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On Nonlinear Implicit Fractional Differential Equations with Integral Boundary Condition Involving p-Laplacian Operator without Compactness

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Abstract : The motive behind this work is to obtain some sufficient conditions for the existence of solution to a nonlinear problem of implicit fractional differential equations (IFDEs) involving integral boundary conditions with p-Laplacian operator, using prior estimate method. The method applied here does not require compactness of the operator, which makes it distinguished from other methods. Besides developing the respective conditions, we also investigate Hyers-Ulam type

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stability for the solution of the problem under study. The validity of the established results are justified by providing a suitable example.

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1 Introduction

In last few decades, fractional differential equations have become an area of interest to researchers due to its high accuracy and applicability in various filed of science and technology. As many physical, dynamical, biological and chemical phenomenons are represented in more realistic way by using fractional differential equations instead of integer order differential equation. More realistic approach is the main reason for attracting the attention of researchers. Fractional differential equations are equally suitable not only to the mathematicians but also to engineers and physicists. The fractional order differential equations have a large numbers of applications in many fields of science and technology, for example, rheology, porous media, electrochemistry, electromagnetism, optics, geology, bioscience, bioengineering, medicine, probability and statistics, etc. The fractional order differential equations are also applicable in ecology, control theory, splines, tomography, control of power electron, converter, polymer science, polymer physics and neural networks. Furthermore, It has many applications in the modeling of other phenomenons, such as nonlinear oscillations due to earthquake, seepage flow in porous media, absorption of drug in blood stream, image processing , mathematical biology, genetic properties and traffic model of fluid dynamic (traffic model). These equations are also be used for the calculations of genetically and chemically acquired properties of different material and phenomenons, (see for details[6, 7, 11, 12, 13, 36, 37]).
Since nonlinear operators have a vital roles in differential equations. One of the most important operator use in differential equations is the classical nonlinear p-Laplacian operators, which is defined as

$$\frac{1}{p}+\frac{1}{q}=1, \phi_p(\nu)=|\nu|^{(p-2)}\nu, p\geq 1 \text{ and } \phi_q(\tau)={\phi_p}^{-1}.$$

For the applications mentioned above, researchers have paid much attention to study (FDEs) with p-Laplacian operators from different aspects. One of the important aspect which has been greatly investigated is devoted to the existence theory of (FDEs) involving p-Laplacian operators. Since p-Lapalcian operators has been greatly applied in the mathematical modeling of large numbers of real world phenomenons devoted to physics, mechanics, dynamical systems, elector dynamics, etc. A considerable number of valuable research articles can be found in literature regarding this topic, (see for detail[8, 9, 10, 14, 17, 18, 19, 24]). Researchers applied various technique of nonlinear analysis including fixed point theory, hybrid fixed point theory, iterative techniques for establishing existence theory, (see for details [27, 28, 30]). The existence theory by using classical fixed point theory has been much explored. In the mentioned theory compactness for the corresponding operator for the fractional integral equations equivalent to FDEs is necessary which need some strong conditions. Using classical fixed point theory need strong conditions to establish necessary and sufficient conditions for existence, uniqueness of solutions to (FDEs) and therefore restrict the applicability to certain classes of (FDEs) and their systems. To relax the criteria, degree theory plays excellent roles for the existence of solution to (FDEs). In many articles, degree approach has been used to replace the stronger conditions with weaker one, see[5, 6, 15, 21]. Some of the degree theories, for example Schauder degree theory, Brouwer's degree theory and topological degree theory are well known. An important degree theory introduced by Mahwin [10], which later on extended by Isaia [11] has been used to establish existence theory for nonlinear differential and integral equations. The mentioned method is also called prior estimate method which need no compactness of the operator and relax much more the conditions for existence and uniqueness of solution to differential and integral equations. The topological degree methods has appeared as a strongest tool in the study of great numbers of problems which occurs in nonlinear analysis. The priori estimate method is frequently used to find out the existence of solutions of nonlinear differential equation or partial differential equation. As in many articles, the existence theory for FDEs with p-Lapalcian operators through classical fixed point theory has been very well studied. For example, Han *et.al*[21], applied Guo-Krasnosel'skii fixed point theorem on cones to study the following problem

$$\begin{cases} D^{\beta}(\phi_p({}^cD^{\alpha}u(t))) = \lambda\mathcal{F}(t,u(t)); t \in J = [0,T], \\ u(t)|_{t=0} = u^{'}(t)|_{t=0} = u^{'}(t)|_{t=1} = 0, \\ \phi_p({}^cD^{\alpha}u(t))|_{t=0} = \phi_p({}^cD^{\alpha}u^{'}(t))|_{t=1} = 0. \end{cases}$$

Where ${}^cD^{\alpha}$ is the Caputo fractional derivative and D^{β} is the Riemann-Liouville fractional derivative, $\alpha \in (2,3]$, $\beta \in (1,2]$. Being inspired from the aforesaid article, L.yunhong [3] investigated the following BVP of (FDEs) with p-Lapalcian operators

$$\begin{cases} D^{\beta}(\phi_p({}^cD^{\alpha}u(t))) + \mathcal{F}(t,u(t)) = 0; t \in J = (0,1), \\ u^{'}(t)|_{t=0} = u^{''}(t)|_{t=1} = 0, \\ \phi_p({}^cD^{\alpha}u(t))|_{t=0} = \phi_p({}^cD^{\alpha}u^{'}(t))|_{t=0} = \phi_p({}^cD^{\alpha}u(t))|_{t=1} = 0, \\ au(t)|_{t=0} + bu^{'}(t)|_{t=0} = \int_0^1 g(t)u(t)dt. \end{cases}$$

Where ${}^cD^{\alpha}$ is the Caputo fractional derivative, D^{β} is the Riemann-Liouville fractional derivative, $\alpha \in (2,3], \beta \in (2,3]$ and $\beta + \alpha \in (5,6]$.

