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On Solving the Split Feasibility Problem and the Fixed Point Problem in Banach Spaces

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Abstract : In this paper, we introduce an iterative method for solving the split feasibility problem and the fixed point problem of countable family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Then, we prove strong convergence theorem of the sequence generated by our iterative scheme with a new way of selecting the step-size which does not require the computation on the norm of the bounded linear operator. The obtained result of this paper complements many recent results in this direction.

Keywords : Split feasibility problem; Banach space; Strong convergence; Iterative method; Fixed point
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1 Introduction

Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also

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uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be its adjoint of A. The *split feasibility problem* (SFP) is to find an element

$$\hat{x} \in C \quad \text{such that} \quad A\hat{x} \in Q. \tag{1.1}$$

The set of solutions of problem (1.1) is denoted by Γ, *i.e.*, $\Gamma := \{x \in C : Ax \in Q\}$. It is well known that if Γ is nonempty then Γ is a closed and convex subset of E_1. The SFP was first introduced, in a finite dimensional Hilbert space, by Censor-Elfving [1] in 1994 for modeling inverse problems in radiation therapy treatment planning which arise from phase retrieval and in medical image reconstruction (see [2]). The SFP has also been studied by numerous authors in both finite and infinite dimensional Hilbert spaces (see, *e.g.*, [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

For solving the SFP in Banach spaces, Schöpfer et al. [14] first introduced the following algorithm for solving the SFP: $x_1 \in E_1$ and

$$x_{n+1} = \Pi_C J_{E_1}^* \left[J_{E_1}(x_n) - \lambda_n A^* J_{E_2}(Ax_n - P_Q(Ax_n)) \right], \ n \geq 1, \tag{1.2}$$

where $\{\lambda_n\}$ is a positive sequence, Π_C denotes the generalized projection on E, P_Q is the metric projection on E_2, J_{E_1} is the duality mapping on E_1 and $J_{E_1}^*$ is the duality mapping on E_1^*. It was proved that the sequence $\{x_n\}$ converges weakly to a solution of SFP, under some mild conditions, in p-uniformly convex and uniformly smooth Banach spaces.

Recently, Shehu et al. [15] introduced an iterative scheme for solving the SFP and the fixed point problem of Bregman strongly nonexpansive mapping T in the framework of p-uniformly convex real Banach spaces which are also uniformly smooth as follows: Let $u \in C$, $u_1 \in E_1$ and

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*} \big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n \big) \\ u_{n+1} = \Pi_C J_q^{E_1^*} \left[\alpha_n J_p^{E_1}(u) + (1-\alpha_n) \big(\beta_n J_p^{E_1}(x_n) + (1-\beta_n) T x_n \big) \right], \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and the step-size λ_n is chosen by $0 < t \leq \lambda_n \leq k < \big(\frac{q}{\kappa_q \|A\|^q} \big)^{\frac{1}{q-1}}$.

They proved that the sequence $\{x_n\}$ and $\{u_n\}$ defined by (1.3) converge strongly to a point in $F(T) \cap \Gamma$ under some mild conditions. However, it is observed that iterative method (1.3) involves step-size that depend on the operator norm $\|A\|$ (matrix in the finite-dimensional space), which may not be calculated easily in general. It makes the implementation of the iteration process inefficient when the computation of the operator norm $\|A\|$ is not explicit (see [16, 17]).

Motivated by the previous works, we introduce an iterative method for solving the split feasibility problem and the fixed point problem of countable family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Then, we prove strong convergence theorem of the sequence generated by our iterative scheme with a new way of selecting the step-size which does not require the computation on the norm of the bounded linear operator. Our result complements the results of Byrne [2], Schöpfer et al. [14], Wang [18], Shehu et al. [15], Shehu et al. [19] and many other recent results in the literature.

2 Preliminaries

Let E and E^* be real Banach spaces and the dual space of E, respectively. Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$ which is defined by

$$\langle A^* \bar{y}, x\rangle := \langle \bar{y}, Ax\rangle, \quad \forall x \in E_1, \quad \bar{y} \in E_2^*.$$

Let $S(E) := \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \tfrac{\|x+y\|}{2} : x, y \in S(E), \|x - y\| \geq \epsilon \right\}.$$

The space E is said to be *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let $p > 1$. Then E is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p \epsilon^p$ for all $\epsilon \in (0, 2]$. Observe that every p-uniformly convex space is uniformly convex. The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup \left\{ \tfrac{\|x+\tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S(E) \right\}.$$

The space E is said to be *uniformly smooth* if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Suppose that $q > 1$, a Banach space E is said to be *q-uniformly smooth* if there exists a $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q \tau^q$ for all $\tau > 0$. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth. Moreover, we note that a Banach space E is p-uniformly convex if and only if E is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$ (see [20]).

