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# Shrinking Projection Methods for a Split Equilibrium Problem and a Hybrid Multivalued Mapping

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**Abstract :** In this paper, we introduce two different hybrid methods by using the Shrinking projection method for a split equilibrium problem and a hybrid multivalued mapping in Hilbert space. We obtain strong convergence theorems under the same conditions. Furthermore, we give examples and numerical results for supporting our main results and compare the rate of convergence of two iteration methods.

**Keywords :** Hybrid multivalued mapping; Shriking projection method; Split equilibrium problem; Strong convergence; Hausdorff metric
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## 1 Introduction

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Let $H$ be real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $C$ be a nonempty subset of $H$. A subset $C \subset H$ is said to be *proximinal* if for each $x \in H$, there exists $y \in C$ such that

$$\|x - y\| = d(x, C) = \inf\{\|x - z\| : z \in C\}.$$

Let $CB(C), K(C)$ and $P(C)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal bounded subset of $C$, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A, B) = \max\left\{ \sup_{x \in A} d(x, B),\ \sup_{y \in B} d(y, A) \right\}$$

for all $A, B \in CB(C)$ where $d(x, B) = \inf_{b \in B} \|x - b\|$.
A singlevalued mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.
A multivalued mapping $T : C \to CB(C)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in C$. An element $z \in C$ is called a *fixed point* of $T : C \to C$ (resp., $T : C \to CB(C)$) if $z = Tz$ (resp., $z \in Tz$). The fixed point set of $T$ is denoted by $F(T)$. If $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|$$

for all $x \in C$ and $p \in F(T)$, then $T$ is said to be $quasi-nonexpansive$.

In 1953, Mann [29] introduced the iteration procedure as follows:

$$x_1 \in C, x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{x_n}, \forall n \in \mathbb{N}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\mathbb{N}$ the set of all positive integers. Recently, many mathematician (see [10, 12, 23]) used Mann's iteration for obtaining weak convergence theorem.

In 2008, Takahashi et al.[29] introduced a new projection method which is called the *shrinking projection method* by using the modification Mann's iteration for obtaining strong convergence theorem for a countable family of nonexpansive singlevalued mapping in Hilbert spaces. They proved the following theorem:

**Theorem 1.1.** [29] *Let $H$ be a Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $\{T_n\}$ and $\tau$ be a family of nonexpansive mappings of $C$ into $H$ such that $F := \cap_{n=1}^{\infty} F(T_n) = F(\tau) \neq \emptyset$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the $NST$-condition $(I)$ with $\tau$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ in $C$ as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n) T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

*where* $0 \leq \alpha_n \leq a < 1$ *for all* $n \in \mathbb{N}$. *Then the sequence* $\{u_n\}$ *converges strongly to a point* $z_0 = P_F x_0$.

In 2008, Kohsaka and Takahashi [19, 29] presented a new mapping which is called a nonspreading mapping and obtained fixed point theorems for a single nonspreading mapping and also a common fixed point point theorem for a commutative family of nonspreading mapping in Banach spaces. Let $H$ be a Hilbert space and $C$ be nonempty closed and convex subset of $H$. Then a mapping $T : C \to C$ is said to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$. Recently, Iemoto and Takahashi [13] showed that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Ty, y - Ty\rangle,\ \forall x, y \in C.$$

Further, Takahashi [28] defined a class of nonlinear mapping which is called *hybrid* as follows:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty\rangle,$$

for all $x, y \in C$. It was shown that a mapping $T : C \to C$ is hybrid if and only if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2,$$

for all $x, y \in C$.

Inspired by Kohsaka and Takahashi [19], Iemoto and Takahashi [13] and Takahashi [28], Cholamjiak and Cholamjiak [3] introduced a new concept of multivalued mapping in Hilbert spaces. A multivalued mapping $T : C \to CB(C)$ is said to be *hybrid* if

$$3H(Tx, Ty)^2 \leq \|x - y\|^2 + d(y, Tx)^2 + d(x, Ty)^2,$$

for all $x, y \in C$. They showed that if $T$ is hybrid and $F(T) \neq \emptyset$, then $T$ is quasi-nonexpansive. The following example shows that $T$ is hybrid but $T$ is not nonexpansine.

**Example 1.2.** [3] Let $H = \mathbb{R}$. Consider $C = [0, 3]$ with the usual norm. Define a multivalued mapping $T : C \to CB(C)$ by

$$Tx = \begin{cases} \{0\}, & x \in [0, 2]; \\ [0, \frac{x}{x+1}], & x \in (2, 3]. \end{cases}$$

Let $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real number. The *equilibrium problem* is the problem of finding a point $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0 \tag{1.2}$$

for all $y \in C$, which has been introduced and studied by Blum and Oettli [2]. The solution set of the equilibrium problem (1.2) is denoted by $EP(F)$.

In 2013, Kazmi and Rizvi [14] introduced and studied the following split equilibrium problem which is a generalization of the equilibrium problem:

Let $H_1$, $H_2$ be real Hilbert spaces. Let $C \subseteq H_1$ and $Q \subseteq H_2$ and let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions. Let $A : H_1 \to H_2$ be a bounded linear operator. The *split equilibrium problem* is to find $\hat{x} \in C$ such that

$$F_1(\hat{x}, x) \geq 0 \text{ for all } x \in C \tag{1.3}$$

and

$$\hat{y} = A\hat{x} \in Q \text{ solves } F_2(\hat{y}, y) \geq 0 \text{ for all } y \in Q. \tag{1.4}$$

Note that the inequality (1.3) is the classical equilibrium problem (1.2) . The problems (1.3) and (1.4) constitute a pair of equilibrium problems which have to find the image $\hat{y} = A\hat{x}$, under a given bounded linear operator $A$, of the solution $\hat{x}$ of the problem (1.3) in $H_1$ which is the solution of the problem (1.4) in $H_2$. It's easy to see that the split equilibrium problem generalize an equilibrium problem. We denote the solution set of the problem (1.4) by $EP(F_2)$. The solution set of the split equilibrium (1.3) and (1.4) is denoted by $\Omega = \{z \in EP(F_1) : Az \in EP(F_2)\}$.

