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Commutativity of Inverse Semirings through $f(xy) = [x, f(y)]$

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In this paper, we introduce an additive mapping $f : S \longrightarrow S$ on additive inverse semiring S, satisfying $f(xy) = [x, f(y)]$. We investigate its properties and establish the commutativity of additive inverse semiring with the help of this mapping

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1 Introduction

Over the last 30 years, several authors have investigated the relationships between the commutativity of the rings and certain types of maps on rings (see [3, 7, 15]). Devinsky [6] proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Posner [16] stated the existence of a non-zero centralizing derivation on prime ring forces the ring to be commutative. Mayne [15] proved the analogous result for centralizing automorphisms. These results were subsequently refined and extended by a number of authors by different mappings on rings (see

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[7, 15-17]).M.Aslam and M.Anjum [9, 10] generalized some of these results through endomorphisms and derivation in additive inverse semirings.

For completeness, we recall some preliminaries which are useful for the development of this paper. Throughout this paper, S represents additive inverse semiring with absorbing zero which satisfies A_2 condition ($a + a' \in$ Z(S)) of Bandlet and Petrich [4], with center Z(S). Karvellas [11] introduced the additive inverse semiring, a semiring S is an inverse semiring if for every $a \in$ S there exist a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$, where a' is called pseudo inverse of a. Karvellas [11] proved that for all $a, b \in$ S, $(a.b)' = a'.b = a.b'$ and $a'b' = ab$. The second author of this paper, M. Aslam and others [10] attached the A_2 condition to inverse semiring and referred it as MA semiring .Commutative inverse semirings and distributive lattices are natural examples of MA-semirings. For further details of MA semirings we recommend [10, 11]. The notion of MA-semirings is indeed crucial for developing the identities of commutators ($[x, y] = xy + y'x$,see [11]).

The Lie derivation is an additive mapping on ring, $L : R \longrightarrow R$ which satisfies $L([x, y]) = [L(x), y] + [x, L(y)]$, which can be canonically extendable in inverse semirings. The origin of Lie derivations has tremendous applications in Differential geometry and Tensor fields, see [14, 19, 20] For more details about Lie derivations in rings, we refer [5, 9, 14].

In this paper, we introduce an additive mapping $f : S \longrightarrow S$, if it satisfies $f(xy) = [x, f(y)]$. This mapping admits Lie derivations. An additive mapping is called centralizing on S if $[[f(x), x], y] = 0$ for all $x, y \in$ S, in special case where $[f(x), x] = 0$ for all x $\in$ S, the mapping f is said to be commuting.The additive mapping f on a additive inverse semiring S is strong commutativity preserving if $[f(x), f(y)] + [x, y]' = 0$. Lie ideal is an additive subsemigroup U of S satisfies $[u, r] = ur + r'u \in U$ for all $u \in U, r \in S$.

Throughout this paper, f stands for additive mapping on S, which satisfies$f(xy) = [x, f(y)]$, unless mentioned otherwise.After examining some properties we investigate the commutativity of MA-semirings by f. We establish that, if S is semiprime MA-semiring and f is centralizing, then f is commuting. Furthermore we inquire more conditions on f which enforces the MA semiring to be commutative (see Theorems 3.4, 3.5, 3.7)
In the following theorem we mention a few fundamental properties of MA-semirings which are useful for this article.

Theorem 1.1. [10, 11] *Let S be a MA-semiring, then*
(1) *If $a + b = 0$, for some $a, b \in S$ then $a = b'$*
(2) *Jacobian identity holds in S*

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z$$

(3) *Let $[x, y] = 0$, for all $x, y \in S$ then S is commutative.*

2 Some Properties of f mapping

The additive mapping f which satisfies $f(xy) = [x, f(y)]$ leads us to

$$f([x, y]) = f(xy) + f'(yx) = [x, f(y)] + [f(x), y]$$

and hence it admits lie derivation in inverse semirings. Therefore every f mapping is Lie derivation, however, the converse is not true in general as follows from the following example.

