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Best Proximity Point Theorems Without The P-Property for New Generalized Weakly Contraction Mappings in Partially Ordered Metric Spaces

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Abstract : In this paper, we define the new concept of a generalized weakly contractive condition for nonlinear nonself-mappings and establish new best proximity point theorems for such mappings with three control functions in partially ordered metric spaces without the P-property. These results generalize the main results of Babu and Leta [1].

Keywords : Best proximity points; weakly contraction mappings; partially ordered metric spaces

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1 Introduction and preliminaries

A fundamental result in fixed point theory is the Banach contraction mapping principle. Several extensions of this result have appeared in the literatures. Let A, B be two nonempty subsets of a metric space (X, d) and $T : A \to B$ be a mapping. If $A \cap B = \emptyset$, then the equation $Tx = x$ might have no solution. Under this circumstance, it is meaningful to find a point $x \in A$ such that $d(x, Tx)$ is minimum. If $d(x, Tx) = d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$, we get $d(x, Tx)$ is the global minimum value $d(A, B)$ and so x is an approximate solution of the equation $Tx = x$ with the least possible error. Such a solution is known as a best proximity point of the mapping T. That is, a point $x \in A$ is called the best proximity point of T if

$$d(x, Tx) = d(A, B).$$

Throughout this paper, (X, d) denotes a metric space, $\preceq$ denotes a partial order on X, $A, B \subseteq X$. We also use the following notations:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

A metric space (X, d) with a partial order $\preceq$ defined on X is called a partially ordered metric space. It is denoted by $(X, d, \preceq)$.

In 2011, Sankar Raj [4] introduced the new property called P-property as follows:

Definition 1.1 ([4]). *Let A and B be two nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, the following condition holds:*

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \implies d(x_1, x_2) = d(y_1, y_2).$$

Example 1.2. *Let $X = \mathbb{R}^2$ with the Euclidian metric. Assume that*

$$A := \left\{\left(\frac{2}{n+1}, 0\right) : n \in \mathbb{N}\right\} \cup (0, 0),$$
$$B := \left\{\left(\frac{2}{n+1}, 1\right) : n \in \mathbb{N}\right\} \cup (0, 1).$$

It is easy to see that $d(A, B) = 1$. Suppose that $(x_1, 0), (x_2, 0) \in A_0$ and $(y_1, 1), (y_2, 1) \in B_0$ such that

$$d((x_1, 0), (y_1, 1)) = d(A, B) \text{ and } d((x_2, 0), (y_2, 1)) = d(A, B).$$

Then we have

$$\sqrt{(x_1 - y_1)^2 + 1} = 1 \text{ and } \sqrt{(x_2 - y_2)^2 + 1} = 1.$$

Therefore, $x_1 = y_1$ *and* $x_2 = y_2$ *and so*

$$d((x_1,0),(x_2,0)) = |x_1 - x_2| = |y_1 - y_2| = d((y_1,1),(y_2,1)).$$

Thus, the pair (A,B) *has the P-property.*

In 2012, Basha [2] introduced the following ideas.

Definition 1.3 ([2])**.** *Let* A, B *be nonempty subsets of a metric space* (X,d) *and* $\preceq$ *be a partial order on* X*. A mapping* $T : A \to B$ *is called proximally increasing on* A *if for all* $x_1, x_2, y_1, y_2 \in A$*, the following condition holds:*

$$\left.\begin{array}{l} y_1 \preceq y_2, \\ d(x_1, Ty_1) = d(A,B), \\ d(x_2, Ty_2) = d(A,B) \end{array}\right\} \implies x_1 \preceq x_2.$$

Definition 1.4 ([2])**.** *Let* A, B *be nonempty subsets of a metric space* (X,d) *and* $\preceq$ *be a partial order on* X*. A mapping* $T : A \to B$ *is called proximally increasing on* A_0 *if for all* $x_1, x_2, y_1, y_2 \in A_0$*, the following condition holds:*

$$\left.\begin{array}{l} y_1 \preceq y_2, \\ d(x_1, Ty_1) = d(A,B), \\ d(x_2, Ty_2) = d(A,B) \end{array}\right\} \implies x_1 \preceq x_2.$$