Since 2013, the study of a split equilibrium problem and a fixed point problem for a singlevalued mapping was introduced by many authors (see [4, 5, 11, 14, 17, 18, 21, 22, 30, 31]) and references therein.

Inspired by above works, we present two different hybrid methods which are the modified Shrinking projection method for a split equilibrium problem and a hybrid multivalued mapping in a Hilbert space by using Hausdorff metric. As application, we give examples and numerical results for supporting our main results and compare the rate of convergence of two iterative methods.

# 2 Preliminaries

We now provide some results for the main results. In a Hilbert space $H_1$, let $C$ be a nonempty closed and convex subset of $H_1$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to $x$ and $x_n \to x$ implies that $\{x_n\}$ converges strongly to $x$. For every point $x \in H_1$, there exists a unique nearest point of $C$, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a $P_C$ is called the *metric projection* from $H_1$ on to $C$. Further, for any $x \in H_1$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

A mapping $T : C \to H$ is said to be $k -$ *Lipschitz continuous* if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is called $\alpha -$ *inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

We know that if $T : C \to C$ is nonexpansive, then $A \doteq I - T$ is $\frac{1}{2}$ - *inverse strongly monotone*; (see [25, 26, 27]) for more details. It is well know that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$ , the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \geq \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2$$

and therefore we get, for all $(x, y) \in H_1 \times F(T)$,

$$\langle (x - T(x)), (y - T(y)), \rangle \geq \frac{1}{2}\|T(x) - x\|^2$$

see, e.g, [8, 9].

**Lemma 2.1.** *Let $H_1$ be a real Hilbert space. Then the following equations hold:*

(1) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H_1$*;*

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H_1$*;*

(3) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$ *for all* $t \in [0, 1]$ *and* $x, y \in H_1$*;*

(4) *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* $H_1$ *which converges weakly to* $z \in H_1$*, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2$$

*for all* $y \in H_1$.

**Lemma 2.2.** [20] *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H_1$ and $P_C : H_1 \to C$ be the metric projection from $H_1$ onto $C$. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H_1, \ \forall y \in C.$$

**Lemma 2.3.** [16] *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H_1$. Given $x, y, z \in H_1$ and also given $a \in \mathbb{R}$, the set*

$$\{v \in C : \ \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

*is convex and closed.*

**Assumption 2.4.** [2] *Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(1) $F_1(x, x) = 0$ *for all* $x \in C$*;*

(2) $F_1$ *is monotone, i.e.,* $F_1(x, y) + F_1(y, x) \leq 0$ *for all* $x \in C$*;*

(3) *For each* $x, y, z \in C$, $\limsup_{t\to 0} F_1(tz + (1 - t)x, y) \leq F_1(x, y)$*;*

(4) *For each* $x \in C$, $y \to F_1(x, y)$ *is convex and lower semi-continuous.*

**Lemma 2.5.** [7] *Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.4. For any $r > 0$ and $x \in H_1$, define a mapping $T_r^{F_1} : H_1 \to C$ as follows:*

$$T_r^{F_1}(x) = \Big\{ z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \, \forall y \in C \Big\}.$$

*Then we have the following:*

(1) *$T_r^{F_1}$ is nonempty and single-value;*
(2) *$T_r^{F_1}$ is firmly nonexpansive, i.e., for any $x, y \in H_1$,*

$$\|T_r^{F_1}x - T_r^{F_1}y\|^2 \leq \langle T_r^{F_1}x - T_r^{F_1}y, x - y\rangle;$$

(3) *$F(T_r^{F_1}) = EP(F_1)$;*
(4) *$EP(F_1)$ is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.4. For each $s > 0$ and $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(w) = \Big\{ d \in Q : F_2(d, e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \, \forall e \in Q \Big\}.$$

Then we have the following:

(5) $T_s^{F_2}$ is nonempty and single-value;
(6) $T_s^{F_2}$ is firmly nonexpansive;
(7) $F(T_s^{F_2}) = EP(F_2, Q)$;
(8) $EP(F_2, Q)$ is closed and convex.

An operator $B : H_1 \to 2^{H_1}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ wherever $y_1 \in Bx_1$ and $y_2 \in Bx_2$. A monotone operator $B$ is said to be maximal if the graph of $B$ is not property contained in the graph of any other monotone operator. It is known that a monotone operator $B$ is maximal if and only if $R(I + rB) = H_1$ for every $r > 0$, where $R(I + rB)$ is the range of $I + rB$. If $B : H_1 \to 2^{H_1}$ is a maximal monotone, then, for each $r > 0$, a mapping $T_r : H_1 \to D(B)$ is defined by $T_r = (I + rB)^{-1}$, where $D(B)$ is the domain of $A$. $T_r$ is called the resolvent of $B$. We also define the Yosida approximation $B_r = (I - T_r)/r$; see ([15, 25, 26]) for more details. We know the following fundamental results:

(i) $B_r x \in BT_r x$ for all $x \in H_1$;

(ii) if $B^{-1}0 = \{z \in H_1 : 0 \in Bz\}$, then $B^{-1}0 = F(T_r)$ for all $r > 0$, where $F(T_r)$ is the set of fixed points of $T_r$;

(iii) $\|T_r x - T_r y\|^2 \leq \|x - y\|^2 - \|(I - T_r)x - (I - T_r)y\|^2$ for all $x, y \in H_1$ and $r > 0$, that is, $T_r$ is a nonexpansive mapping of $H_1$ into $H_1$.