Let $p > 1$ be a real number. The *generalized duality mapping* $J_p^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^p, \|\bar{x}\| = \|x\|^{p-1}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. In particular, $J_p^E = J_2^E$ is called the *normalized duality mapping*.

In this case, we assume that E is a p-uniformly convex and uniformly smooth, which implies that its dual space, E^* is q-uniformly smooth and uniformly convex. It is known that the generalized duality mapping J_p^E is one-to-one, single-valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$, where $J_q^{E^*}$ is the generalized duality mapping of E^*.

Moreover, if E is uniformly smooth then the duality mapping J_p^E is norm-to-norm uniformly continuous on bounded subsets of E. (see [21, 22] for more details).

Definition 2.1. ([23]) *Let $f : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. The function $D_f : E \times E \to [0, +\infty)$ defined by*

$$D_f(x, y) := f(y) - f(x) - \langle f'(x), y - x\rangle,$$

is called the Bregman distance with respect to f.

We remark that the Bregman distance D_f is not satisfy the well-known properties of a metric because D_f is not symmetric and does not satisfy the triangle inequality.

It is well known that the duality mapping J_p^E is the sub-differential of the functional $f_p(\cdot) = \frac{1}{p}\|\cdot\|^p$ for $p > 1$ (see [24]). Then, we have the Bregman distance with respect to f_p that

$$D_p(x, y) = \frac{1}{q}\|x\|^p - \langle J_p^E x, y\rangle + \frac{1}{p}\|y\|^p. \tag{2.1}$$

If $p = 2$, we get

$$D_2(x, y) := \phi(x, y) = \|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2,$$

where ϕ is called the *Lyapunov function* which was introduced by Alber [25, 26]. Moreover, the Bregman distance has the following properties:

$$D_p(x, y) = D_p(x, z) + D_p(z, y) + \langle J_p^E x - J_p^E z, z - y\rangle, \tag{2.2}$$

$$D_p(x, y) + D_p(y, x) = \langle J_p^E x - J_p^E y, x - y\rangle, \tag{2.3}$$

for all $x, y, z \in E$. For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [14]):

$$\tau\|x - y\|^p \leq D_p(x, y) \leq \langle J_p^E x - J_p^E y, x - y\rangle, \tag{2.4}$$

where $\tau > 0$ is some fixed number. In what follows, we shall use the following notations:

- $x_n \to x$ mean that $\{x_n\}$ converges strongly to x;
- $x_n \rightharpoonup x$ mean that $\{x_n\}$ converges weakly to x.

Let C be a closed and convex subset of E and let T be a mapping from C into itself. We denote $F(T)$ by the set of all fixed points of T, *i.e.*, $F(T) = \{x \in C : x = Tx\}$. A point $z \in C$ called an *asymptotic fixed point* of T, if there exists a sequence $\{x_n\}$ in C which $x_n \rightharpoonup z$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ by the set of asymptotic fixed points of T.

Definition 2.2. ([27, 28]) *A mapping $T : C \to C$ is called* Bregman relatively nonexpansive*, if the following conditions are satisfied:*

(R1) $F(T) = \widehat{F}(T) \neq \emptyset$;

(R2) $D_p(Tx, z) \leq D_p(x, z), \quad \forall z \in F(T), \ \forall x \in C.$

Clearly, in a Hilbert space H, Bregman relatively nonexpansive mappings and quasi-nonexpansive mappings are equivalent, for $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$, *i.e.*,

$$\phi(Tx, z) \leq \phi(x, z) \Longleftrightarrow \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C \ \text{ and } \ z \in F(T).$$

Definition 2.3. ([29]) *Let C be a subset of a real p-uniformly convex Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the* AKTT-condition *if, for any bounded subset B of C,*

$$\sum_{n=1}^{\infty} \sup_{z \in B} \{\|J_p^E(T_{n+1}z) - J_p^E(T_n z)\|\} < \infty.$$

As in [30], we can prove the following fact.