Example 2.1. *Every Inner derivation is a lie derivation.*
To show this, firstly we establish the following identity in MA-semiring.

$$[a, [x, y]] = [x, [a, y]] + [[a, x], y] \tag{2.1}$$

By expanding right side, we get $[x, ay]+[ya, x]+[ax, y]+[y, xa]$. By Theorem 1.1, we have

$$a[x, y] + [x, a]y + y[a, x] + [y, x]a + a[x, y] + [a, y]x + [y, x]a + x[y, a]$$

or $axy + ayx' + xay + axy' + yax + yxa' + yxa + xya' + axy + ayx' + ayx + yax' + yxa + xya' + xya + xay'$. By using the properties of MA-semiring and after simplification , we obtain

$$a(xy + y'x) + (xy + y'x)a' \text{ or } [a, [x, y]]$$

If S is non-commutative MA-semiring, then inner derivation satisfies Lie derivation, since $[x, f(y)] + [f(x), y] = [x, [a, y]] + [[a, x], y]$. By identity (2.1), it follows that

$$[x, f(y)] + [f(x), y] = [a, [x, y]] = f([x, y])$$

Hence, every inner derivation is a Lie derivation , but it doesnot satisfy $f(xy) = [x, f(y)]$.
Consider S be a commutative MA-semiring, then an inner derivation f such that $f(x) = [a, x]$ for some fixed $a \in S$, and for all $x \in S$. This satisfies $f(xy) = [x, f(y)]$ and this also admits Lie derivation.

Example 2.2. *Let $M_n(S)$ be the nn matrix ring over the MA-semirings with unity and tr : $M_n(S) \longrightarrow S$ be the trace mapping. Define L : $M_n(S) \longrightarrow M_n(S)$ as*

$$L(A) = [a, A] + tr(A).I_n = aA + Aa' + tr(A).I_n$$

for $A \in M_n(S)$, and some fixed $a \in S$. By using identity (1), we can verify that this additive mapping L is a Lie derivation but doesnot satisfy f mapping in general. For instance, we take S as set of integers Z which is a ring, and $X, Y \in M_2(Z)$. If $X = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, then [X, Y] $= \begin{bmatrix} 4 & -2 \\ 4 & -4 \end{bmatrix}$, whereas, tr([X, Y])= 0. If we fix $a = 1 \in Z$, then we can calculate that L([X, Y]) $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Moreover, [X, L(Y)] $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and [L(X), Y] $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

On the other hand, $L(XY) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq$ [X, L(Y)] $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence this mapping satisfies Lie derivation but doesn't admit f .

In the following Lemma, we collect some useful properties of f which can be verified.

Lemma 2.3. (1) *If S is with nilpotency index 2, then f is commuting.*
(2) *If $f(x^2) = 0, for\ all\ x \in S$ then f is commuting.*
(3) *If S is with unity, then $f(x) = f(x) + f'(x)$*
(4) *In a semiprime MA-semiring S, if $f(x)f(y) = 0, for\ all\ x, y \in S$ then $f = 0$.*
(5) *If f is commuting on S, then $f(x^n) = 0, \forall\ n \geq 2$.*
(6) *If $f(xa) = 0$, for all $x \in S$, then $f(a) \in Z(S)$.*

Proposition 2.4. *Let S be a semiprime MA-semiring and $a, b \in S$. If $f(a)x + x'f(b) = 0$ then $f(xb) = 0$ for all $x \in S$, and $f(a) = f(b) \in Z(S)$.*

Proof. By hypothesis, we have

$$f(a)x + x'f(b) = 0. \tag{2.2}$$

By Theorem (1.1), it gives $f(a)x = xf(b)$ Replace x by xy in (2.2), we get $f(a)xy+x'yf(b) = 0$ or $xf(b)y+x'yf(b) = 0$ or $x[f(b), y] = 0\ for\ all\ x, y \in S$ This implies that $[f(b), y]x[f(b), y] = 0$. By Semiprimeness, we have

$[f(b), y] = 0$ and hence $f(yb) = 0, for\ all\ y \in S$.
Moreover, By Lemma 2.3(6), we get $f(b) \in Z(S)$, therefore (2.2) becomes $(f(a)+f'(b))x = 0$ or $(f(a)+f'(b))x(f(a)+f'(b) = 0$. Again by semiprimeness, we get $f(a) + f'(b) = 0$ and hence $f(a) = f(b)$. □

Let f be commuting on S, then by Lemma 2.3 (4), we have $f(x^2) = [x, f(x)] = 0$, and this leads to the following Proposition.

Proposition 2.5. *(a) If f is injective then S is of index 2.*
(b) If f is injective and S is non-zero semiprime MA-semiring then $f = 0$

Theorem 2.6. *Let f be non-zero on prime MA-semiring, U be a non-zero right ideal of S such that $f(U) = 0$, then $f(S) \subseteq Z(S)$ and also f is commuting on S.*

For the proof of Theorem 2.6 we need the following two Lemmas, which are interesting independently.