**Lemma 2.6.** [1] *Let $H_1$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H_1$. Let $F_1 : C \times C \to \mathbb{R}$ satisfy (A1) -(A4). Let $A_{F_1}$ be a set-valued mapping of $H_1$ into itself defined by*

$$A_{F_1} = \begin{cases} \{z \in H_1 : F_1(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, \ x \in C, \\ \emptyset, x \notin C. \end{cases}$$

*Then, $EP(F_1) = A_{F_1}^{-1}0$ and $A_{F_1}$ is a maximal monotone operator with $dom(A_{F_1}) \subset C$. Furthermore, for any $x \in H_1$ and $r > 0$, the resolvent $T_r$ of $F_1$ coincides with the resolvent of $A_{F_1}$, i.e.,*

$$T_r x = (I + rA_{F_1})^{-1}x.$$

**Lemma 2.7.** [3] *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $T : C \to K(C)$ be a hybrid multivalued mapping. Let $\{x_n\}$ be a sequence in $C$ such that $x_n \rightharpoonup p$ and $lim_{n\to\infty}\|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

**Lemma 2.8.** [3] *Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $T : C \to K(C)$ be a hybrid multivalued mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed.*

**Lemma 2.9.** [3] *Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $T : C \to K(C)$ be a hybrid multivalued mapping with $F(T) \neq \emptyset$. If $T$ satisfies Condition (A) , then $F(T)$ is convex.*

**Condition(A).** Let $H_1$ be a Hilbert space and $C$ be a subset of $H_1$. A multi-valued mapping $T : C \to CB(C)$ is said to satisfy *Condition* $(A)$ if $\|x-p\| = d(x, Tp)$ for all $x \in H_1$ and $p \in F(T)$.

**Remark 2.10.** We see that $T$ satisfies Condition (A) if and only if $Tp = \{p\}$ for all $p \in F(T)$. It is known that the best approximation operator $P_T$, which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$, also satisfies Condition (A).

# 3 Main Results

In this section, we obtain two different strong convergence theorems for finding a common element of solutions of split equilibrium problems and fixed point problems of a hybird multivalued mapping in Hilbert spaces by using the Shrinking projection method.

**Theorem 3.1.** *Let $H_1$, $H_2$ be two real Hilbert spaces and let $C$, $Q$ be nonempty closed and convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a hybrid multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.4 and $F_2$ is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1)$ and $Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n \in \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ \forall n \geq 1 \end{cases} \tag{3.1}$$

*where* $\{\alpha_n\} \subset (0,1)$*,* $r_n \subset (0,\infty)$ *and* $\gamma \in (0, 1/L)$ *such that* $L$ *is the spectral radius of* $A^*A$ *and* $A^*$ *is the adjoint of* $A$*. Assume that the following conditions hold:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$*;*

(ii) $\liminf_{n\to\infty} r_n > 0$*.*

*If* $T$ *satisfies Condition (A), then the sequences* $\{u_n\}$*,* $\{y_n\}$ *and* $\{x_n\}$ *converge strongly to* $P_\Theta x_1$*.*

*Proof.* We split the proof into six steps.

**Step 1.** Show that $P_{C_{n+1}}x_1$ is well-defined for every $x_1 \in H_1$.

By Lemma 2.8 and 2.9, we obtain that $F(T)$ is closed and convex. Since $A$ is a bounded linear operator, it is easy to prove that $\Omega$ is closed and convex. So, $\Theta = F(T) \cap \Omega$ is also closed and convex. From the definition of $C_{n+1}$, it follows from Lemma 2.3 that $C_{n+1}$ is closed and convex for each $n \ge 1$. Since $T^{F_2}_{r_n}$ is firmly nonexpansive and $I - T^{F_2}_{r_n}$ is 1-inverse strongly monotone, we see that

$$\begin{aligned} & \|A^*(I - T^{F_2}_{r_n})Ax - A^*(I - T^{F_2}_{r_n})Ay\|^2 \\ = \ & \langle A^*(I - T^{F_2}_{r_n})(Ax - Ay), A^*(I - T^{F_2}_{r_n})(Ax - Ay)\rangle \\ = \ & \langle (I - T^{F_2}_{r_n})(Ax - Ay), AA^*(I - T^{F_2}_{r_n})(Ax - Ay)\rangle \\ \le \ & L\langle (I - T^{F_2}_{r_n})(Ax - Ay), (I - T^{F_2}_{r_n})(Ax - Ay)\rangle \\ = \ & L\|(I - T^{F_2}_{r_n})(Ax - Ay)\|^2 \\ \le \ & L\langle Ax - Ay, (I - T^{F_2}_{r_n})(Ax - Ay)\rangle \\ = \ & L\langle x - y, A^*(I - T^{F_2}_{r_n})Ax - A^*(I - T^{F_2}_{r_n})Ay\rangle \end{aligned}$$

for all $x, y \in H_1$. This implies that $A^*(I - T^{F_2}_{r_n})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. Since $\gamma \in (0, \frac{1}{L})$, it follows that $I - \gamma A^*(I - T^{F_2}_{r_n})A$ is nonexpansive. Let $p \in \Theta$. Then $p = T^{F_1}_{r_n}p$ and $(I - \gamma A^*(I - T^{F_2}_{r_n})A)p = p$. Thus, we have

$$\begin{aligned} \|u_n - p\| &= \|T^{F_1}_{r_n}(I - \gamma A^*(I - T^{F_2}_{r_n})A)x_n - T^{F_1}_{r_n}(I - \gamma A^*(I - T^{F_2}_{r_n})A)p\| \\ &\le \|(I - \gamma A^*(I - T^{F_2}_{r_n})A)x_n - (I - \gamma A^*(I - T^{F_2}_{r_n})A)p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{3.2}$$

This implies that

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n u_n + (1 - \alpha_n)z_n - p\| \\ &\le \alpha_n\|u_n - p\| + (1 - \alpha_n)\|z_n - p\| \\ &= \alpha_n\|u_n - p\| + (1 - \alpha_n)d(z_n, Tp) \\ &\le \alpha_n\|u_n - p\| + (1 - \alpha_n)H(Tu_n, Tp) \\ &\le \|u_n - p\| \\ &\le \|x_n - p\| \end{aligned}$$

for all $z_n \in Tu_n$. So, we have $p \in C_{n+1}$, thus $\Theta \subset C_{n+1}$. Therefore $P_{C_{n+1}}x_1$ is well defined.