Proposition 2.1. *Let C be a nonempty, closed and convex subset of a real p-uniformly convex Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition. Suppose that for any bounded subset B of C. Then there exists the mapping $T : B \to E$ such that*

$$Tx = \lim_{n \to \infty} T_n x, \quad \forall x \in B, \tag{2.5}$$

and

$$\lim_{n \to \infty} \sup_{z \in B} \|J_p^E(Tz) - J_p^E(T_n z)\| = 0.$$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and T is defined by (2.5) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Recall that the metric projection from E onto C, denote by $P_C x$, satisfying the property

$$\|x - P_C x\| \leq \inf_{y \in C} \|x - y\|, \quad \forall x \in E.$$

It is well known that $P_C x$ is the unique minimizer of the norm distance. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle J_p^E(x - P_C x), y - P_C x \rangle \leq 0, \quad \forall y \in C. \tag{2.6}$$

Similarly, one can define the Bregman projection from E onto C, denote by Π_C, satisfying the property

$$D_p\big(x, \Pi_C(x)\big) = \inf_{y \in C} D_p(x, y), \quad \forall x \in E. \tag{2.7}$$

Lemma 2.2. ([19]) *Let C be a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and let $x \in E$. Then the following assertions hold:*

(i) $z = \Pi_C x$ *if and only if* $\langle J_p^E(x) - J_p^E(z), y - z\rangle \leq 0$, $\forall y \in C$.

(ii) $D_p(\Pi_C x, y) + D_p(x, \Pi_C x) \leq D_p(x, y)$, $\forall y \in C$.

Lemma 2.3. [31] *Let $1 < q \leq 2$ and E be a Banach space. Then the following are equivalent.*

(i) *E is q-uniformly smooth;*

(ii) *There is a constant $\kappa_q > 0$ such that for all $x, y \in E$*

$$\|x - y\|^q \leq \|x\|^q - q\langle j_q(x), y\rangle + \kappa_q \|y\|^q. \tag{2.8}$$

Remark 2.4. *The constant κ_q satisfying* (2.8) *is called the q-*uniform smoothness coefficient of E.

The following Lemma can be obtained from Theorem 2.8.17 of [21] (see also Lemma 5 of [32]).

Lemma 2.5. *Let $p > 1$, $r > 0$ and E be a Banach space. Then the following statements are equivalent:*

(i) *E is uniformly convex;*

(ii) *There exists a strictly increasing convex function $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ with $g_r^*(0) = 0$ such that*

$$\Big\|\sum_{k=1}^{N} \alpha_k x_k\Big\|^p \leq \sum_{k=1}^{N} \alpha_k \|x_k\|^p - \alpha_i \alpha_j g_r^*(\|x_i - x_j\|),$$

for all $i, j \in \{1, 2, ..., N\}$, $x_k \in B_r := \{x \in E : \|x\| \leq r\}$, $\alpha_k \in (0, 1)$ with $\sum_{k=1}^N \alpha_k = 1$, where $k \in \{1, 2, ..., N\}$.

Lemma 2.6. ([19]) *Let E be a real p-uniformly convex and uniformly smooth Banach space. Thus, for all $z \in E$, we have*

$$D_p\left(J_q^{E^*}\left(\sum_{i=1}^{N} t_i J_p^E(x_i)\right), z\right) \leq \sum_{i=1}^{N} t_i D_p(x_i, z),$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

The following lemmas can be found in [15, 19].

Lemma 2.7. *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : E^* \times E \to [0, +\infty)$ be defined by*

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p, \quad \forall x \in E, \quad x^* \in E^*.$$

Then the following assertions hold:

(i) V_p *is nonnegative and convex in the first variable;*

(ii) $D_p(J_q^{E^*}(x^*), x) = V_p(x^*, x), \quad \forall x \in E, \quad x^* \in E^*.$

(iii) $V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x\rangle \leq V_p(x^* + y^*, x), \quad \forall x \in E, \quad x^*, y^* \in E^*.$

Following the proof line as in Proposition 2.5 of [33], we obtain the following result:

Lemma 2.8. *Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $x \in E$ and $\{x_n\}$ is a sequence in E. If $\{D_p(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.9. *Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then the following assertions are equivalent:*