Lemma 2.7. *Let S be a semiprime MA-semiring and $a \in S$ such that $a[a, x] = 0$ for all $x \in S$, then $a \in Z(S)$*

Proof. By hypothesis, $a[a, x] = 0$ for all $x \in S$. Replace x by xy , we get

$$a[a, xy] = 0, \quad for\ x, y \in S \tag{2.3}$$

By Theorem 1.1, we get $a[a, x]y+ax[a, y] = 0$, or $ax[a, y] = 0\ for\ all\ x.y \in S$. This implies that $aS[a, y] = 0$, and by using semiprimeness, we get $[a, y] = 0$ for all $y \in S$. Hence, by Theorem 1.1, we get $a \in Z(S)$ □

Lemma 2.8. *Let S be a semiprime MA-semiring and U be a non-zero right ideal of S. If $[a, x] = 0$ for all $x \in U$, then $a \in Z(S)$.*

Proof. As $[a, x] = 0$, for all $x \in U$, implies that $ax = xa$. Also ar $\in U$, for all $r \in S$. Therefore $[a, ar] = 0$ or $a[a, r] = 0$, for all $r \in S$. By Lemma 2.7, we get the result. □

Proof. **Theorem 2.6** By hypothesis $f(x) = 0, for\ all\ x \in U$, implies that $f(xr) = 0\ for\ some\ r \in S$. By definition of f, we get

$$[x, f(r)] = 0 \quad for\ all\ x \in U \tag{2.4}$$

By Lemma 2.8, we get $f(r) \in Z(S) for\ all\ r \in S$, and hence $f(S) \subseteq Z(S)$. Replace x by xr in (2.4),we get $[xr, f(r)] = 0$ or $x[r, f(r)]+[x, f(r)]r = 0$ By using (2.4), we have $x[r, f(r)] = 0$. Here, U is right ideal, we can replace x by xs, for some $s \in S$, which implies that $xs[r, f(r)] = 0$ or $xS[r, f(r)] = 0$
By using primeness, we have
$[r, f(r)] = 0, for\ all\ r \in S$. This shows that f is commuting on S. □

Proposition 2.9. *Let $f \neq 0$, and U be a Lie ideal of prime MA-semiring S such that $f(U) = 0$, then $f(S) \subseteq Z(U)$. Moreover, if U is not a ring, then f is centralizing on S.*

Proof. As U is a lie ideal, then for all $u \in U, x \in S$, we have $ux + x'u \in U$. Therefore, by hypothesis, we get $f(ux + x'u) = 0$, which implies that $[u, f(x)] + [x', f(u)] = 0$ or

$$[u, f(x)] = 0 \ for\ all\ u \in U, x \in S \tag{2.5}$$

Hence $f(S) \subseteq Z(U)$. Replace u by $uy + y'u$ for some $y \in S$ in (2.5), which implies that $[uy + y'u, f(x)] = 0$, we get that

$$u[y, f(x)] + [y, f(x)]u' = 0 \tag{2.6}$$

Replace y by x in (2.6), which implies that $[u, [x, f(x)]] = 0$ or $[u, f(x^2)] = 0$. Replace u by $ur+r'u for\ some\ r \in S$, we get that $u[r, f(x^2)]+[r, f(x^2)]u' = 0$. By using (2.6), we have $(u+u')[r, f(x^2)] = 0$. By using primeness, either $u + u' = 0$ or $[r, [x, f(x)]] = 0$.As U is not a ring , therefore $[r, [x, f(x)]] = 0\ for\ all\ x, r \in S$. This completes the proof. □

Theorem 2.10. *(a) Let U be a left ideal of S, if f satisfies $f(u) + u' = 0$ for all $u \in U$, then $U \subseteq l(S)$, where $l(S) = \{y \in S, yx = 0\ for\ all\ x \in S\}$, be the left annihilator of S. Moreover, if S is semiprime then $U = \{0\}$*
(b) Let U be right ideal of S, and f satisfies $f(u)+u' = 0$ for all $u \in U$, then $u^2 \in Z(S)$, for all $u \in U$. Moreover, if S is prime then either $U = \{0\}$ or S ia a ring.

Proof. (a) By hypothesis, $f(u) + u' = 0$ which implies that $f(u) = u$ for all $u \in U$. Replace u by ru, we get $f(ru) + ru' = 0$ or $[r, f(u)] + ru' = 0$ or $[r, u] + ru' = 0$. This implies that $ru + u'r + ru' = 0$. As S is MA-semiring, we get $u'r = 0$ for all $u \in U$, $r \in S$. This implies that

$$uS = \{0\}\ for\ all\ u \in U$$

Hence both results can be followed immediately.