**Step 2.** Show that $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Since $\Theta$ is a nonempty closed and convex subset of $H_1$, there exists a unique $v \in \Theta$ such that

$$v = P_\Theta x_1.$$

From $x_n = P_{C_n} x_1$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$, $\forall n \geq 1$, we get

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \quad \forall n \geq 1.$$

On the other hand, as $\Theta \subset C_n$, we obtain

$$\|x_n - x_1\| \leq \|v - x_1\|, \quad \forall n \geq 1.$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

**Step 3.** Show that $x_n \to w \in C$ as $n \to \infty$.

For $m > n$, by the definition of $C_n$, we see that $x_m = P_{C_m} x_1 \in C_m \subset C_n$. By Lemma 2.2, we get

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2.$$

From Step 2, we obtain that $\{x_n\}$ is Cauchy. Hence, there exists $w \in C$ such that $x_n \to w$ as $n \to \infty$.

**Step 4.** Show that $w \in F(T)$.

From Step 3, we get

$$\|x_{n+1} - x_n\| \to 0 \tag{3.3}$$

as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \to 0 \tag{3.4}$$

as $n \to \infty$. Hence, $y_n \to w$ as $n \to \infty$. For $p \in \Theta$, we estimate

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - p\|^2 \\
&= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1} p\|^2 \\
&\leq \|x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\|^2 \\
&\leq \|x_n - p\|^2 + \gamma^2 \|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \\
&\quad +2\gamma\langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - p\|^2 + \gamma^2\langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*(I - T_{r_n}^{F_2})Ax_n\rangle \\
&\quad +2\gamma\langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle.
\end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}
\gamma^2\langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*(I - T_{r_n}^{F_2})Ax_n\rangle &\leq L\gamma^2\langle Ax_n - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n\rangle \\
&= L\gamma^2\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
2\gamma\langle p-x_n, A^*(I-T_{r_n}^{F_2})Ax_n\rangle &= 2\gamma\langle A(p-x_n), Ax_n-T_{r_n}^{F_2}Ax_n\rangle \\
&= 2\gamma\langle A(p-x_n)+(Ax_n-T_{r_n}^{F_2}Ax_n) \\
&\quad -(Ax_n-T_{r_n}^{F_2}Ax_n), Ax_n-T_{r_n}^{F_2}Ax_n\rangle \\
&= 2\gamma\{\langle Ap-T_{r_n}^{F_2}Ax_n, Ax_n-T_{r_n}^{F_2}Ax_n\rangle \\
&\quad -\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2\} \\
&\leq 2\gamma\{\frac{1}{2}\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2-\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2\} \\
&= -\gamma\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2. \qquad (3.7)
\end{aligned}$$

Using (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
\|u_n-p\|^2 &\leq \|x_n-p\|^2+L\gamma^2\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2-\gamma\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2 \\
&= \|x_n-p\|^2+\gamma(L\gamma-1)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2. \qquad (3.8)
\end{aligned}$$

It follows that, for all $z_n\in Tu_n$,

$$\begin{aligned}
\|y_n-p\|^2 &= \|\alpha_n u_n+(1-\alpha_n)z_n-p\|^2 \\
&\leq \alpha_n\|u_n-p\|^2+(1-\alpha_n)\|z_n-p\|^2 \\
&= \alpha_n\|x_n-p\|^2+(1-\alpha_n)d(z_n,Tp)^2 \\
&\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)H(Tu_n,Tp)^2 \\
&\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|u_n-p\|^2 \\
&\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)(\|x_n-p\|^2 \\
&\quad +\gamma(L\gamma-1)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2) \\
&\leq \|x_n-p\|^2+\gamma(L\gamma-1)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-\gamma(L\gamma-1)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2 &\leq \|x_n-p\|^2-\|y_n-p\|^2 \\
&\leq \big(\|x_n-p\|+\|y_n-p\|\big)\|x_n-y_n\|.
\end{aligned}$$

It follows from $\gamma(L\gamma-1)<0$ and (3.4) that

$$\lim_{n\to\infty}\|Ax_n-T_{r_n}^{F_2}Ax_n\|=0. \qquad (3.9)$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive and $I-\gamma A^*(T_{r_n}^{F_2}-I)A$ is nonexpansive, it follows that

$$\begin{aligned}
&\|u_n - p\|^2 \\
= \;& \|T_{r_n}^{F_1}(x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n) - T_{r_n}^{F_1}p\|^2 \\
\leq \;& \langle T_{r_n}^{F_1}(x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n) - T_{r_n}^{F_1}p, x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\rangle \\
= \;& \langle u_n - p, x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\rangle \\
= \;& \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\|^2 \\
& -\|u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\|^2\} \\
\leq \;& \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\|^2\} \\
= \;& \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - (\|u_n - x_n\|^2 + \gamma^2\|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \\
- \;& 2\gamma\langle u_n - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle)\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u_n - p\|^2 \leq \;& \|x_n - p\|^2 - \|u_n - x_n\|^2 \\
& +2\gamma\langle u_n - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle \\
\leq \;& \|x_n - p\|^2 - \|u_n - x_n\|^2 \\
& +2\gamma\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\|. \qquad (3.10)
\end{aligned}$$

It follows from (3.10) that

$$\begin{aligned}
\|y_n - p\|^2 \leq \;& \alpha_n\|u_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 \\
= \;& \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)d(z_n, Tp)^2 \\
\leq \;& \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)H(Tu_n, Tp)^2 \\
\leq \;& \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\
\leq \;& \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)(\|x_n - p\|^2 \\
& -\|u_n - x_n\|^2 + 2\gamma\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\|)
\end{aligned}$$

Therefore, we have

$$(1 - \alpha_n)\|u_n - x_n\|^2 \leq 2\gamma\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\| + \|x_n - p\|^2 - \|y_n - p\|^2.$$

From the condition (i) , (3.4) and (3.9), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \qquad (3.11)$$

We know that $x_n \to w$ as $n \to \infty$, thus $u_n \to w$ as $n \to \infty$. It follows from Lemma