If $\alpha, \beta = 3$, then the above equation becomes the third order ordinary differential equation involving p-Lapalcian operator, which is called jerk,jolt, surge or lurch equation in physics. This equation has general form $(u(t), u^{'}(t), u^{''}(t), u^{'''}(t)) = 0,$. The jerk equation represent the rate of change of acceleration and has vast applications in physics and daily life. It is concerned with the flow of thin film viscous fluid with a free surface in which surface tension effects play a role typically lead to third order ordinary differential equation governing the shape of the free surface of the fluid. The equation $u^{'''}(t) = u(t)^{-2}$ describes the dynamical balance between surface tension and viscous force in a thin layer in absence of gravity. Jerk also plays an important roles in physiological balancing of human body. The changing gears in an average moving car, operated with a foot clutch, offers well known example of jerk, although the accelerating force is limited by engine power, an inexperienced driver lets you experience sever jerk, just because of intermittent force closure over the clutch. Another application of jerk is in accelerated charged particle which emits radiation, which is proportional to the jerk, see [2, 4, 5].
In most of the situations it is quite hard to find the exact solutions of nonlinear problems, in such situations, approximate solutions to the nonlinear problems of (FDEs) is considered. Stability analysis plays a very important role in such situations. Due to this importance of stability analysis, researchers have been studying for decades its various forms to the nonlinear problems. In last few years the researchers gave attentions to study various form of stability including exponential stability, Mittag-Leffler stability and Lyapunov stability etc, see [27, 28, 30] to the nonlinear problems. One of the important form of stability is Ulam type stability. This form of stability for the first time was pointed out by Ulam [29], in 1940 and was introduced by Hyers [30] in 1941. Later on, this topic was much explored for functional, integral and ordinary differential equations, (see[35, 36]). But it has been very rarely investigated for (FDEs). However, some literature on the Hyers-Ulam stability of (FDEs) can be cited recently with initial conditions, (see [31, 32]). The implicit fractional differential equations represent a very important class of fractional differential equations. The implicit fractional differential equations have been studied so far on different standard fixed point theorems. But here, we investigate the implicit fractional differential equations (IFDEs) involving integral boundary conditions with p-Laplacian operator by using topological degree theory

$$\begin{cases} D^{\beta}(\phi_p({}^cD^{\alpha}u(t))) = \mathcal{F}(t, u(t), D^{\beta}(\phi_p({}^cD^{\alpha}u(t)))); t \in J = (0, 1), \\ u^{'}(t)|_{t=0} = u^{''}(t)|_{t=1} = 0, \\ \phi_p({}^cD^{\alpha}u(t))|_{t=0} = \phi_p({}^cD^{\alpha}u^{'}(t))|_{t=0} = \phi_p({}^cD^{\alpha}u(t))|_{t=1} = 0, \\ u(t)|_{t=1} = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu)) d\nu. \end{cases} \tag{1.1}$$

Where ${}^cD^{\alpha}$ is the Caputo fractional derivative, D^{β} is the Riemann-Liouville fractional derivative, $\alpha \in (2, 3], \beta \in (2, 3]$ and $\beta + \alpha \in [5, 6]$. Existence and uniqueness results are developed through topological degree theory which is also called prior estimate method. Also, we establish some adequate conditions about Hyers-Ulam stability for the solutions of considered problem. The main result is also illustrated by providing an example.

2 Preliminaries

Here we remind few basic theorems, lemmas and results from fractional calculus, functional analysis, topological degree theory which can be found in[6, 7, 8, 9, 10, 11, 12, 30].

Definition 2.1. *Let $z : \mathcal{R}^+ \to \mathcal{R}$ is a given function. The fractional integral of order $\alpha > 0$ is defined as below*

$$I^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} z(\nu) d\nu,$$

subject to the condition the integral on right side is pointwise defined on $\mathcal{R}^+$.

Definition 2.2. *The fractional order Caputo derivative of a function $z : \mathcal{R}^+ \to \mathcal{R}$ is given as below*

$${}^cD^\alpha z(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\nu)^{n-\alpha-1} \left(\frac{d}{d\nu}\right)^n z(\nu) d\nu,$$

subject to the condition the integral on right side is pointwise defined on $\mathcal{R}^+$ and $n = [\alpha] + 1$ where $[\alpha]$ represents the integer part of α.

Definition 2.3. *The fractional order Riemann-Liouville derivative of a continuous function $z : \mathcal{R}^+ \to \mathcal{R}$ is given by*

$$D^\beta z(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\nu)^{n-\alpha-1} z(\nu) d\nu,$$

subject to the condition the integral on right side is pointwise defined on $\mathcal{R}^+$, such that $n = [\beta] + 1$ where $[\beta]$ represents the integer part of β.

Lemma 2.1. *Fractional order differential equation*

$${}^cD^\alpha z(t) = 0,$$

satisfies the result given below

$$I^\alpha({}^cD^\alpha z(t)) = z(t) + a_0 + a_1 t + a_2 t^2 + ... + a_{n-1} t^{n-1},$$

for arbitrary $a_i \in \mathcal{R}, \quad i = 0, 1, 2, ..., n-1$.

Definition 2.4. *The space $\mathcal{Z} = C(J, \mathcal{R})$ endowed with a norm*

$$||u||_{\mathcal{Z}} = \sup\{|u(t)| : t \in J\} \text{ is Banach space.}$$

We state here the results given below from [15].

Definition 2.5. *The Kuratowski's measure of non-compactness $\delta : \mathcal{P} \to \mathcal{R}_+$ is given below as*

$$\delta(P) = inf\left\{\varrho > 0 \text{ where } P \in \mathcal{P} \text{ has a finite cover by sets of diameter } \leq \varrho\right\}.$$

Proposition 2.2. *The Kuratowski's measure δ satisfy the following properties:*

(*i*) $\delta(P) = 0$ *if and only if* P *is relatively compact;*

(*ii*) δ *is a semi norm, that is* $\delta(\lambda P) = |\lambda|\delta(P), \lambda \in \mathcal{R}$ *and* $\delta(P_1 + P_2) \leq \delta(P_1) + \delta(P_2)$;

(*iii*) $P_1 \subset P_2$ *implies* $\delta(P_1) \leq \delta(P_2)$; $\delta(P_1 \cup P_2) = \max\{\delta(P_1), \delta(P_2)\}$;

(*iv*) $\delta(convP) = \delta(P)$;

(*v*) $\delta(\bar{P}) = \delta(P)$.

Definition 2.6. *Let $\mathcal{M} : \mathcal{A} \to \mathcal{Z}$ be a continuous bounded map and $\mathcal{A} \subset \mathcal{Z}$. The operator $\mathcal{M}$ is said to be δ-Lipschitz if we can find a constant $k \geq 0$ satisfying the following condition,*

$$\delta(\mathcal{M}(P)) \leq k\delta(P), \text{for all} \quad P \subset \mathcal{A} \quad \text{is bounded.}$$

Moreover, $\mathcal{M}$ is called strict δ-contraction if $k < 1$.

Definition 2.7. *The function $\mathcal{M}$ is called δ-condensing if*

$$\delta(\mathcal{M}(P)) < \delta(P), \text{for all} \quad P \subset \mathcal{A} \quad \text{bounded with} \quad \delta(P) > 0.$$

In other words, $\delta(\mathcal{M}(P)) \geq \delta(P)$ implies $\delta(P) = 0$.

