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Convergence of Strictly Asymptotically Pseudo-Contractions

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Abstract : In the present paper, we first prove a weak convergence theorem for strictly pseudo contractive mapping using modified Mann algorithm. This convergence is in general not strong, therefore as a generalization of modified Mann algorithm we propose a new (CQ) algorithm for strictly pseudo contractive mappings and obtain a strong convergence theorem for this class of mappings.

Keywords : CQ-iteration process, asymptotically strictly pseudocontraction, weak and strong convergence
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1 Introduction

Given a closed convex subset C of a real Hilbert space H and let T be a (possibly) nonlinear mapping from C to C. We now consider following classes :

$$P_0 = \begin{cases} T \text{ is contractive, i.e., there exists a constant } \kappa < 1 \text{ such that} \\ \|Tx - Ty\| \leq \kappa \|x - y\| \text{ for all } x, y \in C \end{cases}$$

$$P_1 = \begin{cases} T \text{ is nonexpansive, i.e.,} \\ \|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C \end{cases}$$

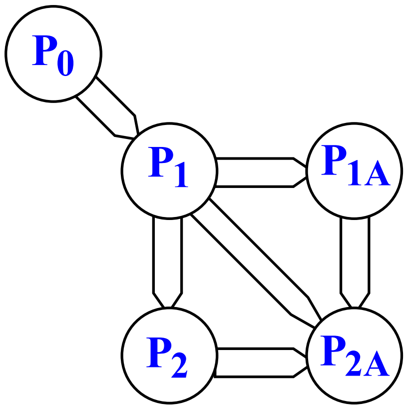
$$P_2 = \begin{cases} T \text{ is strictly pseudo-contractive, i.e., there exists a constant } \kappa \in [0,1) \text{ such that} \\ \|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(x - Tx) - (y - Ty)\|^2 \\ \text{for all } x, y \in C \end{cases}$$

$$P_{1A} = \begin{cases} T \text{ is asymptotically nonexpansive [3], i.e., if there exists a sequence } \{r_n\} \subset [0,\infty) \\ \text{with } \lim_{n\to\infty} r_n = 0 \text{ such that} \\ \|T^n x - T^n y\| \leq (1 + r_n)\|x - y\| \\ \text{for all } x, y \in C \text{ and } n \in \mathbb{N} \end{cases}$$

$$P_{1A} = \begin{cases} T \text{ is asymptotically nonexpansive [3], i.e., if there exists a sequence } \{r_n\} \subset [0,\infty) \\ \text{with } \lim_{n\to\infty} r_n = 0 \text{ such that} \\ \|T^n x - T^n y\| \le (1+r_n)\|x-y\| \\ \text{for all } x, y \in C \text{ and } n \in \mathbb{N} \end{cases}$$

$$P_{2A} = \begin{cases} T \text{ is strictly asymptotically pseudo-contractive [7], i.e., if there exists a sequence} \\ \{r_n\} \subset [0,\infty) \text{ with } \lim_{n\to\infty} r_n = 0 \text{ such that} \\ \|Tx - Ty\|^2 \le (1+r_n^2)\|x-y\|^2 + \kappa\|(x - T^n x) - (y - T^n y)\|^2 \\ \text{for some } \kappa \in [0,1) \text{ for all } x, y \in C \text{ and } n \in \mathbb{N} \end{cases}$$

If follows from the definition that



The class of strictly pseudo-contractive mappings has been studied by several authors. (see, for example [1, 4, 9, 11] and references therein.)

In case of contractive mapping, the Banach Contraction Principle guarantee not only the existence of unique fixed point, but also to obtain the fixed point by successive approximation (or Picard iteration). But for outside the class of contractive mapping, the classical iteration scheme no longer applies. So some other iteration scheme is required.

Two iteration processes are often used to approximate fixed point of nonexpansive and pseudocontractive mappings. The first iteration process is known as Mann's iteration process [8], where $\{x_n\}$ is defined as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n\ , \quad n \ge 0 \tag{1.1}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0,1]$.

The second iteration process is known as Ishikawa iteration process [6] which is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n \ ,$$
$$y_n = \beta_n x_n + (1-\beta_n)Tx_n \,; \qquad n \geq 0 \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$.

Process (1.2) is indeed more general than the process (1.1). But research has been concentrated on the later, probably due to the reason that process (1.1) is simpler and that a convergence theorem for process (1.1) may possibly lead to a convergence theorem for process (1.2), provided that the sequence $\{\beta_n\}$ satisfy certain appropriate conditions.