Example 1.5. *Let* $X = \mathbb{R}^2$ *with the taxicab metric* d *on* X*. We define a partial order* $\preceq$ *on* X *by*

$$(x_1, x_2) \preceq (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2$$

for all $(x_1, y_1), (x_2, y_2) \in X$*. Let*

$$A = \{(x,1) : 1 \leq x \leq 10\},$$
$$B = \{(x,5) : 1 \leq x \leq 10\}.$$

Clearly, $d(A,B) = 4$*. We define* $T : A \to B$ *by*

$$T(x,1) = \left(\frac{x}{5}, 5\right) \quad \text{for all } (x,1) \in [1,10].$$

It is easy to see that $d(A,B) = 4$*. Let* $(x_1,1), (x_2,1), (y_1,1), (y_2,1) \in A$ *with* $(y_1,1) \preceq (y_2,1)$*. Assume that* $d\big((x_1,1), T(y_1,1)\big) = d(A,B)$ *and* $d\big((x_2,1), T(y_2,1)\big) = d(A,B)$*. Then*

$$4 = d\big((x_1,1), T(y_1,1)\big) = d\Big((x_1,1), \Big(\frac{y_1}{5}, 5\Big)\Big) = \Big|x_1 - \frac{y_1}{5}\Big| + 4$$

and

$$4 = d\big((x_2,1), T(y_2,1)\big) = d\Big((x_2,1), \Big(\frac{y_2}{5}, 5\Big)\Big) = \Big|x_2 - \frac{y_2}{5}\Big| + 4,$$

which imply that $x_1 = \frac{y_1}{5}$ *and* $x_2 = \frac{y_2}{5}$*. Since* $(y_1,1) \preceq (y_2,1)$*, we get* $y_1 \leq y_2$ *and then* $\frac{y_1}{5} \leq \frac{y_2}{5}$*. This implies that* $x_1 \leq x_2$ *and so* $(x_1,1) \preceq (x_2,1)$*. Hence,* T *is proximally increasing on* A*.*

In addition, we will give some notations for using in our results.

Let Ψ be the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is continuous and

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Let Θ be the set of all functions $\theta : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) θ is bounded on any bounded interval in $[0,\infty)$ and

(ii) θ is continuous at 0 and $\theta(0) = 0$.

In recently, Babu and Leta [1] introduced the new weak contraction mapping called a $(\psi - \varphi - \theta)$-almost weakly contractive mapping as follows:

Definition 1.6 ([1])**.** *Let $(X, d, \preceq)$ be a partially ordered metric space and A, B be nonempty subsets of X. A nonself-mapping $T : A \to B$ is called a $(\psi - \varphi - \theta)$-almost weakly contractive mapping if there exist $\psi \in \Psi$, $\varphi, \theta \in \Theta$ and $L \geq 0$ such that for all $x, y \in A_0$ with $x \succeq y$*

$$\Longrightarrow \quad \psi(d(Tx,Ty)) \leq \varphi(d(x,y)) - \theta(d(x,y)) + Ln(x,y), \tag{1.1}$$

where

$$\begin{aligned} n(x,y) \;=\; & \min\{d(x,Tx) - d(A,B), d(y,Ty) - d(A,B), d(x,Ty) - d(A,B), \\ & d(y,Tx) - d(A,B)\}. \end{aligned}$$

If $L = 0$ in (1.1), then T is called a $(\psi - \varphi - \theta)$-weakly contractive mapping.

Moreover, they obtained some best proximity point result for mappings satisfying the almost contractive condition with three control functions in partially ordered metric spaces as follows:

Theorem 1.7 ([1])**.** *Let $(X, d, \preceq)$ be a partially ordered complete metric space and (A, B) be a pair of nonempty closed subsets of X such that A_0 is nonempty closed and (A, B) satisfies the P-property. Let $T : A \to B$ be a mapping which satisfies the $(\psi - \varphi - \theta)$-almost weakly contractive condition such that $T(A_0) \subseteq B_0$ and T is proximally increasing on A_0. Suppose that the following condition holds:*

(i) for all $x, y \in [0,\infty)$,

$$\psi(x) \leq \varphi(y) \quad \Longrightarrow \quad x \preceq y; \tag{1.2}$$

(ii) for any sequence $\{x_n\}$ in $[0,\infty)$ with $x_n \to t > 0$,

$$\psi(t) - \overline{\lim_{n\to\infty}}\, \varphi(x_n) + \underline{\lim_{n\to\infty}}\, \theta(x_n) > 0. \tag{1.3}$$