The collection of all strict δ-contractions $\mathcal{M} : \mathcal{A} \to \mathcal{Z}$ is represented by $kC_\delta(\Omega)$ and the collection of all δ-condensing maps $\mathcal{M} : \mathcal{A} \to \mathcal{Z}$ by $C_\delta(\Omega)$.

Remark 2.3. *$kC_\delta(\mathcal{A}) \subset C_\delta(\mathcal{A})$ and every $\mathcal{M} \in C_\delta(\mathcal{A})$ is δ-Lipschitz with constant $k = 1$.*

Moreover, recall that $\mathcal{M} : \mathcal{A} \to \mathcal{Z}$ is Lipschitz if we can find $k > 0$ such that

$$|\mathcal{M}(v) - \mathcal{M}(w)| \leq k|v - w|, \quad \text{for all} \quad v, w \in \mathcal{A},$$

if $k < 1$, $\mathcal{M}$ is said to be strict contraction.

Proposition 2.4. *If $\mathcal{M},\mathcal{N} : \mathcal{A} \to \mathcal{Z}$ are δ-Lipschitz mapping with constants k_1 and k_2 respectively, then $\mathcal{M} + \mathcal{N} : \mathcal{A} \to \mathcal{Z}$ are δ-Lipschitz with constants $k_1 + k_2$.*

Proposition 2.5. *If $\mathcal{M} : \mathcal{A} \to \mathcal{Z}$ is compact, then $\mathcal{M}$ is δ-Lipschitz with constant $k = 0$.*

Proposition 2.6. *If $\mathcal{M} : \mathcal{A} \to \mathcal{Z}$ is Lipschitz with constant k, then $\mathcal{M}$ is δ-Lipschitz with the same constant k.*

The theorem given below due to Isaia [11] is of key importance for the proof of our main result.

Theorem 2.7. *Let $\mathcal{M} : \mathcal{Z} \to \mathcal{Z}$ be δ-condensing and*

$$\mathcal{V} = \{x \in \mathcal{Z} : \textit{there exist} \quad \vartheta \in [0,1] \quad \textit{such that} \quad x = \vartheta \mathcal{M} x\}.$$

If $\mathcal{V}$ is a bounded set in $\mathcal{Z}$, so there exists $r > 0$ such that $\mathcal{V} \subset B_r(0)$, then the degree

$$\deg(I - \vartheta\mathcal{M}, B_r(0), 0) = 1, \quad \textit{for all} \quad \vartheta \in [0,1].$$

Consequently, $\mathcal{M}$ has at least one fixed point and the set of the fixed points of $\mathcal{M}$ lies in $B_r(0)$.

3 Integral representation of BVP (1.1)

In this section, we find an equivalent representation of our considered problem (1.1).

Lemma 3.1. *Let $x \in (J, \mathcal{R})$, then the following BVP of (FDEs) with p-Laplacian operator*

$$\begin{cases} D^{\beta}(\phi_p({}^cD^{\alpha}u(t))) = x(t); t \in J = (0,1), \\ u^{'}(t)|_{t=0} = u^{''}(t)|_{t=1} = 0, \\ \phi_p({}^cD^{\alpha}u(t))|_{t=0} = \phi_p({}^cD^{\alpha}u^{'}(t))|_{t=0} = \phi_p({}^cD^{\alpha}u(t))|_{t=1} = 0, \\ u(t)|_{t=0} = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu. \end{cases} \tag{3.1}$$

has a solution given by

$$u(t)) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-2} g(\nu, u(\nu))d\nu + \int_0^1 \mathcal{H}(t,\nu)\phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)x(\nu)d\tau\bigg)d\nu$$

where $\mathcal{H}(t,\nu)$ is the Green's function given as

$$\mathcal{H}(t,\nu) = \frac{1}{\Gamma(\alpha)}\begin{cases} -\alpha t(1-\nu)^{\alpha-2};\ t \le \nu, \\ -\alpha t(1-\nu)^{\alpha-2} + (t-\nu)^{\alpha-1};\ \nu \le t. \end{cases} \tag{3.2}$$

Proof. From Eq. (3.1), we get

$$\phi_p({}^cD^{\alpha}u(t)) = c_0 + c_1 t + c_2 t^2 + I^{\beta}x(t). \tag{3.3}$$

In view of conditions $\phi_p({}^cD^\alpha u(t))|_{t=0} = \phi_p({}^cD^\alpha u^{'}(t))|_{t=0} = 0$, we get $c_0, c_1 = 0$. Thus (3.3) becomes

$$\phi_p({}^cD^\alpha u(t)) = I^\beta x(t) + c_2t^2. \tag{3.4}$$

Condition $\phi_p({}^cD^\alpha u(t))|_{t=1} = 0$ implies that $c_2 = -I^\beta x(1)$. Thus (3.4) becomes

$$\phi_p({}^cD^\alpha u(t)) = I^\beta x(t) - t^2 I^\beta x(1)$$

Above equation can written as

$$\phi_p({}^cD^\alpha u(t)) = \int_0^1 \mathcal{G}(t,\nu)x(\nu)d\nu. \tag{3.5}$$

Where $\mathcal{G}(t,\nu)$ is called the Green's function such that

$$\mathcal{G}(t,\nu) = \frac{1}{\Gamma(\beta)}\begin{cases} -t^2(1-\tau)^{\beta-1};\ t \le \nu, \\ (t-\tau)^{\beta-1} - t^2(1-\tau)^{\beta-1};\ \nu \le t. \end{cases} \tag{3.6}$$

Therefore Eq.(3.5) implies that ${}^cD^\alpha u(t) = \phi_q\bigg(\int_0^1 \mathcal{G}(t,\nu)x(\nu)d\nu\bigg)$ where ϕ_q is the inverse function of ϕ_p. Putting $\phi_q\bigg(\int_0^1 \mathcal{G}(t,\nu)x(\nu)d\nu\bigg) = y(t)$. We get

$${}^cD^\alpha u(t) = y(t).$$

In view of Lemma 3.1, the above equation implies that,

$$u(t) = d_0 + d_1t + d_2t^2 + I^\alpha y(t).$$

The condition $u^{''}(0) = 0$, implies that $d_2 = 0$, similarly the condition $u^{'}(1) = 0$,implies that

$$d_1 = -\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\nu)^{\alpha-2}y(\nu)d\nu.$$

The condition $u(0) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1}g(\nu,u(\nu))d\nu$, implies that

$$d_0 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1}g(\nu,u(\nu))d\nu.$$

Thus

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1}g(\nu,u(\nu))d\nu - t\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\nu)^{\alpha-2}y(\nu)d\nu \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1}y(\nu)d\nu, \end{aligned}$$

$$u(t)) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1}g(\nu,u(\nu))d\nu + \int_0^1 \mathcal{H}(t,\nu)\phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)x(\nu)\bigg)d\tau d\nu. \tag{3.7}$$