(b) By hypothesis, $f(u) + u' = 0$, this implies that $f(u) = u$ for all $u \in U$. As U is right ideal, therefore $f(ur) + (ur)' = 0$ for $r \in S$. We get that

$$uf(r) + f(r)u' + u'r = 0\ for\ all\ r \in S \tag{2.7}$$

Replace r by ru, which implies that $uf(ru)+f(ru)u'+u'ru=0$ or $u[r,f(u)]+[r,f(u)]u'+u'ru=0$. By using $f(u)=u$, we obtain

$$uru+u^2r'+r'u^2=0 \tag{2.8}$$

Moreover, if we replace r by ur in (2.7), we get that $uf(ur)+f(ur)u'+u^2r'=0$. As U is right ideal, therefore by hypothesis, we get

$$u^2r+uru'+u^2r'=0 \tag{2.9}$$

As S is MA-semirings, then from (2.8) and (2.9), we get that $u^2r+r'u^2=0$. By Theorem 1.1, we get the result.
If S is prime then from $u^2r+r'u^2=0$ we have $u^2r+u^2r'=0$ or $u^2(r+r')=0$, which implies that $u^2S(r+r')=0$. By using primeness, we get that either $u^2=0$ *implies* $u=0$ or $r+r'=0$ *for all* $r\in S$, that the S is a ring. □

From previous theorem we can conclude

Corollary 2.11. *Let U be an ideal in prime MA-semiring S, and f satisfies $f(u)+u'=0$ for all $u\in U$, then $U=\{0\}$*

Proposition 2.12. *If f is injective on a non-zero left ideal U on a prime MA semiring S, then it is injective on S.*

Proof. To show the injectivity of f, we need to show that $kerf=0$. It is observed that $kerf$ is left ideal, since $f(rx)=[r,f(x)]=[r,0]=0, for\ all\ r\in S, x\in kerf$, this implies that

$$(kerf)U\subseteq kerf\cap U$$

As f is injective on U, which implies that $(kerf)U\subseteq kerf\cap U=\{0\}$, and we get $(kerf)U=\{0\}$. As U is left ideal therefore $(kerf)SU=\{0\}$
Now S is prime and $U\neq 0$, therefore $kerf=\{0\}$. This completes the proof. □

3 Main Results

The following theorem provides the Commuting condition of f

Theorem 3.1. *Let U be a non-zero ideal of semiprime MA semiring S. If f is centralizing on U, that f is commuting on U*

Proof. As f is centralizing on U, therefore

$$[[f(x), x], z] = 0 \qquad , for\ all\ x \in U, z \in S \tag{3.1}$$

By Theorem 1.1, we get $[f(x), x]z = z[f(x), x] \quad for\ all\ x \in U, z \in S$ By lineralization of (3.1), we get

$$[[f(x), y] + [f(y), x], z] = 0 \tag{3.2}$$

Replace y by x^2, we get

$$[[f(x), x^2] + [f(x^2), x], z] = 0$$

or $[[f(x), x]x + x[f(x), x] + [[x, f(x)], x], z] = 0$, by using (3.1) we get

$$[2x[f(x), x], z] = 0\ for\ all\ x \in U, z \in S \tag{3.3}$$

Replace z by $f(x)$, we get $[2x[f(x), x], f(x)] = 0$, and therefore

$$2\{x[[f(x), x], f(x)] + [x, f(x)][f(x), x]\} = 0$$

$$2[x, f(x)][f(x), x] = 0$$

$$2[f(x), x]^2 = 0$$

As centre of semiprime inverse semiring does not contain non-zero nilpotent elements, therefore

$$2[f(x), x] = 0 \qquad , for\ all\ x \in U \tag{3.4}$$

By Linearlizing (3.4), we have

$$[y, f(x)] + [x, f(y)] = 0 \tag{3.5}$$

Now consider
$[xy + yx, f(x)] + [x^2, f(y)]$ $=[xy, f(x)] + [yx, f(x)] + x[x, f(y] + [x, f(y)]x$