(a) $\lim_{n\to\infty} D_p(x_n, y_n) = 0$;

(b) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. For the implication $(a) \Longrightarrow (b)$. Suppose that $\lim_{n\to\infty} D_p(x_n, y_n) = 0$. From (2.4), we have

$$0 \leq \tau\|x_n - y_n\|^p \leq D_p(x_n, y_n),$$

where $\tau > 0$ is a fixed number. It follows that $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

For the converse implication $(b) \Longrightarrow (a)$, we assume that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. From (2.4), we observe that

$$\begin{aligned} 0 \leq D_p(x_n, y_n) &\leq \langle J_p^E x_n - J_p^E y_n, x_n - y_n\rangle \\ &\leq \|J_p^E x_n - J_p^E y_n\|\|x_n - y_n\| \\ &\leq \|x_n - y_n\|M, \end{aligned}$$

where $M = \sup_{n\geq 1}\{\|x_n\|^{p-1}, \|y_n\|^{p-1}\}$. It follows that $\lim_{n\to\infty} D_p(x_n, y_n) = 0$. This completes the proof. □

Lemma 2.10. ([34]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11. ([35]) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq ...$ *and* $\tau(n) \to \infty$*;*

(ii) $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Lemma 2.12. *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^{N} \alpha_k = 1$. Then, we have*

$$D_p\bigg(J_q^{E^*}\bigg(\sum_{k=1}^{N}\alpha_k J_p^E(x_k)\bigg), z\bigg) \leq \sum_{k=1}^{N}\alpha_k D_p(x_k, z) - \alpha_i\alpha_j g_r^*\big(\|J_p^E(x_i) - J_p^E(x_j)\|\big),$$

for all $i, j \in \{1, 2, ..., N\}$.

Proof. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^{N} \alpha_k = 1$. Since p-uniformly convex, hence it is uniformly convex. From Lemmas 2.5 and 2.6, we have

$$\begin{aligned}
& D_p\bigg(J_q^{E^*}\bigg(\sum_{k=1}^{N}\alpha_k J_p^E(x_k)\bigg), z\bigg) \\
= \;& V_p\bigg(\sum_{k=1}^{N}\alpha_k J_p^E(x_k), z\bigg) \\
= \;& \frac{1}{q}\Big\|\sum_{k=1}^{N}\alpha_k J_p^E(x_k)\Big\|^q - \Big\langle\sum_{k=1}^{N}\alpha_k J_p^E(x_k), z\Big\rangle + \frac{1}{p}\|z\|^p \\
\leq \;& \frac{1}{q}\sum_{k=1}^{N}\alpha_k\|J_p^E(x_k)\|^q - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|) \\
& -\Big\langle\sum_{k=1}^{N}\alpha_k J_p^E(x_k), z\Big\rangle + \frac{1}{p}\|z\|^p \\
= \;& \frac{1}{q}\sum_{k=1}^{N}\alpha_k\|J_p^E(x_k)\|^q - \sum_{k=1}^{N}\alpha_k\langle J_p^E(x_k), z\rangle + \frac{1}{p}\|z\|^p \\
& -\alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|) \\
= \;& \sum_{k=1}^{N}\alpha_k D_p(x_k, z) - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|),
\end{aligned}$$

for all $i, j \in \{1, 2, ..., N\}$. This completes the proof. □

3 Main Results

Theorem 3.1. *Let E_1 and E_2 be two real p-uniformly convex and uniformly smooth Banach spaces and let C and Q be nonempty, closed and convex subsets* *of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be its adjoint of A. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of Bregman relatively nonexpansive mappings of C into E_1 such that $F(T_n) = \widehat{F}(T_n)$ for all $n \geq 1$. Suppose that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \Gamma \neq \emptyset$. For given $u \in E_1$, let $\{u_n\}$ be a sequence generated by $u_1 \in C$ and*

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}\big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n\big) \\ u_{n+1} = \Pi_C J_q^{E_1^*}\big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\big(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)J_p^{E_1}(T_n x_n)\big)\big], \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that the step-size $\{\lambda_n\}$ is a bounded sequence chosen in such a way that for small enough $\epsilon > 0$,

$$0 < \epsilon < \lambda_n < \left(\frac{q\|(I - P_Q)Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - P_Q)Au_n\|^q} - \epsilon\right)^{\frac{1}{q-1}}, \quad n \in N, \tag{3.2}$$

where the index set $N := \{n \in \mathbb{N} : (I - P_Q)Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < a \leq \beta_n \leq b < 1$.