Furthermore, assume that either

(a) T *is continuous or*

(b) if $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n \to x$*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*.*

Also, suppose that there exist elements $x_0, x_1 \in A_0$ *such that* $d(x_1, Tx_0) = d(A, B)$ *and* $x_0 \preceq x_1$*. Then* T *has a best proximity point in* A_0*, that is, there exists an element* $x^* \in A_0$ *such that* $d(x^*, Tx^*) = d(A, B)$*.*

In this paper, we introduce a generalized weakly contractive mapping and utilize such mapping to establish some best proximity point results in partially ordered metric spaces without the P-property. Our results generalize the main theorem of Babu and Leta [1].

2 Main Results

We establish new best proximity point theorems for some weak contraction mapping in partially ordered metric spaces as follows:

Theorem 2.1. *Let* $(X, d, \preceq)$ *be a partially ordered complete metric space and* (A, B) *be a pair of nonempty closed subsets of* X *such that* A_0 *is nonempty and closed. Suppose that* $T : A \to B$ *is a mapping such that* $T(A_0) \subseteq B_0$ *and* T *is proximally increasing on* A_0*. Assume that there exist* $L \geq 0$*,* $\psi \in \Psi$ *and* $\phi, \theta \in \Theta$ *satisfying the following conditions:*

(i) for all $x, y \in [0, \infty)$*,*

$$\psi(x) \leq \phi(y) \quad \Longrightarrow \quad x \preceq y; \tag{2.1}$$

(ii) for all sequence $\{x_n\}$ *in* $[0, \infty)$ *with* $x_n \to t > 0$*,*

$$\psi(t) - \overline{\lim_{n\to\infty}}\, \phi(x_n) + \underline{\lim_{n\to\infty}}\, \theta(x_n) > 0; \tag{2.2}$$

(iii) for all $x, y, u, v \in A_0$*,*

$$\left.\begin{array}{r} x \preceq y, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{array}\right\} \Longrightarrow \psi\big(d(u, v)\big) \leq \begin{array}{l} \phi\big(M(x, y, u, v)\big) \\ -\theta\big(M(x, y, u, v)\big) \\ +Ln(x, y), \end{array} \tag{2.3}$$

where

$$M(x, y, u, v) = \max\Big\{d(x, y), \frac{d(x, u) + d(y, v)}{2}, \frac{d(y, u) + d(x, v)}{2}\Big\}$$

and

$$\begin{aligned} n(x, y) \quad = \quad & \min\{d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty) - d(A, B), \\ & d(y, Tx) - d(A, B)\}; \end{aligned}$$

(iv) there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$.

Furthermore, suppose that either

(a) T is continuous or

(b) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a best proximity point in A_0, that is, there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$.

Proof. From (iv), there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B). \tag{2.4}$$

Since $T(A_0) \subseteq B_0$, there exists an element $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{2.5}$$

As T is proximally increasing on A_0, using (2.4) and (2.5), we have $x_1 \preceq x_2$. By continuing this process, we can constract a sequence $\{x_n\}$ in A_0 such that

$$x_n \preceq x_{n+1} \tag{2.6}$$

and

$$d(x_{n+1}, Tx_n) = d(A, B) \tag{2.7}$$

for all $n \in \mathbb{N} \cup \{0\}$. By using the hypothesis (iii), we obtain

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) \quad \leq \quad & \phi\big(M(x_n, x_{n+1}, x_{n+1}, x_{n+2})\big) - \theta\big(M(x_n, x_{n+1}, x_{n+1}, x_{n+2})\big) \\ & + Ln(x_n, x_{n+1}),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+1}, x_{n+2}) &= \max\Big\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \\ &\quad \frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})}{2}\Big\} \\ &= \max\Big\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \\ &\quad \frac{d(x_n, x_{n+2})}{2}\Big\} \\ &= \max\Big\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\Big\} \end{aligned}$$

and

$$\begin{aligned} n(x_n, x_{n+1}) &= \min\Big\{d(x_{n+1}, Tx_{n+1}) - d(A, B), d(x_n, Tx_n) - d(A, B), \\ &\quad d(x_{n+1}, Tx_n) - d(A, B), d(x_n, Tx_{n+1}) - d(A, B)\Big\} \\ &= 0. \end{aligned}$$