The adaptation of Mann's iteration (1.1) to asymptotically nonexpansive and asymptotically strictly pseudocontractive mappings is as below [7, 14] :

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \ \ n \geq 0. \tag{1.3}$$

where the initial guess $x_0 \in C$ is arbitrary.

Modified Mann algorithm has been extensively used to approximate fixed points of asymptotically nonexpansive mappings.

Liu [7] proved the following result for the convergence of the sequence $\{x_n\}$ generated by (1.3) :

Theorem L: *Let H be a Hilbert space, $C \subset H$ nonempty closed bounded and convex; $T : C \to C$* ***completely continuous and uniformly L-Lipschitzian****, κ-strict asymptotically pseudocontractive with sequence $\{r_n\}$, $r_n \in [0,\infty)$, $\sum_{n=0}^{\infty} r_n^2 < \infty$, $\varepsilon \leq \alpha_n \leq 1-\kappa-\varepsilon$, for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Then, the sequence $\{x_n\}$ generated by (1.3) converges strongly to some fixed point of T.*

In this paper, we first prove weak convergence theorem for strictly asymptotically pseudocontractive mappings using modified Mann iteration process (1.3), but before this we need some results :

Lemma 1.1 *Let H be a real Hilbert space. There holds the following identities :*

(i) $\|x-y\|^2 = \|x\|^2 - \|y^2\| - 2\langle x-y, y\rangle \quad \forall\, x, y \in H$.

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, $\forall\, t \in [0,1]$, $\forall\, x, y \in H$.

(iii) If $\{x_n\}$ be a sequence in H weakly convergent to z, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2 \quad \forall\, y \in H\,.$$

Lemma 1.2 *[13] Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$\{v \in C \;:\; \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Lemma 1.3 *Let K be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $x \in K$. Then $z = P_K x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leq 0 \quad \forall\, y \in K.$$

where P_K is the nearest point projection from H on to K, i.e. $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \leq \|x - y\| \quad \forall\, x \in K\,.$$

We use following notation:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x \,:\, \exists\, x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Lemma 1.4 *[13] Let K be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - u\| = \|u - q\| \quad \forall\, n\,. \tag{1.4}$$

Then $x_n \to q$.

Lemma 1.5 *[12] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n \quad n \geq 1$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

2 Weak Convergence for Modified Mann's algorithm

In this section we prove a weak convergence theorem for κ-strictly asymptotically pseudo-contractive mappings using modified Mann algorithm :

Theorem 2.1 *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a κ-strictly asymptotically pseudo-contraction for some $0 \leq \kappa < 1$, $\sum_{n=1}^{\infty} r_n < \infty$ and assume that $F(T) \neq \emptyset$, $I - T$ is demiclosed at zero. Let $\{x_n\}$ be the sequence generated by algorithm (1.3). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\kappa < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - \kappa)(1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $p \in F(T)$, using Lemma 1.1 (ii), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(T^nx_n-p)^2\| \\
&= \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|T^nx_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2 \\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\left[(1+r_n)^2\|x_n-p\|^2+\kappa\|x_n-T^nx_n\|^2\right] \\
&\quad -\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2 \\
&\le \left[\alpha_n(1+r_n)^2+(1-\alpha_n)(1+r_n)^2\right]\|x_n-p\|^2 \\
&\quad -(1-\alpha_n)(\alpha_n-\kappa)\|x_n-T^nx_n\|^2 \\
&\le (1+d_n)\|x_n-p\|^2-(1-\alpha_n)(\alpha_n-\kappa)\|x_n-T^nx_n\|^2 \qquad (2.1)
\end{aligned}$$

where $d_n = r_n^2+2r_n$, since $\sum_{n=1}^{\infty} r_n < \infty$ thus $\sum_{n=1}^{\infty} d_n < \infty$ and since $\kappa < \alpha_n < 1$, we get

$$\|x_{n+1}-p\|^2 \le (1+d_n)\|x_n-p\|^2 \qquad (2.2)$$

that means the sequence $\{\|x_n-p\|\}$ is decreasing. Now, since $\sum_{n=1}^{\infty} d_n < \infty$ it follows that $\prod_{i=1}^{\infty}(1+d_i) < \infty$, from (2.1), we have

$$\sum_{n=0}^{\infty}(\alpha_n-\kappa)(1-\alpha_n)\|x_n-T^nx_n\|^2 \le \prod_{i=1}^{\infty}(1+d_i)\|x_0-p\|^2 < \infty\,. \qquad (2.3)$$

Since $\sum_{n=0}^{\infty}(\alpha_n-\kappa)(1-\alpha_n) = \infty$, (2.3) implies that

$$\liminf_{n\to\infty}\|x_n-T^nx_n\| = 0\,. \qquad (2.4)$$

We now prove $\lim_{n\to\infty}\|x_n-Tx_n\|$ exists, for this we show that the sequence $\{\|x_n-Tx_n\|\}$ is decreasing.