2.1 and (3.2), we have

$$\begin{aligned}\|y_n-p\|^2 &= \|\alpha_n u_n+(1-\alpha_n)z_n-p\|^2\\ &\le \alpha_n\|u_n-p\|^2+(1-\alpha_n)\|z_n-p\|^2-\alpha_n(1-\alpha_n)\|u_n-z_n\|^2\\ &= \alpha_n\|u_n-p\|^2+(1-\alpha_n)d(z_n,Tp)^2-\alpha_n(1-\alpha_n)\|u_n-z_n\|^2\\ &\le \alpha_n\|u_n-p\|^2+(1-\alpha_n)H(Tu_n,Tp)^2-\alpha_n(1-\alpha_n)\|u_n-z_n\|^2\\ &\le \|u_n-p\|^2-\alpha_n(1-\alpha_n)\|u_n-z_n\|^2\\ &\le \|x_n-p\|^2-\alpha_n(1-\alpha_n)\|u_n-z_n\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\alpha_n(1-\alpha_n)\|u_n-z_n\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2\\ &\le \big(\|x_n-p\|+\|y_n-p\|\big)\|x_n-y_n\|.\end{aligned}$$

From the condition (i) and (3.4) that

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.12}$$

By Lemma 2.7, we obtain $w\in F(T)$.

**Step 5.** Show that $w\in EP(F)$.

From $u_n=T^{F_1}_{r_n}(I+\gamma A^*(I-T^{F_2}_{r_n})A)x_n$, we have

$$F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n-\gamma A^*(I-T^{F_2}_{r_n})Ax_n\rangle \quad\ge\quad 0$$

for all $y\in C$, which implies that

$$F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle-\frac{1}{r_n}\langle y-u_n,\gamma A^*(I-T^{F_2}_{r_n})Ax_n\rangle \quad\ge\quad 0$$

for all $y\in C$. By Assumption 2.4 (2), we have

$$\frac{1}{r_{n_i}}\langle y-u_{n_i},u_{n_i}-x_{n_i}\rangle-\frac{1}{r_{n_i}}\langle y-u_{n_i},\gamma A^*(I-T^{F_1}_{r_{n_i}})Ax_{n_i}\rangle \quad\ge\quad F_1(y,u_{n_i})$$

for all $y\in C$. From $\liminf_{n\to\infty}r_n>0$, from (3.8), (3.10) and the Assumption 2.4 (4), we obtain

$$F_1(y,w)\quad\le\quad 0$$

for all $y\in C$. For any $0<t\le 1$ and $y\in C$, let $y_t=ty+(1-t)w$. Since $y\in C$ and $w\in C$, $y_t\in C$ and hence $F_1(y_t,w)\le 0$. So, by Assumption 2.4 (1) and (4), we have

$$0=F_1(y_t,y_t)\le tF_1(y_t,y)+(1-t)F_1(y_t,w)\le tF_1(y_t,y)$$

and hence $F_1(y_t,y)\ge 0$. So $F_1(w,y)\ge 0$ for all $y\in C$ and hence $w\in EP(F_1)$. Since $A$ is a bounded linear operator, $Ax_{n_i}\rightharpoonup Aw$. Then it follows from (3.9) that

$$T^{F_2}_{r_{n_i}}Ax_{n_i}\rightharpoonup Aw \tag{3.13}$$

as $i \to \infty$. By the definition of $T^{F_2}_{r_{n_i}} Ax_{n_i}$, we have

$$F_2(T^{F_2}_{r_{n_i}} Ax_{n_i}, y) + \frac{1}{r_{n_i}} \langle y - T^{F_2}_{r_{n_i}} Ax_{n_i}, T^{F_2}_{r_{n_i}} Ax_{n_i} - Ax_{n_i} \rangle \geq 0$$

for all $y \in C$. Since $F_2$ is upper semi-continuous in the first argument and (3.13), it follows that

$$F_2(Aw, y) \geq 0$$

for all $y \in C$. This shows that $Aw \in EP(F_2)$. Hence $w \in \Omega$.

**Step 6.** Show that $w = v = P_\Theta x_1$.

Since $x_n = P_{C_n} x_1$ and $\Theta \subset C_n$, we obtain

$$\big\langle x_1 - x_n, x_n - p \big\rangle \geq 0 \quad \forall p \in \Theta. \tag{3.14}$$

By taking the limit in (3.14), we obtain

$$\big\langle x_1 - w, w - p \big\rangle \geq 0 \quad \forall p \in \Theta.$$

This shows that $w = P_\Theta x_1 = v$.

From Step 4, we obtain that $\{x_n\}, \{y_n\}$ and $\{u_n\}$ converge strongly to $v = P_\Theta x_1$. This completes the proof. □

If $Tp = \{p\}$ for all $p \in F(T)$, then $T$ satisfies Condition (A). We then obtain the following result:

**Theorem 3.2.** *Let $H_1, H_2$ be two real Hilbert spaces and let $C$ , $Q$ be nonempty closed and convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a hybrid multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.4 and $F_2$ is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T^{F_1}_{r_n}(I - \gamma A^*(I - T^{F_2}_{r_n})A)x_n, \\ y_n \in \alpha_n u_n + (1 - \alpha_n) T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ \forall n \geq 1 \end{cases} \tag{3.15}$$

*where $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0, \infty)$ and $\gamma \in (0, 1/L)$ such that $L$ is the spectral radius of $A^*A$ and $A^*$ is the adjoint of $A$. Assume that the following conditions hold:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*;*

(ii) $\liminf_{n\to\infty} r_n > 0$.

*If $Tp = \{p\}$ for all $p \in F(T)$, then the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to $P_\Theta x_1$.*

Since $P_T$ satisfies Condition (A), we also obtain the following result:

**Theorem 3.3.** *Let $H_1, H_2$ be two real Hilbert spaces and let $C$ , $Q$ be nonempty closed and convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to P(C)$ a multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.4 and $F_2$ is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n \in \alpha_n u_n + (1 - \alpha_n)P_T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ \forall n \geq 1 \end{cases} \tag{3.16}$$

*where $\{\alpha_n\} \subset (0, 1)$, $r_n \subset (0, \infty)$ and $\gamma \in (0, 1/L)$ such that $L$ is the spectral radius of $A^*A$ and $A^*$ is the adjoint of $A$. Assume that the following conditions hold:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\liminf_{n\to\infty} r_n > 0$.