$\mathcal{H}(t,\nu)$ is called the Green's function, defined as given in (3.2). □

In view of Lemma 3.1, our considered BVP (1.1) is given by the following second kind of Fredholm integral equation

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu)) d\nu \\ &\quad + \int_0^1 \mathcal{H}(t,\nu)\phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^\beta(\phi_p({}^cD^\alpha u(\tau)))d\tau \bigg) d\nu, \ t \in J. \end{aligned} \tag{3.8}$$

Lemma 3.2. *The Green's function $\mathcal{G}(t,\tau)$ and $\mathcal{H}(t,\nu)$ defined in Lemma 3.1, satisfies the properties given below:*

(i) $\mathcal{G}(t,\nu)$ *and* $\mathcal{H}(t,\nu)$ *are continuous over* $J \times J$*;*

(ii) $\max|\mathcal{G}(t,\nu)| = \frac{1}{\Gamma(\beta)}(1-\tau)^{\beta-1}$,
$\max|\mathcal{H}(t,\nu)| = \frac{1}{\Gamma(\alpha)}(1-\nu)^{\alpha-2}(1-\nu+\alpha)$.

Proof. (i) and (ii) can be proved easily. □

Definition 3.1. *The solution of the considered problem* (1.1) *is Hyers-Ulam stable if we can find a real number $c_{\mathcal{F}} > 0$ with the property that for each $\mu > 0$ and for each solution $u \in C(J, \mathcal{R})$ of the inequality*

$$|D^\beta(\phi_p({}^cD^\alpha u(t))) - \mathcal{F}(t, u(t), D^\beta(\phi_p({}^cD^\alpha u(t))))| \le \mu, \ t \in J, \tag{3.9}$$

there exists a unique solution $w \in C(J, \mathcal{R})$ of the considered equation (1.1) *with a constant $c_{\mathcal{F}} > 0$ such that*

$$|u(t) - w(t))| \le c_{\mathcal{F}}\mu, \ t \in J.$$

Definition 3.2. *The solution of the equation* (1.1) *is said to be generalized Hyers-Ulam stable, if we can find*

$$\Upsilon_{\mathcal{F}} \in C(\mathcal{R}_+, \mathcal{R}_+), \ \Upsilon_{\mathcal{F}}(0) = 0,$$

such that for each solution $u \in C(J, \mathcal{R})$ of the inequality (3.9) *we can find a unique solution $w \in C(J, \mathcal{R})$ of the equation* (1.1) *such that*

$$|u(t) - w(t))| \le \Upsilon_{\mathcal{F}}(\mu), \ t \in J.$$

Remark 3.3. *A function $u \in C(J, \mathcal{R})$ is said to be the solution of inequality given in* (3.9) *if and only if, we can find a function $h \in C(J, \mathcal{R})$ depends on v only then*

(i) $|h(t)| \le \mu,$ *for all* $t \in J$;

(ii) $D^\beta(\phi_p({}^cD^\alpha u(t))) = \mathcal{F}(t, u(t), D^\beta(\phi_p({}^cD^\alpha u(t)))) + h(t),$ *for all* $t \in J$.

4 Existence theory for at least one solution of BVP (1.1)

This section is devoted to the proof of some results required for existence of solution of the considered problem (1.1). Thank to Lemma 3.1, the considered problem (1.1) can be represented by the second kind of Fredholm integral equation given below,

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu + \int_0^1 \mathcal{H}(t,\nu)\phi_q\Big(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^\beta(\phi_p({}^cD^\alpha u(\tau)))\Big)d\tau d\nu,\ t\in J. \tag{4.1}$$

Below we list some assumptions:

(P_1) There exist constant $L_g > 0$, such that for each $u, v \in (J,\mathcal{R})$,

$$\|g(t,u(t)) - g(t,v(t))\| \leq L_g\|u-v\|_{\mathcal{Z}}.$$

(P_2) There exist constants $D_g,\ N_g > 0$, such that for each $u \in (J,\mathcal{R})$,

$$\|g(t,u(t))\| \leq D_g\|u\|_{\mathcal{Z}} + N_g.$$

(P_3) There exist $p, q, r \in C(J,\Re_+)$ and, $u, w \in \mathcal{R}, t \in J$ such that

$$|\mathcal{F}(t,u,w)| \leq p(t) + q(t)|u| + r(t)|w|,$$

where

$$r^* = sup\{|r(t)| : t\in J\} < 1,\ p^* = sup\{|p(t)| : t\in J\},\ q^* = sup\{|q(t)| : t \in J\}.$$

(P_4) There exist constants $L_{\mathcal{F}} > 0$, $0 < N_{\mathcal{F}} < 1$ and $u_1, w_1, u_2, w_2 \in \mathcal{R}$ such that,

$$|\mathcal{F}(t,u_1,w_1) - \mathcal{F}(t,u_2,w_2)| \leq L_{\mathcal{F}}\|u_1-u_2\|_{\mathcal{Z}} + N_{\mathcal{F}}\|w_1-w_2\|_{\mathcal{Z}}.$$

Assume that (P_1) to (P_4) hold, here it will be shown that the fractional integral equation (4.1) has a unique solution $u \in C(J,\mathcal{R})$. We consider two operators $\mathcal{M},\mathcal{N}$ on $C(J,\mathcal{R})$

$$\mathcal{M}:\ \mathcal{Z}\to\mathcal{Z} \text{ defined by } (\mathcal{M}(u(t)) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu,u(\nu))d\nu,$$

and

$$\mathcal{N}:\ \mathcal{Z}\to\mathcal{Z} \text{ defined by } (\mathcal{N}(u(t)) = \int_0^1 \mathcal{H}(t,\nu)\phi_q\Big(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau,u(\tau),D^\beta(\phi_p({}^cD^\alpha u(\tau)))\Big)d\tau d\nu.$$

Let us consider another operator $\mathcal{U}$ on $C(J,\mathcal{R})$, such that

$$\mathcal{U}:\ \mathcal{Z}\to\mathcal{Z} \text{ defined by } \mathcal{U}(u)=\mathcal{M}u+\mathcal{N}u.$$

$\mathcal{U}$ is well defined because $\mathcal{M}$ and $\mathcal{N}$ are well define. Moreover, $\mathcal{U}u=u$. Thus to find the solution of BVP (1.1) is equivalent to find fixed point for operator $\mathcal{U}$ in $\mathcal{Z}$.