Now,

$$\begin{aligned}
\|x_{n+1}-T^nx_{n+1}\|^2 &\le \|\alpha_n(x_n-T^nx_{n+1})+(1-\alpha_n)(T^nx_n-T^nx_{n+1})\|^2 \\
&= \alpha_n\|x_n-T^nx_{n+1}\|^2+(1-\alpha_n)\|T^nx_n-T^nx_{n+1}\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2 \\
&\le \alpha_n\|(x_n-x_{n+1})+(x_{n+1}-T^nx_{n+1})\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2 \\
&\quad +(1-\alpha_n)[(1+r_n)^2\|x_n-x_{n+1}\|^2 \\
&\quad +\kappa\|(x_n-T^nx_n)-(x_{n+1}-T^nx_{n+1})\|^2] \\
&= \alpha_n\left(\|x_n-x_{n+1}\|^2+\|x_{n+1}-T^nx_{n+1}\|^2\right. \\
&\quad \left.+\,2\langle x_n-x_{n+1},x_{n+1}-T^nx_{n+1}\rangle\right) \\
&\quad -\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2
\end{aligned} \qquad (2.5)$$

$$
\begin{aligned}
&+ (1-\alpha_n)[(1+r_n)^2\|x_n - x_{n+1}\|^2 \\
&+ \kappa\{\|x_n - T^n x_n\|^2 + \|x_{n+1} - T^n x_{n+1}\|^2 \\
&- 2\langle x_n - T^n x_n, x_{n+1} - T^n x_{n+1}\rangle\}] \\
= {}& \alpha_n(1-\alpha_n)^2\|x_n - T^n x_n\|^2 + \alpha_n\|x_{n+1} - T^n x_{n+1}\|^2 \\
&+ 2\alpha_n(1-\alpha_n)\langle x_n - T^n x_n, x_{n+1} - T^n x_{n+1}\rangle \\
&- \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 + (1-\alpha_n)(1+r_n)^2[(1-\alpha_n)^2\|x_n - T^n x_n\|^2] \\
&+ (1-\alpha_n)\kappa(\|x_n - T^n x_n\|^2 + \|x_{n+1} - T^n x_{n+1}\|^2) \\
&- 2(1-\alpha_n)\kappa\langle x_n - T^n x_n, x_{n+1} - T^n x_{n+1}\rangle \\
= {}& [\alpha_n(1-\alpha_n)^2 - \alpha_n(1-\alpha_n) + \kappa(1-\alpha_n) + (1-\alpha_n)^3(1+r_n)^2]\|x_n - T^n x_n\|^2 \\
&+ [\alpha_n + (1-\alpha_n)\kappa]\|x_{n+1} - T^n x_{n+1}\|^2 \\
&+ [2\alpha_n(1-\alpha_n) - 2\kappa(1-\alpha_n)]\|x_n - T^n x_n\|\|x_{n+1} - T^n x_{n+1}\| \\
\le {}& (1-\alpha_n)[\alpha_n(1-\alpha_n) - \alpha_n + \kappa + (1-\alpha_n)^2(1+d_n)]\,\|x_n - T^n x_n\|^2 \\
&+ [\alpha_n(1-\alpha_n)\kappa]\,\|x_{n+1} - T^n x_{n+1}\|^2 \\
&+ 2(1-\alpha_n)(\alpha_n - \kappa)\,\|x_n - T^n x_n\|\|x_{n+1} - T^n x_{n+1}\|
\end{aligned}
$$

We may assume $\|x_n - T^n x_n\| > 0$, setting $\gamma_n = \frac{\|x_{n+1} - T^n x_{n+1}\|}{\|x_n - T^n x_n\|}$, above inequality gives,

$$(1-\kappa)\gamma_n^2 - 2(\alpha_n - \kappa)\,\gamma_n - [1 - 2\alpha_n + \kappa + d_n(1-\alpha_n)^2] \le 0$$

gives,

$$\gamma_n \le \frac{\alpha_n - \kappa + \sqrt{(\alpha_n - \kappa)^2 + (1-\kappa)[1 - 2\alpha_n + \kappa + d_n(1-\alpha_n)^2]}}{1-\kappa}\,. \tag{2.6}$$