Let $\alpha_n := d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Case 1: Assume that $M(x_n, x_{n+1}, x_{n+1}, x_{n+2}) = d(x_n, x_{n+1})$ for some $n \in \mathbb{N}\cup\{0\}$. It follows from (2.8) that

$$\psi(d(x_{n+1}, x_{n+2})) \leq \phi(d(x_n, x_{n+1})) - \theta(d(x_n, x_{n+1})),$$

that is,

$$\psi(\alpha_{n+1}) \leq \phi(\alpha_n) - \theta(\alpha_n), \tag{2.9}$$

which implies that $\psi(\alpha_{n+1}) \leq \phi(\alpha_n)$. By the hypothesis (i), we obtain $\alpha_{n+1} \leq \alpha_n$.

Case 2: Assume that $M(x_n, x_{n+1}, x_{n+1}, x_{n+2}) = \dfrac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} = \dfrac{\alpha_n + \alpha_{n+1}}{2} =: \beta_n$. It follows from (2.8) that

$$\psi(\alpha_{n+1}) \leq \phi(\beta_n) - \theta(\beta_n), \tag{2.10}$$

which implies that $\psi(\alpha_{n+1}) \leq \phi\Big(\dfrac{\alpha_{n+1} + \alpha_n}{2}\Big)$. By the hypothesis (i), we obtain $\alpha_{n+1} \leq \dfrac{\alpha_n + \alpha_{n+1}}{2}$, that is, $\alpha_{n+1} \leq \alpha_n$.

From Case 1 and Case 2, we obtain $\{\alpha_n\}$ is a monotone decreasing sequence of nonnegative real numbers. Since $\{\alpha_n\}$ is bounded below by zero, there exists $t \geq 0$ such that

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = t \tag{2.11}$$

and so

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} = \frac{t+t}{2} = t. \tag{2.12}$$

Taking the limit superior in both sides of the inequality (2.8), using (2.11), the continuity of ψ, and the property of ϕ and θ, we get

$$\psi(t) \leq \overline{\lim_{n\to\infty}}\, \phi(\max\{\alpha_n, \beta_n\}) + \overline{\lim_{n\to\infty}}\, (-\theta(\max\{\alpha_n, \beta_n\})).$$

Since $\overline{\lim_{n\to\infty}}\, (-\theta(\max\{\alpha_n, \beta_n\})) = -\underline{\lim_{n\to\infty}}\, \theta(\max\{\alpha_n, \beta_n\})$, it follows that

$$\psi(t) \leq \overline{\lim_{n\to\infty}}\, \phi(\max\{\alpha_n, \beta_n\}) - \underline{\lim_{n\to\infty}}\, (\theta(\max\{\alpha_n, \beta_n\})),$$

that is,

$$\psi(t) - \overline{\lim_{n\to\infty}}\, \phi(\max\{\alpha_n, \beta_n\}) + \underline{\lim_{n\to\infty}}\, \theta(\max\{\alpha_n, \beta_n\}) \leq 0.$$

By the hypothesis (ii), (2.11) and (2.12), it is a contradiction unless $t = 0$. Therefore,

$$\alpha_n = d(x_n, x_{n+1}) \to 0 \quad \text{as } n \to \infty. \tag{2.13}$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\delta > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for each positive integer k,

$$n_k > m_k > k \quad \text{and} \quad d(x_{m_k} x_{n_k}) \geq \delta.$$

Assuming that n_k is the smallest such positive integer, we get

$$d(x_{m_k}, x_{n_k-1}) < \delta.$$

Using the triangle inequality, we get

$$\delta \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) < \delta + d(x_{n_k-1}, x_{n_k}). \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \delta. \tag{2.15}$$

Using the triangle inequality again, we get

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})$$

and

$$d(x_{m_k+1}, x_{n_k+1}) \leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}).$$

The above two inequalities imply that

$$\begin{aligned} d(x_{m_k}, x_{n_k}) - d(x_{m_k}, x_{m_k+1}) - d(x_{n_k+1}, x_{n_k}) &\leq d(x_{m_k+1}, x_{n_k+1}) \\ &\leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) \\ &\quad + d(x_{n_k}, x_{n_k+1}). \end{aligned}$$