Lemma 4.1. *$\mathcal{M}$ is Lipschitz with constant k_g. Moreover, $\mathcal{M}$ satisfies the growth condition given below*

$$\|\mathcal{M}u\|_{\mathcal{Z}}\le\frac{1}{\Gamma(\alpha+1)}(D_g\|u\|+N_g). \tag{4.2}$$

Proof. Let $u,w\in C(J,\mathcal{R})$, then consider

$$\begin{aligned}|\mathcal{M}(u)-\mathcal{M}(w)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^1(1-\nu)^{\alpha-1}g(\nu,u(\nu))d\nu\right.\\ &\qquad \left.-\frac{1}{\Gamma(\alpha)}\int_0^1(1-\nu)^{\alpha-1}g(\nu,w(\nu))d\nu\right|\\ &\le\frac{L_g}{\Gamma(\alpha+1)}|u-w|.\end{aligned}$$

Putting $k_g=\frac{L_g}{\Gamma(\alpha+1)}\le 1$, which yields

$$\begin{aligned}|\mathcal{M}(u)-\mathcal{M}(w)| &\le k_g|u-w|\\ \|\mathcal{M}(u)-\mathcal{M}(w)\|_{\mathcal{Z}} &\le k_g\|u-w\|_{\mathcal{Z}}.\end{aligned}$$

Consequently $\mathcal{M}$ is μ-Lipschitz with some constant k_g. The growth condition is a simple consequences of (P_2) as given by

$$\|\mathcal{M}u\|_{\mathcal{Z}}\le\frac{1}{\Gamma(\alpha+1)}(D_g\|u\|+N_g).$$

□

Lemma 4.2. *The operator $\mathcal{N}$ is continuous and satisfies the growth condition given as below,*

$$\|\mathcal{N}(u)\|_{\mathcal{Z}}\le\left(\frac{(q-1)m^{q-2}(\alpha^2+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha-1)(1-r^*)}\right)(p^*+q^*\|u\|_{\mathcal{Z}}). \tag{4.3}$$

Proof. To prove that $\mathcal{N}$ is continuous. Let $\{u_n\}$ be any sequence in bounded set $\mathfrak{B}_\gamma$, such that $\mathfrak{B}_\gamma=\{u_n:\|v_n\|_{\mathcal{Z}}\le\gamma\}$ and $u_n\to u$ as $n\to\infty$ in $\mathfrak{B}_\gamma$. We are required to prove that $|\mathcal{N}u_n-\mathcal{N}u|\to 0,\ n\to\infty$. As $\mathcal{F}(t,u(t),D^\beta(\phi_p(^cD^\alpha u(t))))$ is continuous, thus it follows that

$$\mathcal{F}(\nu,u_n(\nu),D^\beta(\phi_p(^cD^\alpha u_n(t))))\to\mathcal{F}(\nu,u(\nu),D^\beta(\phi_p(^cD^\alpha u(t)))),\ as\ n\to\infty$$

. Using assumption (P_3), we get the relation given below

$$\begin{aligned}
&|\mathcal{N}u_n - \mathcal{N}u| \\
&\leq \int_0^1 |\mathcal{H}(t,\nu)| \bigg| \phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u_n(\tau), D^{\beta}(\phi_p({}^cD^{\alpha}u_n(\tau)))) \bigg) \\
&\quad - \phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^{\beta}(\phi_p({}^cD^{\alpha}u(\tau)))) \bigg) \bigg| d\nu d\tau \\
&\leq \frac{(q-1)m^{q-2}}{\Gamma(\alpha)\Gamma(\beta)} \bigg[\int_0^1 (1-\nu)^{\alpha-2}(1-\nu+\alpha)d\nu \int_0^1 (1-\tau)^{\beta-1}d\tau \bigg] \\
&\quad \times \bigg| \mathcal{F}(\tau, u_n(\tau), D^{\beta}(\phi_p({}^cD^{\alpha}u_n(\tau)))) - \mathcal{F}(\tau, u(\tau), D^{\beta}(\phi_p({}^cD^{\alpha}u(\tau)))) \bigg|.
\end{aligned}$$

Since $|\mathcal{F}(t, u_n(t), D^{\beta}(\phi_p({}^cD^{\alpha}u_n(t)))) - \mathcal{F}(t, u(t), D^{\beta}(\phi_p({}^cD^{\alpha}u(t)))| \leq \frac{L_{\mathcal{F}}}{1-N_{\mathcal{F}}}|u_n - u|$. Therefore, we get

$$||\mathcal{N}u_n - \mathcal{N}u|| \quad \leq \frac{(q-1)m^{q-2}(\alpha^2+\alpha+1)L_{\mathcal{F}}}{(\alpha-1)\Gamma(\alpha+1)\Gamma(\beta+1)((1-N_{\mathcal{F}})}||u_n - u|| \tag{4.4}$$

the inequality (4.4) clearly implies that the right hand side tends to 0 as n tends to infinity. Therefore $\mathcal{N}(u_n) \to \mathcal{N}(u),$ as $n \to \infty$. This means that the operator $\mathcal{N}$ is continuous. For the growth condition, we use (P_4) and obtain as

$$\|\mathcal{N}(u)\|_{\mathcal{Z}} \leq \left(\frac{(q-1)m^{q-2}(\alpha^2+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha-1)(1-r^*)} \right)(p^* + q^*\|u\|_{\mathcal{Z}}). \tag{4.5}$$

□

Lemma 4.3. *The operator $\mathcal{N} : \mathcal{Z} \to \mathcal{Z}$ is compact and $\delta-$lipschitz with constant* 0.

Proof. In order to show that $\mathcal{N}$ is compact. Let us take a bounded set $\mathfrak{B} \subset \mathfrak{B}_{\gamma}$. We are required to show that $\mathcal{N}(\mathfrak{B})$ is relatively compact in $\mathcal{Z}$. For arbitrary $u_n \in \mathfrak{B} \subset \mathfrak{B}_{\gamma}$, the growth condition is given as

$$\|\mathcal{N}(u)\|_{\mathcal{Z}} \leq \left(\frac{(q-1)m^{q-2}(\alpha^2-\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha-1)(1-r^*)} \right)(p^* + q^*\|u\|_{\mathcal{Z}}).$$