Now,

$$
\begin{aligned}
(\alpha_n - \kappa)^2 &+ (1-\kappa)[1 - 2\alpha_n + \kappa + d_n(1-\alpha_n)^2] \\
&\le (\alpha_n - \kappa)^2 + (1-\kappa)(1 - 2\alpha_n + \kappa) + d_n(1-\alpha_n)^2 \\
&= (1-\alpha_n)^2 + d_n(1-\alpha_n)^2 \\
&\le (1-\alpha_n)^2 + 2d_n(1-\alpha_n) + d_n^2 \\
&= (1 - \alpha_n + d_n)^2
\end{aligned} \tag{2.7}
$$

Therefore, from (2.6) and (2.7), we have

$$\|x_{n+1} - T^n x_{n+1}\| \le \left(1 + \frac{d_n}{1-\kappa}\right)\|x_n - T^n x_n\|$$

hence by Lemma 1.5, $\lim_{n\to\infty}\|x_n - T^n x_n\|$ exists.

Now, by (2.4), we get

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0\,. \tag{2.8}$$

$$\|x_{n+1} - x_n\| = (1-\alpha_n)\|x_n - T^n x_n\| \to 0 \quad \text{as} \quad n \to \infty \tag{2.9}$$

Since $I - T$ is demiclosed at zero, (2.9) imply that $\omega_w(x_n) \subset F(T)$. Now we show that $\{x_n\}$ is weakly convergent. Let $p, q \in \omega_w(x_n)$ and $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ which converges weakly to some p and q respectively.

Since $\lim_{n\to\infty} \|x_n - z\|$ exists for every $z \in F(T)$ and since $p, q \in F(T)$, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\|^2 &= \lim_{j\to\infty} \|x_{m_j} - p\|^2 \\ &= \lim_{j\to\infty} \|x_{m_j} - q\|^2 + \|q-p\|^2 \\ &= \lim_{i\to\infty} \|x_{n_i} - p\|^2 + 2\|q-p\|^2 \\ &= \lim_{n\to\infty} \|x_n - p\|^2 + 2\|q-p\|^2\end{aligned}$$

Hence $p = q$.

This completes the proof.

3 Strong Convergence Theorem

The weakness of Mann's iteration is that, its convergence is, in general, not strong even in Hilbert space (see [2] for counter example). So in order to get strong convergence some modification in Mann algorithm is needed.

Nakajo and Takahashi [10] proposed a modification in Mann's algorithm for nonexpansive mappings in Hilbert spaces, so that the strong convergence is guaranteed.

The algorithm is as below:

Consider the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C; \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{3.1}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H.

Algorithm (3.1) is called a (CQ) algorithm for the Mann iteration process because at each step the Mann iterate (denoted by y_n in (3.1)) is used to construct the sets C_n and Q_n which are in turn used to construct the next iterate x_{n+1}.

In algorithm (3.1) the initial guess x_0 is projected onto the intersection of two suitably constructed closed convex subsets C_n and Q_n.

Recently Kim and Xu [5] adapted the iteration (3.1) to asymptotically nonexpansive mappings. They introduced the following iteration process for asymptotically nonexpansive mappings in Hilbert space H :

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C ; \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{3.2}$$

where

$$\theta_n = (1-\alpha_n)\left((1+r_n)^2 - 1\right)(\text{diam } C)^2 \to 0 \quad \text{as} \quad n \to \infty ,$$

Recently, Marino and Xu [9] extended (CQ) algorithm from nonexpansive mappings to strict pseudo-contractive mappings.

It is important to note that the set C_n in the (CQ) algorithm differs among distinct classes of mappings.

Now, we propose a new (CQ) algorithm for strictly asymptotically pseudocontractive mappings :

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily ,} \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \qquad\qquad + (\kappa - \alpha_n)\|x_n - T^n x_n\|^2\} + \theta_n, \\ Q_n = \{z \in C ; \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{3.3}$$

where $\theta_n = \left((1+r_n)^2 - 1\right)(\text{diam } C)^2 \to 0$ as $n \to \infty$.

We now prove strong convergence of strictly pseudocontractive mapping using algorithm (3.3) :

Theorem 3.1 *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a κ-strictly asymptotically pseudo-contraction for some $0 \leq \kappa < 1$, $\sum_{n=1}^{\infty} r_n < \infty$ and assume that $F(T) \neq \emptyset$, $I - T$ is demiclosed at zero. Let $\{x_n\}$*

be the sequence generated by the algorithm (3.3). Assume that the sequence $\{\alpha_n\}$ *is chosen so that* $\sup_{n\geq 0}\alpha_n < 1$. *Then* $\{x_n\}$ *converges strongly to* $P_{Fix(T)}x_0$.