Suppose in addition that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then, $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to an element $x^* = \Pi_\Omega u$, where Π_Ω is the Bregman projection from C onto Ω.

Proof. By the choice of λ_n, we observe that

$$\begin{aligned} & \lambda_n^{q-1} < \frac{q\|(I - P_Q)Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - P_Q)Au_n\|^q} - \epsilon \\ \iff \quad & \kappa_q \lambda_n^{q-1} \|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\ & < q\|(I - P_Q)Au_n\|^p - \epsilon\kappa_q \|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\ \iff \quad & \frac{\epsilon\kappa_q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\ < \quad & \|(I - P_Q)Au_n\|^p - \frac{\kappa_q \lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q. \end{aligned} \tag{3.3}$$

For each $n \geq 1$, we put $x_n = \Pi_C v_n$, where

$$v_n := J_q^{E_1^*}\big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - P_Q(Au_n))\big).$$

Let $z \in \Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \Gamma$. From (2.6), we observe that

$$\begin{aligned}
& \langle J_p^{E_2}(Au_n - P_Q(Au_n)), Au_n - Az\rangle \\
= & \langle J_p^{E_2}(Au_n - P_Q(Au_n)), Au_n - P_Q(Au_n)\rangle \\
& +\langle J_p^{E_2}(Au_n - P_Q(Au_n)), P_Q(Au_n) - Az\rangle \\
= & \|Au_n - P_Q(Au_n)\|^p \\
& +\langle J_p^{E_2}(Au_n - P_Q(Au_n)), P_Q(Au_n) - Az\rangle \\
\geq & \|Au_n - P_Q(Au_n)\|^p.
\end{aligned} \tag{3.4}$$

Then from Lemma 2.3 and (3.4), we have

$$\begin{aligned}
& D_p(x_n, z) \\
\leq & D_p\big(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n), z\big) \\
= & \frac{1}{q}\|J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n)\|^p \\
& -\langle J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n, z\rangle + \frac{1}{p}\|z\|^p \\
= & \frac{1}{q}\|J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\
& -\langle J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n, z\rangle + \frac{1}{p}\|z\|^p \\
\leq & \frac{1}{q}\|J_p^{E_1}(u_n)\|^q - \lambda_n\langle J_p^{E_2}(I - P_Q)Au_n, Au_n\rangle \\
& +\frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q - \langle J_p^{E_1}(u_n), z\rangle \\
& +\lambda_n\langle J_p^{E_2}(I - P_Q)Au_n, Az\rangle + \frac{1}{p}\|z\|^p \\
= & \frac{1}{q}\|u_n\|^p - \langle J_p^{E_1}(u_n), z\rangle + \frac{1}{p}\|z\|^p + \lambda_n\langle J_p^{E_2}(I - P_Q)Au_n, Az - Au_n\rangle \\
& +\frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\
= & D_p(u_n, z) + \lambda_n\langle J_p^{E_2}(I - P_Q)Au_n, Az - Au_n\rangle \\
& +\frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\
\leq & D_p(u_n, z) - \lambda_n\left(\|(I - P_Q)Au_n\|^p - \frac{\kappa_q\lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q\right),
\end{aligned} \tag{3.5}$$

which implies that

$$D_p(x_n, z) \leq D_p(u_n, z).$$

Now, we put

$$y_n := J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1 - \beta_n)J_p^{E_1}(T_n x_n))$$

for all $n \geq 1$. From Lemma 2.12, we have

$$\begin{aligned}
& D_p(y_n, z) \\
= \;& D_p(J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1-\beta_n) J_p^{E_1}(T_n x_n)), z) \\
\leq \;& \beta_n D_p(x_n, v) + (1-\beta_n) D_p(T_n x_n, z) - \beta_n(1-\beta_n) g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|) \\
\leq \;& D_p(x_n, z) - \beta_n(1-\beta_n) g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|) \qquad (3.6) \\
\leq \;& D_p(x_n, z) \qquad (3.7)
\end{aligned}$$

It follows from (3.7) that

$$\begin{aligned}
D_p(x_{n+1}, z) &\leq D_p(u_{n+1}, z) \\
&\leq D_p\big(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(y_n)), z\big) \\
&\leq \alpha_n D_p(u, z) + (1-\alpha_n) D_p(y_n, z) \\
&\leq \alpha_n D_p(u, z) + (1-\alpha_n) D_p(x_n, z) \\
&\leq \max\{D_p(u, z), D_p(x_n, z)\} \\
&\;\;\vdots \\
&\leq \max\{D_p(u, z), D_p(x_1, z)\}. \qquad (3.8)
\end{aligned}$$

Hence, $\{D_p(x_n, z)\}$ is bounded, which implies by Lemma 2.8 that $\{x_n\}$ is bounded.