From this it is clear that $\mathcal{N}(u_n)$ is uniformly bounded. For equi-continuity of $\mathcal{N}$, we have,

$$
\begin{aligned}
&|\mathcal{N}(u_n)(t_1)-\mathcal{N}(u_n)(t_2)|\\
&\le \int_0^1 |(\mathcal{H}(t_1,\nu)-\mathcal{H}(t_2,\nu))|\left|\phi_q\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau,u_n(\tau),D^\beta(\phi_p^(cD^\alpha u_n(t))))\right|d\nu d\tau\\
&\le \left(\frac{(q-1)m^{q-2}}{\Gamma(\beta+1)(1-r^*)}\right)(p^*+q^*\|u\|_{\mathcal{Z}})\bigg(\int_0^{t_1}(t_1-\nu)^{\alpha-1}d\nu-\int_0^{t_2}(t_2-\nu)^{\alpha-1}d\nu\\
&\quad+\int_{t_1}^{t_2}(t_1-\nu)^{\alpha-1}-(t_2-\nu)^{\alpha-1}d\nu+\alpha(t_2-t_1)\int_0^1(1-\nu)d\nu\bigg)\\
&\le \left(\frac{(q-1)m^{q-2}}{\Gamma(\alpha+1)\Gamma(\beta+1)(1-r^*)}\right)(p^*+q^*\|u\|_{\mathcal{Z}})\\
&\quad\times[t_1^\alpha-t_2^\alpha-(t_1-t_2)^\alpha-(t_2-t_1)^\alpha+\alpha(t_1-t_2)].
\end{aligned}
$$

From above relation , it follows clearly that $\|\mathcal{N}(u_n)(t_1)-\mathcal{N}(u_n)(t_2)\|\to 0,\ as\ t_1\to t_2$, which implies that $\mathcal{N}(u)$ is equi-continuous.
Hence by Arzelá-Ascoli theorem $\mathcal{N}(u)$ is compact and thus by proposition 2.4 $\mathcal{N}$ is $\delta-$Lipschitz with constant 0. □

Theorem 4.4. *Suppose that $(P_2)-(P_4)$ are satisfied, then the (BVP) (1.1) has at least one solution $u\in C(J,\mathcal{R})$ and the set of the solutions is bounded in $C(J,\mathcal{R})$.*

Proof. Let $\mathcal{M},\mathcal{N},\mathcal{U}$ are the operators defined in the start of this section. These operators are continuous and bounded. Moreover, by Lemma 4.1, $\mathcal{M}$ is $\delta-$Lipschitz with constant K_g and by Lemma 4.2, $\mathcal{N}$ is $\delta-$Lipschitz with constant 0. Thus, $\mathcal{U}$ is $\delta-$Lipschitz with constant K_g. Let us take the set given below

$$\mathcal{W}=\left\{u:\text{there exist }\xi\in[0,1],\text{such that }u=\xi\mathcal{U}(u)\right\}.$$

We will show that the set $\mathcal{W}$ is bounded. For $u\in\mathcal{W}$, we have $u=\xi\mathcal{U}u=\xi(\mathcal{M}(u)+\mathcal{N}(u))$, which implies that

$$
\begin{aligned}
|u| &\le \xi(|\mathcal{M}u|+|\mathcal{N}u|),\ \text{from which we have}\\
\|u\| &\le \frac{1}{\Gamma(\alpha+1)}(D_g\|u\|+N_g)+\left(\frac{(q-1)m^{q-2}(\alpha^2+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha-1)(1-r*)}\right)(p^*+q^*\|u\|_{\mathcal{Z}})
\end{aligned}
$$

From the above inequalities, we conclude that $\mathcal{W}$ is bounded in $C(J,\mathcal{R})$. If it is not bounded, then dividing the above inequality by $a=\|u\|$ and letting $a\to\infty$, we arrive at

$$1\le\frac{\frac{1}{\Gamma(\alpha+1)}(D_g\|v\|+N_g)+\left(\frac{(q-1)m^{q-2}(\alpha^2+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha-1)(1-r*)}\right)(p^*+q^*\|u\|_{\mathcal{Z}})}{a}\le 0,$$

which is a contradiction. Thus the set $\mathcal{W}$ is bounded and the operator $\mathcal{U}$ has at least one fixed point which represent the solution of (BVP) (1.1). □

Theorem 4.5. *Under the assumptions* $(P_1)-(P_4)$, *the (BVP)* (1.1) *has a unique solution if*

$$\left(\frac{1}{\Gamma(\alpha+1)}L_g+\frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\beta+1)\Gamma(\alpha+1)(1-N_{\mathcal{F}})}\right)<1.$$

Proof. By application of Banach contraction theorem we will show that equation (1.1) has a unique solution. Consider $u(.)$ and $w(.)$ be the solutions of (BVP) (1.1), then

$$\begin{aligned}
|\mathcal{U}(u)(t)-\mathcal{U}(w)(t)| &\leq \Bigg|\frac{1}{\Gamma(\alpha)}\int_0^1(1-\nu)^{\alpha-1}g(\nu,u(\nu))d\nu \\
&\quad+\int_0^1\mathcal{H}(t,\nu)\phi_q\Big(\int_0^1\mathcal{G}(\nu,\tau)\mathcal{F}(\tau,u(\tau),D^\beta(\phi(^cD^\alpha w(\tau)))d\tau d\nu\Big) \\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^1(1-\nu)^{\alpha-1}g(\nu,w(\nu))d\nu \\
&\quad+\int_0^1\mathcal{H}(t,\nu)\phi_q\Big(\int_0^1\mathcal{G}(\nu,\tau)\mathcal{F}(\tau,w(\tau),D^\beta(\phi(^cD^\alpha w(\tau)))\Big)d\tau d\nu\Bigg| \\
&\leq \frac{1}{\Gamma(\alpha+1)}|g(\nu,u(\nu))-g(\nu,w(\nu))| \\
&\quad+\left|\int_0^1\mathcal{H}(t,\nu)\right|\Bigg|\phi_q\Big(\int_0^1\mathcal{G}(\nu,\tau)\mathcal{F}(\tau,u(\tau),D^\beta(^cD^\alpha u(t)))\Big)d\tau d\nu \\
&\quad-\phi_q\Big(\int_0^1\mathcal{G}(\nu,\tau)\mathcal{F}(\tau,w(\tau),D^\beta(^cD^\alpha w(t)))\Big)d\tau d\nu\Bigg| \\
&\leq \frac{1}{\Gamma(\alpha+1)}L_g|v-w|+\frac{(q-1)m^{q-2}}{\Gamma(\beta+1)\Gamma(\alpha+1)} \\
&\quad\times|\mathcal{F}(\tau,u(\tau),D^\beta(^cD^\alpha u(t)))-\mathcal{F}(\tau,w(\tau),D^\beta(^cD^\alpha w(t)))| \\
&\leq \frac{1}{\Gamma(\alpha+1)}L_g|v-w|+\frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\beta+1)\Gamma(\alpha+1)(1-N_{\mathcal{F}})}|u-w| \\
&\leq \left(\frac{1}{\Gamma(\alpha+1)}L_g+\frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\beta+1)\Gamma(\alpha+1)(1-N_{\mathcal{F}})}\right)||u-w||.
\end{aligned}$$

