $\lambda_i g_i(\bar{x}, v_i) = 0$ for all $i \in I$ such that*
$$\bar{x} = P_C\left(x - \sum_{i \in I} \lambda_i l_i\right);$$

(iii) *There exist $\lambda_i \geq 0$, $v_i \in \mathcal{V}_i$ with $\lambda_i g_i(\bar{x}, v_i) = 0$ for all $i \in I$ such that*
$$0 \in \partial \|x - \cdot\|(\bar{x}) + \sum_{i \in I} \lambda_i \partial_x^o g_i(\bar{x}, v_i) + (C - \bar{x})^\circ.$$

Proof. Suppose that $\{C, K\}$ has the G-RS strong CHIP at $\bar{x}$.

[(i)⇔(ii)]. This equivalence easily follows from Lemma 5.1 together with the G-RS strong CHIP at $\bar{x}$ on noting that the statement (i) is, in turn, equivalent to the fact that,

$$\begin{aligned}
\bar{x} = P_\Omega(x) &\Leftrightarrow x - \bar{x} \in (\Omega - \bar{x})^\circ \\
&\Leftrightarrow \exists \lambda_i \geq 0,\ v_i \in \mathcal{V}_i,\ l_i \in \partial_x^o g_i(\bar{x}, v_i) \text{ with } \lambda_i g_i(\bar{x}, v_i) = 0,\ i \in I \\
&\qquad \text{such that } \left[x - \sum_{i \in I} \lambda_i l_i\right] - \bar{x} \in (C - \bar{x})^\circ \\
&\Leftrightarrow \exists \lambda_i \geq 0,\ v_i \in \mathcal{V}_i,\ l_i \in \partial_x^o g_i(\bar{x}, v_i) \text{ with } \lambda_i g_i(\bar{x}, v_i) = 0,\ i \in I \\
&\qquad \text{such that } \bar{x} = P_C\left(x - \sum_{i \in I} \lambda_i l_i\right).
\end{aligned}$$

[(i)⇔(iii)]. Let $f(w) := \|w - x\|$, $\forall w \in \mathbb{R}^n$. Then, f is a continuous convex function. Now, $\bar{x} = P_\Omega(x)$ if and only if $\bar{x}$ is a global minimizer of the problem (RP_f), which is equivalent to the statement, as Ω is a closed convex set, that
$$0 \in \partial \|x - \cdot\|(\bar{x}) + (\Omega - \bar{x})^\circ.$$

Therefore, (4.1) gives to the equivalence (i)⇔(iii).

□

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