Put $u_{n+1} = \Pi_C z_n$, where $z_n := J_q^{E_1^*}\big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(y_n)\big]$ for all $n \geq 1$. From Lemma 2.7 and (3.6), we have

$$\begin{aligned}
& D_p(x_{n+1}, z) \\
\leq \;& D_p(u_{n+1}, z) \\
\leq \;& D_p(z_n, z) \\
= \;& V_p\big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(y_n), z\big) \\
\leq \;& V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(y_n) - \alpha_n(J_p^{E_1}(u) - J_p^{E_1}(z), z)) \\
& + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \\
= \;& V_p(\alpha_n J_p^{E_1}(z) + (1-\alpha_n) J_p^{E_1}(y_n), z) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \\
\leq \;& \alpha_n V_p(J_p^{E_1}(z), z) + (1-\alpha_n) V_p(J_p^{E_1}(y_n), z) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - v\rangle \\
= \;& \alpha_n D_p(z, z) + (1-\alpha_n) D_p(y_n, z) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \\
\leq \;& (1-\alpha_n)[D_p(x_n, z) - \beta_n(1-\beta_n) g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|)] \\
& + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - z\rangle \\
\leq \;& (1-\alpha_n) D_p(x_n, z) - \beta_n(1-\beta_n) g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|)] \\
& + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \qquad (3.9) \\
\leq \;& (1-\alpha_n) D_p(x_n, z) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle. \qquad (3.10)
\end{aligned}$$

Next, we will divide the proof into two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_p(x_n, z)\}_{n=n_0}^{\infty}$ is non-increasing. By the boundedness of $\{D_p(x_n, z)\}_{n=1}^{\infty}$, we have $\{D_p(x_n, z)\}_{n=1}^{\infty}$ is convergent. Furthermore, we have

$$D_p(x_n, z) - D_p(x_{n+1}, z) \to 0 \quad \text{as} \quad n \to \infty.$$

Then, from (3.9), we have

$$\begin{aligned} 0 &\leq a(1-b)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|) \\ &\leq \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|) \\ &\leq D_p(x_n, z) - D_p(x_{n+1}, z) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies by the property of g_r^* that

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0. \tag{3.11}$$

Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*, then

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.12}$$

From Lemma 2.9, we also have

$$\lim_{n\to\infty} D_p(T_n x_n, x_n) = 0. \tag{3.13}$$

Since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0.$$

By Proposition 2.1, we observe that

$$\begin{aligned} &\|J_p^{E_1}(x_n) - J_p^{E_1}(T x_n)\| \\ \leq\ &\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \|J_p^{E_1}(T_n x_n) - J_p^{E_1}(T x_n)\| \\ \leq\ &\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_n x) - J_p^{E_1}(T x)\| \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

By the reflexivity of a Banach space E and the boundedness of $\{x_n\}$, without loss of generality, we may assume that $x_{n_i} \rightharpoonup v \in C$ as $i \to \infty$. Then, we get $v \in \widehat{F}(T_n) = F(T_n)$ for all $n \geq 1$, *i.e.*, $v \in \bigcap_{n=1}^{\infty} F(T_n)$. Further, we show that $v \in \Gamma$. From (3.3) and (3.5), we have

$$\begin{aligned} &\frac{\epsilon^2\kappa_q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\ <\ &\lambda_n\left(\|(I - P_Q)Au_n\|^p - \frac{\kappa_q \lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q\right) \\ \leq\ &D_p(u_n, v) - D_p(x_n, v) \\ \leq\ &\alpha_{n-1}D_p(u, v) + D_p(x_{n-1}, v) - D_p(x_n, v), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|Au_n - P_Q(Au_n)\| = 0. \tag{3.14}$$