From the above inequality, (2.13) and (2.15), we have

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \delta. \tag{2.16}$$

Again, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})$$

and

$$d(x_{m_k}, x_{n_k+1}) \leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}).$$

The above two inequalities imply that

$$\begin{aligned} d(x_{m_k}, x_{n_k}) - d(x_{n_k+1}, x_{n_k}) &\leq d(x_{m_k}, x_{n_k+1}) \\ &\leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}). \end{aligned}$$

From the above inequality, (2.13) and (2.15), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \delta. \tag{2.17}$$

Similarly, we can prove that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \delta. \tag{2.18}$$

By the construction of the sequence $\{x_n\}$, we have

$$x_{m_k} \preceq x_{n_k}, \quad d(x_{m_k+1}, Tx_{m_k}) = d(A,B) \quad \text{and} \quad d(x_{n_k+1}, Tx_{n_k}) = d(A,B),$$

which, by the hypothesis (iii), imply that

$$\begin{aligned} \psi(d(x_{m_k+1}, x_{n_k+1})) \leq\ & \phi(M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1})) \\ & -\theta(M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1})) \\ & +Ln(x_{m_k}, x_{n_k}), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1}) = \max\Big\{ & d(x_{m_k}, x_{n_k}), \\ & \frac{d(x_{m_k}, x_{m_k+1}) + d(x_{n_k}, x_{n_k+1})}{2}, \\ & \frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}\Big\} \end{aligned}$$

and

$$\begin{aligned} n(x_{m_k}, x_{n_k}) = \min\Big\{ & d(x_{m_k}, Tx_{m_k}) - d(A,B), d(x_{n_k}, Tx_{n_k}) - d(A,B), \\ & d(x_{n_k}, Tx_{m_k}) - d(A,B), d(x_{m_k}, Tx_{n_k}) - d(A,B)\Big\}. \end{aligned}$$

Using the triangle inequality, it follows that

$$\begin{aligned} n(x_{m_k}, x_{n_k}) &= \min\{d(x_{m_k}, Tx_{m_k}) - d(A,B), d(x_{n_k}, Tx_{n_k}) - d(A,B), \\ &\quad d(x_{n_k}, Tx_{m_k}) - d(A,B), d(x_{m_k}, Tx_{n_k}) - d(A,B)\} \\ &\leq \min\{d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) - d(A,B), \\ &\quad d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, Tx_{m_k}) - d(A,B), d(x_{n_k}, x_{m_k+1}) \\ &\quad +d(x_{m_k+1}, Tx_{m_k}) - d(A,B), d(x_{m_k}, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) - d(A,B)\} \\ &= \min\{d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{m_k+1}), d(x_{m_k}, x_{n_k+1})\}. \end{aligned}$$

Therefore, we get

$$\begin{aligned}\psi(d(x_{m_k+1},x_{n_k+1})) \quad &\leq \quad \phi(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))-\theta(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))\\ &\quad +L\min\{d(x_{n_k},x_{n_k+1}),d(x_{m_k},x_{m_k+1}),d(x_{n_k},x_{m_k+1}),\\ &\quad d(x_{m_k},x_{n_k+1})\}, \qquad (2.20)\end{aligned}$$

Consider

$$\begin{aligned}M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}) \quad &= \quad \max\Big\{d(x_{m_k},x_{n_k}),\frac{d(x_{m_k},x_{m_k+1})+d(x_{n_k},x_{n_k+1})}{2},\\ &\quad \frac{d(x_{n_k},x_{m_k+1})+d(x_{m_k},x_{n_k+1})}{2}\Big\}.\end{aligned}$$

From (2.13), (2.15), (2.22), and (2.18), it follows that

$$\lim_{k\to\infty} M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1})=\delta. \qquad (2.21)$$

Taking the limit superior in both sides of the inequality (2.20), using (2.16), (2.21), the continuity of ψ, and the property of ϕ and θ, we obtain

$$\psi(\delta)\leq \overline{\lim_{n\to\infty}}\,\phi(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))+\overline{\lim_{n\to\infty}}\,(-\theta(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))).$$