Since $\left(\frac{1}{\Gamma(\alpha+1)}L_g+\frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\beta+1)\Gamma(\alpha+1)(1-N_{\mathcal{F}})}\right)<1$. Thus $\mathcal{U}$ is a contraction mapping and by Banach contraction principle $\mathcal{U}$ has a unique fixed point. □

5 Hyers-Ulam stability

Theorem 5.1. *Let $u \in C(J, \mathcal{R})$ be a solution of*

$$\begin{aligned} D^{\beta}(\phi_p({}^cD^{\alpha}u(t))) &= \mathcal{F}(t, u(t), D^{\beta}(\phi_p({}^cD^{\alpha}u(t)))) + h(t),\ \ t \in J, \alpha, \beta \in (2,3], \\ & u^{'}(t)|_{t=0} = u^{''}(t)|_{t=1} = 0, \\ & \phi_p({}^cD^{\alpha}u(t))|_{t=0} = \phi_p({}^cD^{\alpha}u^{'}(t))|_{t=0} = \phi_p({}^cD^{\alpha}u(t))|_{t=1} = 0, \\ & u(t)|_{t=0} = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu. \end{aligned}$$

Then the following result holds,

$$\begin{aligned} &\Bigg| u(t) - \Bigg(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu \\ &+ \int_0^1 \mathcal{H}(t,\nu)\phi_q\Bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^{\beta}(\phi_p({}^cD^{\alpha}u(\tau))))d\nu d\tau \Bigg)\Bigg)\Bigg| \\ &\le \frac{(\alpha^2+\alpha-1)(q-1)m^{q-2}}{(\alpha-1)\Gamma(\alpha+1)\Gamma(\beta+1)}\delta. \end{aligned}$$

Proof. The solution of the problem

$$\begin{aligned} D^{\beta}(\phi_p({}^cD^{\alpha}u(t))) &= \mathcal{F}(t, u(t), D^{\beta}(\phi_p({}^cD^{\alpha}u(t)))) + h(t),\ \ t \in J, \alpha, \beta \in (2,3], \\ & u^{'}(t)|_{t=0} = u^{''}(t)|_{t=1} = 0, \\ & \phi({}^cD^{\alpha}u(t))|_{t=0} = \phi_p({}^cD^{\alpha}u^{'}(t))|_{t=0} = \phi_p({}^cD^{\alpha}u(t))|_{t=1} = 0, \\ & u(t)|_{t=0} = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu. \end{aligned}$$

is given by

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu \\ &+ \int_0^1 \mathcal{H}(t,\nu)\phi_q\Bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^{\beta}(\phi_p({}^cD^{\alpha}u(\tau))))d\tau \Bigg)d\nu \\ &+ \int_0^1 \mathcal{H}(t,\nu)\phi_q\Bigg(\int_0^1 \mathcal{G}(\nu,\tau)h(\tau)d\tau \Bigg)d\nu. \end{aligned}$$

From, which we have

$$\begin{aligned} &\Bigg| u(t) - \Bigg(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu \\ &+ \int_0^1 \mathcal{H}(t,\nu)\phi_q\Bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^{\beta}(\phi_p^c D^{\alpha}u(\tau)))d\tau d\nu \Bigg)\Bigg)\Bigg| \\ &\le \Bigg| \int_0^1 \mathcal{H}(t,\nu)\phi_q\Bigg(\int_0^1 \mathcal{G}(\nu,\tau)h(\tau)d\tau d\nu \Bigg)\Bigg|, \end{aligned}$$

which implies that

$$\begin{aligned}
&\bigg|u(t) - \bigg(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu \\
&+ \int_0^1 \mathcal{H}(t,\nu)\phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^\beta(\phi_p^c D^\alpha u(\tau)))d\tau d\nu\bigg)\bigg)\bigg| \\
&\leq \frac{(q-1)m^{q-2}(\alpha^2+\alpha-1)}{(\alpha-1)\Gamma(\alpha+1)\Gamma(\beta+1)}\mu.
\end{aligned}$$

□

Theorem 5.2. *If assumption* (P_3) *and* $\Gamma(\alpha+1)(1-N_{\mathcal{F}}) \neq (q-1)m^{q-2}L_{\mathcal{F}}$ *holds. Then the solution of BVP* (1.1) *is Hyers-Ulam stable.*

Proof. Let $u \in C(J, \mathcal{R})$ be the solution of (3.9) and $w \in C(J, \mathcal{R})$ be the unique solution of

$$\begin{aligned}
D^\beta(\phi_p({}^cD^\alpha w(t)))) &= \mathcal{F}(t, w(t), D^\beta(\phi_p({}^cD^\alpha w(t)))),\ t \in J, \alpha, \beta \in (2,3], \\
\phi_p({}^cD^\alpha w(t))|_{t=0} &= \phi_p({}^cD^\alpha w^{'}(t))|_{t=0} = \phi_p({}^cD^\alpha w(t))|_{t=1} = 0, \\
w(t)|_{t=0} &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, w(\nu))d\nu = u(t)|_{t=0} \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu.
\end{aligned}$$

Consider

$$\begin{aligned}
&|u(t) - w(t)| \\
&\leq \bigg|u(t) - \bigg(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1} g(\nu, u(\nu))d\nu \\
&\quad + \int_0^1 \mathcal{H}(t,\nu)\phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^\beta(\phi_p({}^cD^\alpha u(\tau))))d\tau d\nu\bigg)\bigg)\bigg| \\
&\quad + \bigg|\int_0^1 \mathcal{H}(t,\nu)\phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, u(\tau), D^\beta(\phi_p^(cD^\alpha u(\tau))))d\tau d\nu\bigg) \\
&\quad - \int_0^1 \mathcal{H}(t,\nu)\phi_q\bigg(\int_0^1 \mathcal{G}(\nu,\tau)\mathcal{F}(\tau, w(\tau), D^\beta(\phi_p({}^cD^\alpha w(\tau))))d\tau d\nu\bigg)\bigg| \\
&\leq \frac{(q-1)m^{q-2}(\alpha^2+\alpha-1)}{(\alpha-1)\Gamma(\alpha+1)\Gamma(\beta+1)}\mu + \frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\alpha+1)(1-N_{\mathcal{F}})}|u(t) - w(t)|.
\end{aligned}$$