Since $v_n := J_q^{E_1^*}\big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - P_Q(Au_n))\big)$ for all $n \geq 1$, it follows that

$$\begin{aligned} 0 \leq \|J_p^{E_1}(v_n) - J_p^{E_1}(u_n)\| &\leq \lambda_n \|A^*\| \|J_p^{E_2}(Au_n - P_Q(Au_n))\| \\ &\leq \left(\frac{q}{\kappa_q \|A\|^q}\right)^{\frac{1}{q-1}} \|A^*\| \|Au_n - P_Q(Au_n)\|^{p-1}, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|J_p^{E_1}(v_n) - J_p^{E_1}(u_n)\| = 0, \tag{3.15}$$

and hence

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.16}$$

By Lemma 2.2 (*ii*) and (3.6), we have

$$\begin{aligned} D_p(v_n, x_n) &= D_p(v_n, \Pi_C v_n) \leq D_p(v_n, v) - D_p(x_n, xv) \\ &\leq D_p(u_n, v) - D_p(x_n, v) \\ &\leq \alpha_{n-1} D_p(u, v) + D_p(x_{n-1}, v) - D_p(x_n, v) \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

By Lemma 2.9, we get

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.17}$$

Then from (3.16) and (3.17), we have

$$\|x_n - u_n\| \leq \|v_n - u_n\| + \|v_n - x_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{3.18}$$

Since $x_{n_i} \rightharpoonup v \in C$ and from (3.18), we also get $u_{n_i} \rightharpoonup v \in C$. From (2.6), we have

$$\begin{aligned} &\|(I - P_Q)Av\|^p \\ = \;& \langle J_p^{E_2}(Av - P_Q(Av)), Av - P_Q(Av)\rangle \\ = \;& \langle J_p^{E_2}(Av - P_Q(Av)), Av - Au_{n_i}\rangle + \langle J_p^{E_2}(Av - P_Q(Av)), Au_{n_i} - P_Q(Au_{n_i})\rangle \\ &+ \langle J_p^{E_2}(Av - P_Q(Av)), P_Q(Au_{n_i}) - P_Q(Av)\rangle \\ \leq \;& \langle J_p^{E_2}(Av - P_Q(Av)), Av - Au_{n_i}\rangle + \langle J_p^{E_2}(Av - P_Q(Av)), Au_{n_i} - P_Q(Au_{n_i})\rangle. \end{aligned}$$

Since A is continuous, we have $Au_{n_i} \rightharpoonup Av$ as $i \to \infty$. From (3.14), we obtain

$$\|(I - P_Q)Av\| = 0,$$

i.e., $Av = P_Q(Av)$, this shows that $Av \in Q$. Thus $v \in \Omega := F(T) \cap \Gamma$. From Lemma 2.6 and (3.13), we have

$$\begin{aligned} D_p(y_n, x_n) &= D_p(J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)J_p^{E_1}(T_n x_n)), x_n) \\ &\le \beta_n D_p(x_n, x_n) + (1-\beta_n) D_p(T_n x_n, x_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that

$$\begin{aligned} D_p(z_n, x_n) &= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n)), x_n) \\ &\le \alpha_n D_p(u, x_n) + (1-\alpha_n) D_p(y_n, x_n) \to 0 \text{ as } n \to \infty, \end{aligned}$$

and hence

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.19}$$

Next, we show that

$$\limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_n - x^* \rangle \le 0,$$

where $x^* = \Pi_\Omega u$. From (3.19), we have

$$\begin{aligned} \limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_n - x^* \rangle &= \limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^* \rangle \\ &= \lim_{i\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^* \rangle. \end{aligned}$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. It follows from Lemma 2.2 that

$$\begin{aligned} \limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^* \rangle &= \lim_{i\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^* \rangle \\ &= \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), v - x^* \rangle \le 0. \end{aligned} \tag{3.20}$$

Applying Lemma 2.10 to (3.10) and (3.20), we can conclude that $D_p(x_n, x^*) \to 0$ as $n \to \infty$. Therefore, $x_n \to x^*$ as $n \to \infty$.
Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Let us define a mapping $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then, by Lemma 2.11, we obtain

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1} \text{ and } \Gamma_n \le \Gamma_{\tau(n)+1}.$$