$=[x, f(x)]y + x[y, f(x)] + y[x, f(x)] + [y, f(x)]x + x[x, f(y)] + [x, f(y)]x$
By using (3.2),

$$= 2y[x, f(x)] + 2x\{[y, f(x)] + [x, f(y)]\}$$

by using (3.4) and (3.5), we get

$$[xy + yx, f(x)] + [x^2, f(y)] = 0 \tag{3.6}$$

If we substitute $y = f(x)x$ in (3.6) we obtain

$$[xf(x)x + f(x)x.x, f(x)] + [x^2, f(f(x).x)] = 0$$

$$[(xf(x) + f'(x)x + f(x)x + f(x)x)x, f(x)] + [x^2, [f(x), f(x)]] = 0$$

$$[([x, f(x)] + 2f(x)x, f(x)] + [x^2, [f(x), f(x)]] = 0 \text{ , which gives}$$

$$[[x, f(x)]x, f(x)] + [2f(x)x^2, f(x)] + x^2[f(x), f(x)] + [f(x), f(x)]x^2 = 0$$

$$[x, f(x)]^2 + [[x, f(x)], f(x)]x + 2f(x)[x^2, f(x)] + 2[f(x), f(x)]x^2 + \\ x^2[f(x), f(x)] + x^2[f(x), f(x)]' = 0$$

As $[f(x), f(x)] \in Z(S)$ and $[f(x), f(x)] = [f(x), f(x)]'$, therefore we get

$$[x, f(x)]^2 + [x, f(x)]f(x)x + f'(x)[x, f(x)]x + 2f(x)[x^2, f(x)] \\ + [f(x), f(x)](2(x^2)' + 2(x^2)) = 0 \tag{3.7}$$

As $[x^2, f(x)] = x[x, f(x)] + [x, f(x)]x = 2x[x, f(x)] = 0$, so we have

$$[x, f(x)]^2 + [x, f(x)]f(x)(x + x') + [f(x), f(x)]((x^2)' + (x^2)) = 0$$

Consequently, we get

$$[x, f(x)]^2 + 2[x, f(x)]f(x)(x + x') = 0 \tag{3.8}$$

By using (3.4) , it becomes $[x, f(x)]^2 = 0$. As S is semiprime MA-semiring therefore $[x, f(x)] = 0$ and hence f is commuting on S □

Corollary 3.2. *Let S be a semiprime MA semiring, if f is centralizing on S then f is commuting on S.*

By Proposition 2.9 and Corollary 3.2, we get the following result

Corollary 3.3. *Let U be a Lie ideal of prime MA-semiring S. If $f(x) = 0$, $for\ all\ x \in U$, then either U is a ring or f is commuting on S.*

Proposition 3.4. *Let S be a prime MA-semiring and U be its non-zero right ideal. If f is surjective and $f(uv) = 0$ for all $u, v \in U$ then U is ring or S is commutative.*

Proof. Consider $f(uv) = 0$ implies that [u,$f(v)$] = 0, Substituting ur in place of v, we get [u, $f(ur)$] = [u, [u, $f(r)$]] = 0
Here f is surjective therefore we can replace $f(r)$ with r , implies

$$[\text{u}, [\text{u}, \text{r}]] = 0 \text{ ,for all r} \in \text{S ,u} \in U$$

Therefore $u[u, r] + [u, r]u' = 0$ or $u[u, r] + u'[u, r] = 0$ or $(u + u')[u, r] = 0$ or $(u + u')S[u, r] = 0$,consequently we have either $(u + u') = 0$ *or* $[u, r] = 0$ for all r ∈ S, u $\in U$. If $(u + u') = 0$ for all $u \in U$ then U is ring, otherwise [u, r] = 0 which implies that $U \subseteq Z(S)$. If we replace u by us, for some s∈ S, we get $u[s, r] + [u, r]s = 0 or u[s, r] = 0$ which implies that $uS[s, r] = 0$. By primeness of S we have $[s, r] = 0$ for all $s, r \in S$. By Theorem 1.1, we get $sr = rs$ for all $s, r \in S$. Hence S is commutative □

Theorem 3.5. *Let S be semiprime MA-semiring and f be centralizing and surjective, then f forces the S to be commutative.*

Proof. As f is centralizing so by Theorem (3.1), it is commuting, therefore

$$[f(x), x] = 0 \text{ } for \text{ } all \text{ } x \in S \tag{3.9}$$

By linearlization of (3.9), we get

$$[f(x), y] = [x, f(y)] \text{ } for \text{ } all \text{ } x, y \in S \tag{3.10}$$