*If $P_T$ is hybrid multivalued mapping and $I - T$ is demiclosed at 0, then the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to $P_\Theta x_1$.*

*Proof.* By the same proof as in Theorem 3.1, we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0$$

where $z_n \in P_T u_n \subseteq T u_n$. From $I - T$ is demiclosed at 0, so we obtain the result. □

**Theorem 3.4.** *Let $H_1, H_2$ be two real Hilbert spaces and let $C$ , $Q$ be nonempty closed and convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a hybrid multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.4 and $F_2$ is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n \in \alpha_n x_n + (1 - \alpha_n)T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ \forall n \geq 1 \end{cases} \tag{3.17}$$

*where $\{\alpha_n\} \subset (0, 1)$, $r_n \subset (0, \infty)$ and $\gamma \in (0, 1/L)$ such that $L$ is the spectral radius of $A^*A$ and $A^*$ is the adjoint of $A$. Assume that the following conditions hold:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$.

*If $T$ satisfies Condition (A), then the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to $P_\Theta x_1$.*

*Proof.* As the same proof in Step 1 of Theorem 3.1, we have

$$\begin{aligned}\|u_n - p\| &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)p\| \\ &\leq \|(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - (I - \gamma A^*(I - T_{r_n}^{F_2})A)p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.18}$$

This implies that

$$\begin{aligned}\|y_n - p\| &= \|\alpha_n x_n + (1-\alpha_n)z_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|z_n - p\| \\ &= \alpha_n\|x_n - p\| + (1-\alpha_n)d(z_n, Tp) \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)H(Tu_n, Tp) \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|u_n - p\| \\ &\leq \|x_n - p\|\end{aligned}$$

for all $z_n \in Tu_n$. So, we have $p \in C_{n+1}$, thus $\Theta \subset C_{n+1}$. Therefore $P_{C_{n+1}}x_1$ is well defined.

From Step 2-3 in Theorem 3.1, we know that $\{x_n\}$ is Cauchy. Hence, there exists $w \in C$ such that $x_n \to w$ as $n \to \infty$. Since $\{x_n\}$ is Cauchy, we get

$$\|x_{n+1} - x_n\| \to 0 \tag{3.19}$$

as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \to 0 \tag{3.20}$$

as $n \to \infty$. Hence, $y_n \to w$ as $n \to \infty$. For $p \in \Theta$, as the same proof in Step 4 of Theorem 3.1, we have

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 + L\gamma^2\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 - \gamma\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma(L\gamma - 1)\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2.\end{aligned} \tag{3.21}$$

Since $T$ satisfies condition $(A)$, for $z_n \in Tu_n$,

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)z_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(z_n, Tp)^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(Tu_n, Tp)^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 \\ &\quad + \gamma(L\gamma - 1)\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2) \\ &\leq \|x_n - p\|^2 + \gamma(L\gamma - 1)\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned} -\gamma(L\gamma-1)\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 &\le \|x_n-p\|^2 - \|y_n-p\|^2 \\ &\le \big(\|x_n-p\| + \|y_n-p\|\big)\|x_n-y_n\|. \end{aligned}$$

It follows from $\gamma(L\gamma-1)<0$ and (3.20) that

$$\lim_{n\to\infty}\|Ax_n - T_{r_n}^{F_2}Ax_n\| = 0. \tag{3.22}$$

From Step 4 in Theorem 3.1, we also have

$$\begin{aligned} \|y_n-p\|^2 &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|z_n-p\|^2 \\ &= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)d(z_n,Tp)^2 \\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)H(Tu_n,Tp)^2 \\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|u_n-p\|^2 \\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)(\|x_n-p\|^2 \\ &\quad -\|u_n-x_n\|^2 + 2\gamma\|u_n-x_n\|\|A^*(I-T_{r_n}^{F_2})Ax_n\|). \end{aligned}$$

Therefore, we have

$$(1-\alpha_n)\|u_n-x_n\|^2 \le 2\gamma\|u_n-x_n\|\|A^*(I-T_{r_n}^{F_2})Ax_n\| + \|x_n-p\|^2 - \|y_n-p\|^2.$$

From the condition (i) , (3.20) and (3.22), we have

$$\lim_{n\to\infty}\|u_n-x_n\| = 0. \tag{3.23}$$

We know that $x_n \to w$ as $n\to\infty$, thus $u_n\to w$ as $n\to\infty$. It follows from Lemma 2.1 and (3.18), we have

$$\begin{aligned} \|y_n-p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)z_n - p\|^2 \\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|z_n-p\|^2 - \alpha_n(1-\alpha_n)\|u_n-z_n\|^2 \\ &= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)d(z_n,Tp)^2 - \alpha_n(1-\alpha_n)\|u_n-z_n\|^2 \\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)H(Tu_n,Tp)^2 - \alpha_n(1-\alpha_n)\|u_n-z_n\|^2 \\ &\le \|x_n-p\|^2 - \alpha_n(1-\alpha_n)\|u_n-z_n\|^2 \\ &\le \|x_n-p\|^2 - \alpha_n(1-\alpha_n)\|u_n-z_n\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} \alpha_n(1-\alpha_n)\|u_n-z_n\|^2 &\le \|x_n-p\|^2 - \|y_n-p\|^2 \\ &\le \big(\|x_n-p\| + \|y_n-p\|\big)\|x_n-y_n\|. \end{aligned}$$

From the condition (i) and (3.20) that

$$\lim_{n\to\infty}\|u_n-z_n\| = 0. \tag{3.24}$$

By Lemma 2.7, we obtain $w\in F(T)$. As the same proof in Step 5-6 of Theorem 3.1, we can conclude that $\{x_n\},\{y_n\}$ and $\{u_n\}$ converge strongly to $v = P_\Theta x_1$. This completes the proof. □

If $Tp = \{p\}$ for all $p \in F(T)$, then $T$ satisfies Condition (A). We then obtain the following result:

**Theorem 3.5.** *Let $H_1, H_2$ be two real Hilbert spaces and let $C$ , $Q$ be nonempty closed and convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a hybrid multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.4 and $F_2$ is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n \in \alpha_n x_n + (1 - \alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ \forall n \geq 1 \end{cases} \tag{3.25}$$

*where $\{\alpha_n\} \subset (0, 1)$, $r_n \subset (0, \infty)$ and $\gamma \in (0, 1/L)$ such that $L$ is the spectral radius of $A^*A$ and $A^*$ is the adjoint of $A$. Assume that the following conditions hold:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\liminf_{n\to\infty} r_n > 0$.