Put $\Gamma_n := D_p(x_n, x^*)$ for all $n \in \mathbb{N}$. Then, we have from (3.8) that

$$\begin{aligned} 0 &\le \lim_{n\to\infty} (D_p(x_{\tau(n)+1}, x^*) - D_p(x_{\tau(n)}, x^*)) \\ &\le \lim_{n\to\infty} (D_p(u, x^*) + (1-\alpha_{\tau(n)}) D_p(x_{\tau(n)}, x^*) - D_p(x_{\tau(n)}, x^*)) \\ &= \lim_{n\to\infty} \alpha_{\tau(n)} \big(D_p(u, x^*) - D_p(x_{\tau(n)}, x^*) \big) = 0, \end{aligned}$$

which implies that

$$\lim_{n\to\infty}(D_p(x_{\tau(n)+1},x^*)-D_p(x_{\tau(n)},x^*))=0. \tag{3.21}$$

Following the proof line in **Case 1**, we can show that

$$\lim_{n\to\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\|=0,$$

$$\lim_{n\to\infty}\|Au_{\tau(n)}-P_Q(Au_{\tau(n)})\|=0.$$

Further, we can show that

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),z_{\tau(n)}-x^*\rangle\leq 0.$$

From (3.10), we have

$$\begin{aligned}D_p(x_{\tau(n)+1},x^*) &\leq (1-\alpha_{\tau(n)})D_p(x_{\tau(n)},x^*)\\ &\quad +\alpha_{\tau(n)}\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),z_{\tau(n)}-x^*\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\alpha_{\tau(n)}D_p(x_{\tau(n)},x^*) &\leq D_p(x_{\tau(n)},x^*)-D_p(x_{\tau(n)+1},x^*)\\ &\quad +\alpha_{\tau(n)}\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),z_{\tau(n)}-x^*\rangle.\end{aligned}$$

Since $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)}>0$, we get

$$D_p(x_{\tau(n)},x^*) \leq \langle J_p^{E_1}(u)-J_p^{E_1}(x^*),z_{\tau(n)}-x^*\rangle.$$

Hence, $\lim_{n\to\infty}D_p(x_{\tau(n)},x^*)=0$. From (3.21), we have

$$\begin{aligned}D_p(x_n,x^*) &\leq D_p(x_{\tau(n)+1},x^*)=D_p(x_{\tau(n)},x^*)+(D_p(x_{\tau(n)+1},x^*)-D_p(x_{\tau(n)},x^*))\\ &\to 0 \text{ as } n\to\infty,\end{aligned}$$

which implies that $D_p(x_n,x^*)\to 0$. Therefore $x_n\to x^*$ as $n\to\infty$. Thus from above two cases, we conclude that $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^*=\Pi_\Omega u$. This completes the proof. □

We consequently obtain the following result in Hilbert spaces.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces and let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A:E_1\to E_2$ be a bounded linear operator and $A^*:E_2^*\to E_1^*$ be its adjoint of A. Let $\{T_n\}_{n=1}^\infty$ be a countable family of quasi-nonexpansive mappings of C into E_1 such that $F(T_n)=\widehat{F}(T_n)$ for all $n\geq 1$. Suppose that $\Omega:=\bigcap_{n=1}^\infty F(T_n)\cap\Gamma\neq\emptyset$. For given $u\in E_1$, let $\{u_n\}$ be a sequence generated by $u_1\in C$ and*

$$\begin{cases} x_n=P_C(u_n-\lambda_nA^*(I-P_Q)Au_n)\\ u_{n+1}=P_C(\alpha_nu+(1-\alpha_n)(\beta_nx_n+(1-\beta_n)T_nx_n)), \quad \forall n\geq 1,\end{cases} \tag{3.22}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that the step-size $\{\lambda_n\}$ is a bounded sequence chosen in such a way that for small enough $\epsilon > 0$,

$$0 < \epsilon < \lambda_n < \frac{2\|(I-P_Q)Au_n\|^2}{\|A^*(I-P_Q)Au_n\|^2} - \epsilon, \quad n \in N, \tag{3.23}$$

where the index set $N := \{n \in \mathbb{N} : (I-P_Q)Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < a \leq \beta_n \leq b < 1$.

Suppose in addition that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then, $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to an element $x^* = P_\Omega u$, where P_Ω is the metric projection from C onto Ω.

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