As $\overline{\lim_{n\to\infty}}\,(-\theta(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1})))=-\underline{\lim_{n\to\infty}}\,\theta(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))$, it follows that

$$\psi(\delta)\leq \overline{\lim_{n\to\infty}}\,\phi(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))-\underline{\lim_{n\to\infty}}\,\theta(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1})),$$

that is,

$$\psi(\delta)-\overline{\lim_{n\to\infty}}\,\phi(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))+\underline{\lim_{n\to\infty}}\,\theta(M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))\leq 0,$$

which, by the hypothesis (ii) and (2.16), it is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in A_0. Since X is complete and A_0 is a closed subset of X and hence complete. From the completeness of A_0, there exists $z\in A_0$ such that

$$\lim_{n\to\infty} x_n=z, \text{ that is, } \lim_{n\to\infty} d(x_n,z)=0. \qquad (2.22)$$

First, we assume that T is continuous. On taking limit as $n\to\infty$ in (2.7) and using the continuity of T, we obtain $d(z,Tz)=d(A,B)$. Therefore z is the best proximity point of T.

We now assume that the condition (b) holds. By (2.6) and (2.22), we have

$$x_n\preceq z \quad \text{for all } n\in\mathbb{N}. \qquad (2.23)$$

Since $z\in A_0$ and $T(A_0)\subseteq B_0$, there exists a point $w\in A_0$ for which

$$d(w,Tz)=d(A,B). \qquad (2.24)$$

By (2.7), (2.23) and (2.24), we have

$$x_n \preceq z, \quad d(x_{n+1}, Tx_n)$$

for all $n \in \mathbb{N}$ and

$$d(w, Tz) = d(A, B),$$

which, by the hypothesis (iii), imply that

$$\psi(d(x_{n+1}, w)) \leq \phi(M(x_n, z, x_{n+1}, w)) - \theta(M(x_n, z, x_{n+1}, w)) + Ln(x_n, z), \quad (2.25)$$

where

$$M(x_n, z, x_{n+1}, w) = \max\Big\{d(x_n, z), \frac{d(x_n, x_{n+1}) + d(z, w)}{2}, \frac{d(z, x_{n+1}) + d(x_n, w)}{2}\Big\}$$

and

$$\begin{aligned} n(x_n, z) &= \min\{d(z, Tz) - d(A, B), d(x_n, Tx_n) - d(A, B), d(z, Tx_n) - d(A, B), \\ &\quad d(x_n, Tz) - d(A, B)\}. \end{aligned}$$

Using the triangle inequality, it follows that

$$\begin{aligned} n(x_n, z) &= \min\{d(z, Tz) - d(A, B), d(x_n, Tx_n) - d(A, B), d(z, Tx_n) - d(A, B), \\ &\quad d(x_n, Tz) - d(A, B)\} \\ &\leq \min\{d(z, Tz) - d(A, B), d(x_n, Tx_n) - d(A, B), \\ &\quad d(z, x_{n+1}) + d(x_{n+1}, Tx_n) - d(A, B), d(x_n, Tz) - d(A, B)\} \\ &= \min\{d(z, Tz) - d(A, B), d(x_n, Tx_n) - d(A, B), \\ &\quad d(z, x_{n+1}), d(x_n, Tz) - d(A, B)\}. \end{aligned}$$

Therefore

$$\begin{aligned} \psi(d(x_{n+1}, w)) &\leq \phi(M(x_n, z, x_{n+1}, w)) - \theta(M(x_n, z, x_{n+1}, w)) \\ &\quad + L\min\{d(z, Tz) - d(A, B), d(x_n, Tx_n) - d(A, B), \\ &\quad d(z, x_{n+1}), d(x_n, Tz) - d(A, B)\}. \end{aligned} \quad (2.26)$$

From (2.22), we obtain that

$$\lim_{n\to\infty} M(x_n, z, x_{n+1}, w) = \frac{d(z, w)}{2}. \quad (2.27)$$

Taking the limit superior in both sides of the inequality (2.26), using (2.22), (2.27), the properties of ψ, and the property of ϕ and θ, we obtain

$$\psi\Big(\frac{d(z, w)}{2}\Big) \leq \psi(d(z, w)) \leq \overline{\lim_{n\to\infty}}\, \phi(M(x_n, z, x_{n+1}, w)) + \overline{\lim_{n\to\infty}}\, (-\theta(M(x_n, z, x_{n+1}, w))).$$

Argument similarly as discussed above, we have

$$\psi\Big(\frac{d(z,w)}{2}\Big) - \overline{\lim_{n\to\infty}}\,\phi(M(x_n,z,x_{n+1},w)) + \underline{\lim}_{n\to\infty}\,\theta(M(x_n,z,x_{n+1},w)) \le 0,$$

which, by the hyprothesis (ii) and (2.27), it is a contraction unless $d(z,w) = 0$, that is, $z = w$. By (2.24), we have $d(z,Tz) = d(A,B)$, that is, z is a best proximity point of T. □

By using the same technique in the proof of Theorem 2.1, we get the following result.