Replace x by xy, we get

$[f(xy), x] = [xy, f(x)]$ or $[f(xy), x] = x[y, f(x)] + [x, f(x)]y$, which gives

$$[f(xy), x] = x[y, f(x)] \tag{3.11}$$

By definition of f,

$$[f(xy), x] = [[x, f(y)], x] \tag{3.12}$$

From (3.11) and (3.12), we get

$$[f(xy), x] + [f(xy), x]' = [[x, f(y)], x] + x[yf(x)]' \tag{3.13}$$

$$= [x, [x, f(y)] + x[y, f(x)]]$$

This leads the following

$$[f(xy), x] + [f(xy), x]' = x[x, f(y)] + [x, f(y)]'x + x[y, f(x)] \tag{3.14}$$

On the other hand, by using (3.10)

$$[f(xy), x]+[f(xy), x]' = [xy, f(x)]+[xy, f(x)]' = x\{[y, f(x)]+[f(y), x]'\}$$

$$= x\{[f(x), y] + [f(y), x]\}' = 0$$

(3.14) becomes

$$x[f(x), y] + [x, f(y)]'x + x[f(x), y]' = 0.$$

This implies that

$$[x, f(y)]x = 0$$

Since f is surjective, therefore we can replace $f(y)$ by y and we obtain

$$[x, y]x = 0 for\ all\ x, y \in S$$

Replace y by yr, we get $[x, yr]x = 0$ or $[x, y]rx + y[x, r]x = 0$ or$[x, y]Sx = 0$, consequently we have, $[x, y]S[x, y] = 0$
By using semiprimeness, we have $[x, y] = 0$ $\quad , for\ all\ x, y \in S$
Hence by using Theorem 1.1, we have S is commutative. □

Proposition 3.6. *(a) If f satisfies $f(xy) + [x, y]' = 0$ $for\ all\ x, y \in S$ then $f + I'$ maps S into Z(S)*
(b) Let S be prime MA semiring, and f is centralizing which satisfies $f(xy) + f'(x)y = 0$,then $f = 0$ or S is a ring.

Proof. (a) Suppose $f(xy) + [x, y]' = 0$, which implies that

$$[x, f(y)] + [x, y]' = 0$$

or $[x, f(y) + I'(y)] = 0$ for all x, y $\in S$. Hence $(f + I')(y) \in Z(S)\ for\ all\ y \in S$ or $f + I'$ maps S into Z(S).
(b) Let $f(xy) + f'(x)y = 0$, this implies that

$$[x, f(y)] + f'(x)y = 0 \tag{3.15}$$

Also f is centralizing, therefore by Theorem 3.1, it is commuting ,and by using (3.10) , it becomes

$$[f(x), y] + f'(x)y = 0 \tag{3.16}$$

or $f(x)y + y'f(x) + f'(x)y = 0$ which gives $f(x)y + [f(x), y] = 0$. From (3.15) we get

$$f(x)y + [x, f(y)] = 0 \tag{3.17}$$

From (3.16) and (3.17), we get$f(x)(y + y') = 0$, for all x,y $\in S$. Replace y by ry , we have $f(x)r(y + y')$= 0, and by the primeness of S, we get the result. □

Theorem 3.7. *Let S be a prime MA-semiring and U be a nonzero left ideal of S. If f is centralizing and strong commutativity preserving map on U, then S is commutative*

Proof. As f is Strong commutativity preserving on U therefore by definition

$$[f(x), f(y)] + [x, y]' = 0 \ for \ all \ x \in S \tag{3.18}$$

Replace y by xy, we get

$$[f(x), f(xy)] + [x, xy]' = 0 \text{ or } [f(x), [x, f(y)]] + x[x, y]' = 0$$

or $x[f(x), f(y)] + [f(x), x]f(y) + [f(x), f(y)]x' + f(y)[f(x), x] + x[x, y]' = 0$
By using Theorem 3.1 and (3.18), we obtain

$$x[x, y] + [x, y]x' + x[x, y]' = 0$$

Again by Using (3.18) we get , $[x, y]x = 0$ for all $x, y \in U$ or $[x, y]SU = 0$. By using primeness we get $[x, y] = 0$. Replace y by ry for $r \in S$, we get $[x, r]y = 0$, or $[x, r]Sy = \{0\}$ and by using primeness we get$[x, r] = 0$ for $x \in U, r \in S$,which shows that U is central ideal. If we replace x by xs, for some s∈ S, we get $x[s, r] + [x, r]s = 0$ or $x[s, r] = 0$, as U is central ideal this implies that $Sx[s, r] = xS[s, r] = 0$. By using primeness of S we get that $[s, r] = 0$ for all $s, r \in S$. Hence S is commutative □

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