*If $Tp = \{p\}$ for all $p \in F(T)$, then the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to $P_\Theta x_1$.*

Since $P_T$ satisfies Condition (A), we also obtain the following result:

**Theorem 3.6.** *Let $H_1, H_2$ be two real Hilbert spaces and let $C$ , $Q$ be nonempty closed and convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to P(C)$ a hybrid multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.4 and $F_2$ is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n \in \alpha_n x_n + (1 - \alpha_n)P_T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ \forall n \geq 1 \end{cases} \tag{3.26}$$

*where $\{\alpha_n\} \subset (0, 1)$, $r_n \subset (0, \infty)$ and $\gamma \in (0, 1/L)$ such that $L$ is the spectral radius of $A^*A$ and $A^*$ is the adjoint of $A$. Assume that the following conditions hold:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\liminf_{n\to\infty} r_n > 0$.

*If $P_T$ is hybrid multivalued mapping and $I-T$ is demiclosed at 0, then the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to $P_\Theta x_1$.*

*Proof.* By the same proof as in Theorem 3.4, we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0$$

where $z_n \in P_T u_n \subseteq T u_n$. From $I - T$ is demiclosed at 0, so we obtain the result. $\square$

We then apply our main theorems to solve the proximal point problems.

**Remark 3.7.** *If, we set* $T_{r_n}^{F_1} = (I + r_n A_{F_1})^{-1}$ *and* $T_{r_n}^{F_2} = (I + r_n A_{F_2})^{-1}$ *where*

$$A_{F_1} = \begin{cases} \{f_1 \in H_1 : F_1(x,y) \geq \langle y - x, f_1\rangle,\ \forall y \in C\}, & x \in C \\ \emptyset & x \notin C \end{cases}$$

*and*

$$A_{F_2} = \begin{cases} \{f_2 \in H_2 : F_2(x,y) \geq \langle y - x, f_2\rangle,\ \forall y \in Q\}, & x \in Q \\ \emptyset & x \notin Q. \end{cases}$$

*Then the sequences* $\{u_n\}$, $\{y_n\}$ *and* $\{x_n\}$ *generated in Theorem 3.1-3.6 converge strongly to* $P_\Theta x_1$, *where* $\Theta = F(T) \cap \Omega$ *and* $\Omega = \{z \in C : \ z \in A_{F_1}^{-1}0$ *and* $Az \in A_{F_2}^{-1}0\}$.

# 4 Examples and Numerical Results

In this section, we give examples and numerical results for supporting our main theorem.

**Example 4.1.** Let $H_1 = H_2 = \mathbb{R}$, $C = [1,4]$ and $Q = [0,\infty)$. Let $F_1(u,v) = 2(u-4)(v-u)$ for all $u,v \in C$ and $F_2(x,y) = 2(x-8)(y-x)$ for all $x,y \in Q$. Define two mappings $A : \mathbb{R} \to \mathbb{R}$ and $T : C \to K(C)$ by $Ax = 2x$ for all $x \in \mathbb{R}$ and

$$Tx = \begin{cases} \{4\}, & x \in [2,4]; \\ \big[(x-4)\cdot \frac{tan^{-1}(20x-45)}{2} + x, 4\big], & x \notin [2,4]. \end{cases}$$

Choose $\alpha_n = \frac{n}{5n+3}$, $r_n = \frac{n}{200n+2}$ and $\gamma = \frac{1}{200}$. It is easy to check that $F_1$ and $F_2$ satisfy all conditions in Theorem 3.1 and $T$ satisfies Condition (A) such that $F(T) = \{4\}$ .

Now, we show that $T$ is hybrid. In fact, we have the following case:

Case 1: If $x,y \in [2,4]$, then $H(Tx,Ty) = 0$.

Case 2: If $x \in [2,4]$ and $y \notin [2,4]$, then $Tx = \{4\}$ and $Ty = \big[(y-4)\cdot \frac{tan^{-1}(20y-45)}{2} + y, 4\big]$. This implies that

$$3H(Tx,Ty)^2 = 3\Big((y-4)\cdot\frac{tan^{-1}(20y-45)}{2} + y - 4\Big)^2 < 3 < \|x-y\|^2 + d(x,Ty)^2 + d(y,Tx)^2.$$

Case 3: If $x, y \notin [2,4]$, then $Tx = \big[(y-4)\cdot\frac{tan^{-1}(20x-45)}{2} + x, 4\big]$ and $Ty = \big[(y-4)\cdot\frac{tan^{-1}(20y-45)}{2} + y, 4\big]$. This implies that

$$
\begin{aligned}
3H(Tx,Ty)^2 &= 3\Big((x-4)\cdot\frac{tan^{-1}(20x-45)}{2} + x - (y-4)\cdot\frac{tan^{-1}(20y-45)}{2} + y\Big)^2 \\
&< 3 \\
&< \|x-y\|^2 + d(x,Ty)^2 + d(y,Tx)^2.
\end{aligned}
$$