Which on simplification implies that

$$||u-w||_{\mathcal{Z}} \le \frac{(q-1)m^{q-2}(\alpha^2+\alpha-1)}{(\alpha-1)\Gamma(\alpha+1)\Gamma(\beta+1)}\mu + \frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\alpha+1)(1-N_{\mathcal{F}})}||u-w||_{\mathcal{Z}}$$

$$\left(1-\frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\alpha+1)(1-N_{\mathcal{F}})}\right)||w-u||_{\mathcal{Z}} \le \frac{(q-1)m^{q-2}(\alpha^2+\alpha-1)}{(\alpha-1)\Gamma(\alpha+1)\Gamma(\beta+1)}\mu$$

$$||w-u||_{\mathcal{Z}} \le \frac{\frac{(q-1)m^{q-2}(\alpha^2+\alpha-1)}{(\alpha-1)\Gamma(\alpha+1)\Gamma(\beta+1)}\mu}{\left(1-\frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\alpha+1)(1-N_{\mathcal{F}})}\right)}$$

$$||w-u||_{\mathcal{Z}} \le \frac{(q-1)m^{q-2}(\alpha^2+\alpha-1)(1-N_{\mathcal{F}})}{(\alpha-1)\Gamma(\beta+1)\left(\Gamma(\alpha+1)(1-N_{\mathcal{F}})-(q-1)m^{q-2}L_{\mathcal{F}}\right)}\mu,$$

where $(\Gamma(\alpha+1)(1-N_{\mathcal{F}}) \neq (q-1)m^{q-2}L_{\mathcal{F}}$.

this implies that the solution of BVP (1.1) is Hyer-Ulam stable. Further taking $\Upsilon(\mu) = \frac{\frac{(q-1)m^{q-2}(\alpha^2+\alpha-1)}{(\alpha-1)\Gamma(\alpha+1)\Gamma(\beta+1)}\mu}{\left(1-\frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\alpha+1)(1-N_{\mathcal{F}})}\right)}$. Then clearly $\Upsilon(0)=0$. So the solution of (BVP) (1.1) is Generalized Hyers-Ulam stable. □

6 Example

Example 6.1. *Consider the following (BVP)*

$$\begin{aligned} D^{\frac{5}{2}}(\phi_2(^cD^{\frac{5}{2}}u(t)) &= \frac{|\sin(u(t))|+|\sin(D^{\frac{5}{2}}(\phi_2^c D^{\frac{1}{2}}u(t)))|}{49+9e^t},\ t\in J,\\ \phi_2(^cD^{\alpha}u(t))|_{t=0} &= \phi_2(^cD^{\alpha}u^{'}(t))|_{t=0} = \phi_2(^cD^{\alpha}u(t))|_{t=1} = 0, \qquad (6.1)\\ u(t)|_{t=0} &= \frac{1}{\Gamma(\frac{5}{2})}\int_0^1 (1-\nu)^{\frac{5}{2}}\frac{\cos u(\nu)}{9e^{\nu}})d\nu. \end{aligned}$$

We have $\alpha=\frac{5}{2}, \beta=\frac{5}{2}, p=2, q=2$ *and the nonlinear function* $\mathcal{F}(t,u(t),D^{\beta}(\phi_2(^cD^{\alpha}u(t)))) = \frac{|\sin(u(t))|+|\sin(D^{\frac{5}{2}}(^cD^{\frac{5}{2}}u(t)))|}{49+9e^t}$.
In view of Theorem 3.1, the Green functions of(BVP) (6.1) *is given by*

$$\mathcal{G}(t,\nu) = \frac{1}{\Gamma(\frac{5}{2})}\begin{cases} -t^2(1-\tau)^{\frac{3}{2}};\ t\le\nu,\\ (t-\tau)^{\frac{3}{2}} - t^2(1-\tau)^{\frac{3}{2}};\ \nu\le t, \end{cases} \qquad (6.2)$$

$$\mathcal{H}(t,\nu) = \frac{1}{\Gamma(\frac{5}{2})}\begin{cases} -\frac{5}{2}t(1-\nu)^{\frac{1}{2}};\ t\le\nu,\\ (t-\nu)^{\frac{3}{2}} - \frac{5}{2}t(1-\nu)^{\frac{1}{2}};\ \nu\le t \end{cases} \qquad (6.3)$$

Now let $u, w \in \mathcal{R},\ t \in J$, *we have*

$$\begin{aligned}&\Big|\mathcal{F}(t,u(t),D^{\beta}(\phi_2({}^cD^{\alpha}u(t)))) - \mathcal{F}(t,w(t),D^{\beta}(\phi_2({}^cD^{\alpha}w(t))))\Big|\\ &= \left|\frac{|\sin(u(t))| + |\sin(D^{\frac{5}{2}}(\phi_2({}^cD^{\frac{1}{2}}u(t))))|}{49+9e^t} - \frac{|\sin(w(t))| + |\sin(D^{\frac{5}{2}}(\phi_2({}^cD^{\frac{1}{2}}w(t))))|}{49+9e^t}\right|\\ &\le \frac{1}{58}\Big|u(t)-w(t)\Big| + \frac{1}{58}\Big|D^{\frac{5}{2}}(\phi_2({}^cD^{\frac{5}{2}}u(t)) - D^{\frac{5}{2}}(\phi_2({}^cD^{\frac{5}{2}}w(t))\Big|.\end{aligned}$$

So we have $L_{\mathcal{F}} = \frac{1}{58}$ *and* $N_{\mathcal{F}} = \frac{1}{58}, L_g = \frac{1}{9}$. *Also* $\left(\frac{1}{\Gamma(\alpha+1)}L_g + \frac{(q-1)m^{q-2}L_{\mathcal{F}}}{\Gamma(\beta+1)\Gamma(\alpha+1)(1-N_{\mathcal{F}})}\right) = \frac{124}{1000} < 1$. *Hence by Theorem 4.5, the BVP* (6.1) *has a unique solution. Further* $(q-1)m^{q-2}L_{\mathcal{F}} = \frac{1}{58}$ *and* $\Gamma(\alpha+1)(1-N_{\mathcal{F}}) = \frac{181}{58}$,*which implies that* $\Gamma(\alpha+1)(1-N_{\mathcal{F}}) \neq (q-1)m^{q-2}L_{\mathcal{F}}$. *So by Theorem* (5.2) *the solution of the considered BVP* (6.1) *is Hyer-Ulam stable and hence Generalized Hyer-Ulam stable.*

7 Conclusion

By applying Arzela Ascoli's theorem and Banach contraction theorem coupled with topological degree theory, we have developed sufficient conditions for existence and uniqueness of solution to the considered BVP(1.1) of (IFDEs) involving p-Laplacian operator. Also some results for Hyers-Ulam and generalized Hyers-Ulam types stability of the solutions for the considered BVP (1.1) are obtained.

Conflict of interests

We declare that there do not exist any conflict of interest.

Authors contribution

All authors equally contributed to this paper and approved the final version.

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