Proof. By Lemma 1.2, we observe that C_n is convex.

Now, for all $p \in F(T)$, using Lemma 1.1(ii), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T^n x_n - p)\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T^n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\left[(1+r_n)^2\|x_n - p\|^2 + \kappa\|x_n - T^n x_n\|^2\right] \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\
&\leq (1+r_n)^2\|x_n - p\|^2 + (1-\alpha_n)\kappa\|x_n - T^n x_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\
&\leq \|x_n - p\|^2 + (\kappa - \alpha_n)\|x_n - T^n x_n\|^2 + \theta_n,
\end{aligned}$$

so $p \in C_n$ for all n. So $F(T) \subset C_n$ for all n.

Next we show that $F(T) \subset Q_n$ for all $n \geq 0$, for this we use induction.

For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$.

Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 1.3, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0 \quad \forall z \in C_n \cap Q_n\,.$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence $F(T) \subset Q_n$ for all $n \geq 0$.

Now, since $x_n = P_{Q_n}x_0$ (by the definition of Q_n), and since $F(T) \subset Q_n$, we have

$$\|x_n - x_0\| \leq \|p - x_0\| \quad \forall\, p \in F(T).$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \quad \text{where } q = P_{F(T)}x_0\,. \tag{3.4}$$

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. This together with Lemma 1.1 (i), implies that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2
\end{aligned}$$

It follows that,

$$\|x_{n+1} - x_n\| \to 0. \tag{3.5}$$

By the fact $x_{n+1} \in C_n$ we get

$$\begin{aligned} \|x_{n+1} - y_n\|^2 \leq \|x_{n+1} - x_n\|^2 \\ + (\kappa - \alpha_n)\|x_n - T^n x_n\|^2 + \theta_n \end{aligned} \tag{3.6}$$

Moreover, since $y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n$, we deduce that

$$\begin{aligned} \|x_{n+1} - y_n\|^2 = \alpha_n\|x_{n+1} - x_n\|^2 \\ + (1 - \alpha_n)\|x_{n+1} - T^n x_n\|^2 \\ - \alpha_n(1 - \alpha_n)\|x_n - T^n x_n\|^2. \end{aligned} \tag{3.7}$$

Substitute (3.7) into (3.6) to get

$$(1 - \alpha_n)\|x_{n+1} - T^n x_n\|^2 \leq (1 - \alpha_n)\|x_{n+1} - x_n\|^2 + \kappa\|x_n - T^n x_n\|^2 + \theta_n$$

Since $\alpha_n < 1$ for all n, the last inequality becomes,

$$\|x_{n+1} - T^n x_n\|^2 \leq \|x_{n+1} - x_n\|^2 + \kappa\|x_n - T^n x_n\|^2 + \frac{\theta_n}{\tau} \tag{3.8}$$

for some positive number $\tau > 0$, such that $\alpha_n < \tau < 1$.

But on the otherhand, we compute

$$\begin{aligned} \|x_{n+1} - T^n x\|^2 = \|x_{n+1} - x_n\|^2 + 2\langle x_{n+1} - x_n, x_n - T^n x_n\rangle \\ + \|x_n - T^n x_n\|^2 . \end{aligned} \tag{3.9}$$

By (3.8) and (3.9), we get

$$(1 - \kappa)\|x_n - T^n x_n\|^2 \leq \frac{\theta_n}{\tau} - 2\langle x_{n+1} - x_n, x_n - T^n x_n\rangle \tag{3.10}$$

Therefore

$$\|x_n - T^n x_n\|^2 \leq \frac{\theta_n}{\tau(1 - \kappa)} - \frac{2}{1 - \kappa}\langle x_{n+1} - x_n, x_n - T^n x_n\rangle \to 0 \tag{3.11}$$

Now,

$$\begin{aligned} \|x_n - Tx_n\| &= \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\ &\leq \|x_n - T^n x_n\| + (1 + r_1)\|T^{n-1}x_n - x_n\| \\ &\to 0 \end{aligned} \tag{3.12}$$

Now, since $(I - T)$ is demiclosed at zero, (3.11) guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T. That is, $\omega_w(x_n) \subset F(T)$. This fact, the inequality (3.4) and Lemma 1.4 implies that $\{x_n\} \to q = P_{F(T)}x_0$.

This completes the proof.

Remark: It is of interest to extend the results of this paper to asymptotically pseudocontractive mappings, which are mappings satisfying P_{2A} with $\kappa = 1$. For such a mapping, Mann's algorithm does not converge even though C is assumed to be compact.

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