On the other hand, $T$ is not nonexpansive since for $x = 1.83$ and $y = 2.18$, we have $Tx = \big[3.41, 4\big]$ and $Ty = \{4\}$. This implies that

$$H(Tx,Ty) = 4 - 3.41 = 0.39 > 0.35 = |1.83 - 2.18| = \|x-y\|.$$

Choosing $x_1 = 2$ and taking randomly $y_n \in \alpha_n u_n + (1-\alpha_n)Tu_n$, we obtain the numerical results of iteration (3.1) as follows :

| n | Randomized in the 1st | | | Randomized in the 2nd | | |
|---|---|---|---|---|---|---|
| | $u_n$ | $y_n$ | $x_n$ | $u_n$ | $y_n$ | $x_n$ |
| 1 | 1.980296 | 3.238563 | 2.000000 | 1.990245 | 3.309093 | 2.000000 |
| 2 | 2.600318 | 3.784664 | 2.619281 | 2.635056 | 3.790009 | 2.654546 |
| 3 | 3.174306 | 3.980938 | 3.201973 | 3.194307 | 3.865718 | 3.222278 |
| 4 | 3.499574 | 3.984921 | 3.532179 | 3.511217 | 3.914994 | 3.543998 |
| 5 | 3.687122 | 3.987276 | 3.722574 | 3.693940 | 3.945346 | 3.729496 |
| 6 | 3.796242 | 3.988661 | 3.833351 | 3.800251 | 3.963682 | 3.837421 |
| 7 | 3.860073 | 3.989470 | 3.898152 | 3.862437 | 3.974659 | 3.900552 |
| 8 | 3.897540 | 3.989939 | 3.936188 | 3.898936 | 3.981197 | 3.937605 |
| 9 | 3.919580 | 3.990206 | 3.958563 | 3.920406 | 3.985076 | 3.959401 |
| 10 | 3.932562 | 3.990354 | 3.971742 | 3.933051 | 3.987368 | 3.972239 |
| ... | ... | ... | ... | ... | ... | |
| 40 | 3.950556 | 3.990257 | 3.990010 | 3.950558 | 3.990258 | 3.990012 |

**Table 1.** Numerical results of iteration (3.1) being randomized in two times.

Choosing $x_1 = 2$ and taking randomly $y_n \in \alpha_n x_n + (1-\alpha_n)Tu_n$, we also obtain the numerical results of iteration (3.17) as follows :

| n | Randomized in the 1st | | | Randomzied in the 2nd | | |
|---|---|---|---|---|---|---|
| | $u_n$ | $y_n$ | $x_n$ | $u_n$ | $y_n$ | $x_n$ |
| 1 | 1.990245 | 3.591220 | 2.000000 | 1.990245 | 3.741293 | 2.000000 |
| 2 | 2.774011 | 3.814709 | 2.795610 | 2.847926 | 3.826253 | 2.870646 |
| 3 | 3.275949 | 3.884193 | 3.305159 | 3.318592 | 3.891408 | 3.348450 |
| 4 | 3.561137 | 3.929509 | 3.594676 | 3.586012 | 3.933901 | 3.619929 |
| 5 | 3.726049 | 3.957517 | 3.762093 | 3.740650 | 3.960163 | 3.776915 |
| 6 | 3.822299 | 3.974510 | 3.859805 | 3.830903 | 3.976098 | 3.868539 |
| 7 | 3.878794 | 3.984739 | 3.917157 | 3.883878 | 3.985690 | 3.922319 |
| 8 | 3.912079 | 3.990874 | 3.950948 | 3.915089 | 3.991443 | 3.954004 |
| 9 | 3.311743 | 3.994546 | 3.790911 | 9.933528 | 3.994886 | 3.972724 |
| 10 | 3.943384 | 3.996741 | 3982729 | 3.944444 | 3.996944 | 3.983805 |
| ... | ... | ... | ... | ... | ... | ... |
| 30 | 3.960396 | 4.000000 | 3.999999 | 3.960396 | 4.000000 | 3.999999 |

**Table 2.** Numerical results of iteration (3.17) being randomized in two times.

From Table 1 , we see that 4 is the solution in Example 4.1.

We next show error plots for comparing the convergence of iterations (3.1) and (3.17).

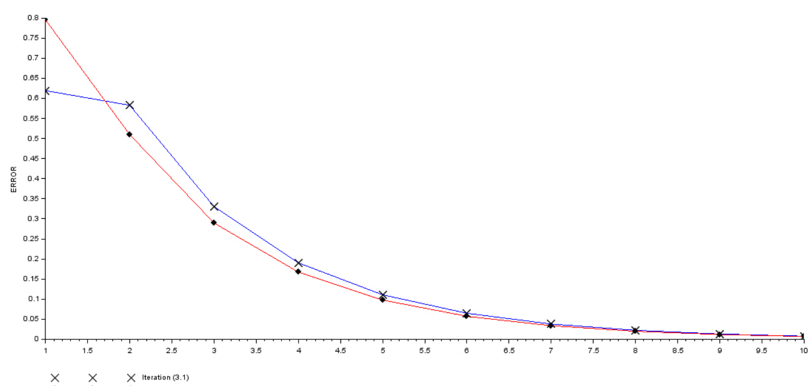


**Figure 1.** Error plots for all sequences $\{x_n\}$ in Table 1 and Table 2 being randomized in the first time.

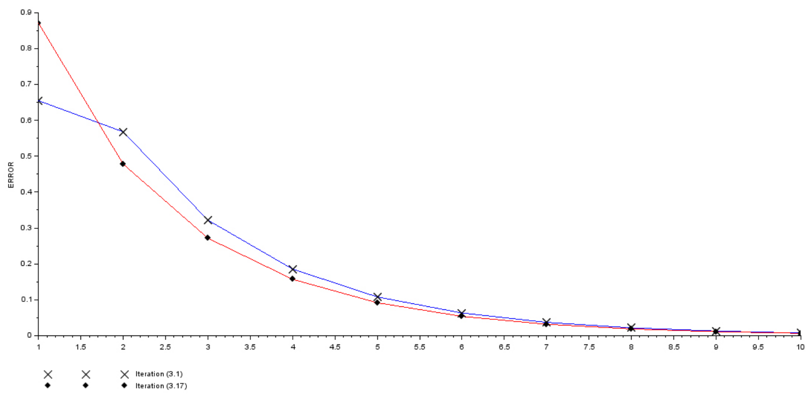


**Figure 2.** Error plots for sequences $\{x_n\}$ in Table 1 and Table 2 being randomized in the second time .

**Remark 4.2.** We see that the iteration (3.17) converges faster than the iteration (3.1) under the same conditions.

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