Theorem 2.2. *Let $(X,d,\preceq)$ be a partially ordered complete metric space and (A,B) be a pair of nonempty closed subsets of X such that A_0 is nonempty and closed. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$ and T is proximally increasing on A_0. Assume that there exist $L \ge 0$, $\psi \in \Psi$ and $\phi,\theta \in \Theta$ satisfying the following conditions:*

(i) for all $x,y \in [0,\infty)$,

$$\psi(x) \le \phi(y) \quad \Longrightarrow \quad x \preceq y; \tag{2.28}$$

(ii) for all sequence $\{x_n\}$ in $[0,\infty)$ with $x_n \to t > 0$,

$$\psi(t) - \overline{\lim_{n\to\infty}}\,\phi(x_n) + \underline{\lim}_{n\to\infty}\,\theta(x_n) > 0; \tag{2.29}$$

(iii) for all $x,y,u,v \in A_0$,

$$\left.\begin{array}{r} x \preceq y,\\ d(u,Tx) = d(A,B),\\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow \psi\big(d(u,v)\big) \le \begin{array}{l} \phi\big(d(x,y)\big)\\ -\theta\big(d(x,y)\big)\\ +Ln(x,y), \end{array} \tag{2.30}$$

where

$$\begin{aligned} n(x,y) \;&=\; \min\{d(x,Tx) - d(A,B), d(y,Ty) - d(A,B), d(x,Ty) - d(A,B),\\ &\quad d(y,Tx) - d(A,B)\}; \end{aligned}$$

(iv) there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1,Tx_0) = d(A,B)$.

Furthermore, suppose that either

(a) T is continuous or

(b) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a best proximity point in A_0, that is, there exists an element $z \in A_0$ such that $d(z,Tz) = d(A,B)$.

Next, we apply Theorem 2.2 which is the best proximity point result without the P-property for proving the best proximity point with the P-property via the following useful lemma due to Gabeleh [3].

Lemma 2.3 ([3]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and (A, B) has the P-property. Then (A_0, B_0) is a closed pair of subsets of X.*

Corollary 2.4 ([1]). *Let $(X, d, \preceq)$ be a partially ordered complete metric space and (A, B) be a pair of nonempty closed subsets of X such that A_0 is nonempty and (A, B) satisfies the P-property. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$ and T is proximally increasing on A_0. Assume that there exist $L \geq 0$, $\psi \in \Psi$ and $\phi, \theta \in \Theta$ satisfying the following conditions:*

(i) for all $x, y \in [0, \infty)$,

$$\psi(x) \leq \phi(y) \quad \Longrightarrow \quad x \preceq y; \tag{2.31}$$

(ii) for all sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \to t > 0$,

$$\psi(t) - \overline{\lim_{n\to\infty}} \phi(x_n) + \underline{\lim_{n\to\infty}} \theta(x_n) > 0; \tag{2.32}$$

(iii) T satisfies the $(\psi - \varphi - \theta)$-almost weakly contractive condition, that is,

$$\psi(d(Tx, Ty)) \leq \phi\big(d(x, y)\big) - \theta\big(d(x, y)\big) + Ln(x, y), \tag{2.33}$$

for all $x, y, u, v \in A_0$ with $x \preceq y$, where

$$\begin{aligned} n(x, y) \quad = \quad & \min\{d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty) - d(A, B), \\ & d(y, Tx) - d(A, B)\}; \end{aligned}$$

(iv) there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$.

Furthermore, assume that either

(a) T is continuous or

(b) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a best proximity point in A_0, that is, there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$.

Proof. Since (A, B) satisfies the P-property, the contractive condition (2.33) implies the condition (2.30). By using Lemma 2.3 and applying Theorem 2.2, we get this result. □

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