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Some Results for Generalized Suzuki Type $\mathcal{Z}$-Contraction in $\theta-$Metric Spaces

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Abstract : In this paper, we introduce the concept of Suzuki type $\mathcal{Z}$-contraction and prove the existence of fixed point results for this contraction in $\theta-$metric spaces. As an application, we apply our result to show the solution of nonlinear Hammerstein integral equations.

Keywords : Simulation function; Suzuki type $\mathcal{Z}$-contraction; $\theta-$metric spaces
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1 Introduction and Preliminaries

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A metric space (X, d) is called complete if every Cauchy sequence in X converges to a point in X. Banach's contraction principle [1] is very important to show the existence of solutions for some nonlinear equations, differential and integral equations, and other nonlinear problems. Since Banach's contraction principle, many authors have studied in several ways in [2, 3, 4, 5, 6, 7]. We can see this principle as follows.

Theorem 1.1. *If (X, d) is a complete metric space and $T : X \to X$ be a mapping such that, for some $\alpha \in [0, 1)$,*

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1}$$

for each $x, y \in X$, then T has a unique fixed point in X.

Note that the mapping T satisfying the Banach contraction condition is continuous, but the mappings T satisfying the following contractions conditions are not continuous.

(1) In 1968, Kannan's contraction ([3]): for some $\beta \in [0, \frac{1}{2})$,

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

for each $x, y \in X$.

(2) In 1971, Reich's contraction ([4]): for some $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$,

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(y, Ty) \tag{1.3}$$

for each $x, y \in X$.

(3) In 1971, Ciric's contraction ([5]): for some $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\delta < 1$,

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta[d(x, Ty) + d(y, Tx)] \tag{1.4}$$

for each $x, y \in X$

(4) In 1972, Chatterjea's contraction ([6]): for some $\beta \in [0, \frac{1}{2})$,

$$d(Tx, Ty) \leq \beta[d(x, Ty) + d(y, Tx)] \tag{1.5}$$

for each $x, y \in X$.

(5) In 1973, Hardy and Rogers's contraction ([7]): for some $\alpha, \beta, \gamma, \delta, \eta \geq 0$ with $\alpha + \beta + \gamma + \delta + \eta < 1$,

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + \eta d(y, Tx) \tag{1.6}$$

for each $x, y \in X$.

On the other way, in 2009, Suzuki defined the new generalized Banach contraction principle and proved the existence of a unique fixed point for this contraction in compact metric spaces as the following theorem.

Theorem 1.2. *[8] Let (X,d) be a compact metric space and $T:X\to X$ be a mapping. Assume that*

$$\frac{1}{2}d\left(x,Tx\right)<d\left(x,y\right)\Rightarrow d\left(Tx,Ty\right)<d\left(x,y\right) \text{ for all distinct } x,y\in X.$$

Then T has a unique fixed point in X.

Later, in 2015, the new generalized Banach contraction is introduced by Khojasteh et al. [9] which they defined a simulation function and $\mathcal{Z}$-contraction as follows.

Definition 1.3. *Let $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the assumptions as follows:*

(Δ_1) $\zeta(0,0)=0$;

(Δ_2) $\zeta(t,s)<s-t$ *for* $t,s>0$;

(Δ_3) *if $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ so that*

$$\lim_{n\to\infty}t_n=\lim_{n\to\infty}s_n>0,$$

then

$$\limsup_{n\to\infty}\zeta(t_n,s_n)<0.$$

The all simulation functions set were denoted as $\mathcal{Z}$.

Definition 1.4. *Let (X,d) be a metric space, T is a self-mapping and $\zeta\in\mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction by respect to ζ if the following condition holds:*

$$\zeta(d(Tx,Ty),d(x,y))\geq 0 \tag{1.7}$$

where $x,y\in X$, with $x\neq y$.

Recently, in 2017, Kumam et al. [11] introduced the concept of Suzuki type $\mathcal{Z}$-contraction as follows.

Definition 1.5. *[11] Let (X,d) be a metric space, $T\colon X\to X$ be a mapping and $\zeta\in\mathcal{Z}$. Then T is called a Suzuki type $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied*

$$\frac{1}{2}d\left(x,Tx\right)<d\left(x,y\right)\Rightarrow\zeta\left(d\left(Tx,Ty\right),d\left(x,y\right)\right)\geq 0$$

for all distinct $x,y\in X$.

Remark 1.6. *[11] It is clear from the definition of simulation function that $\zeta\left(t,s\right)<0$ for all $t\geq s>0$. Therefore if T is a Suzuki type $\mathcal{Z}$-contraction with respect to ζ, then*

$$\frac{1}{2}d\left(x,Tx\right)<d\left(x,y\right)\Rightarrow d\left(Tx,Ty\right)<d\left(x,y\right)$$

for all for all distinct $x,y\in X$.

Theorem 1.7. *[11] Let (X,d) be a metric space and $T : X \to X$ be a Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then T has at most one fixed point.*

Remark 1.8. *[9] Every $\mathcal{Z}$-contraction is contractive and hence Banach contraction.*

On the other hand, Khojasteh et al. [10] introduced the concept of B-action and a θ-metric spaces as follows.

Definition 1.9. *Let $\theta : [0,\infty) \times [0,\infty) \to [0,\infty)$ be a continuous mappings with respect to both the variables. Let $Im(\theta) = \{\theta(s,t) : s \geq 0, t \geq 0\}$. The mapping θ is said to be B-action if and only if the following conditions hold:*

(B1) $\theta(0,0) = 0$ *and* $\theta(s,t) = \theta(t,s)$ *for all* $s,t \geq 0$*;*

(B2)

$$\theta(s,t) = \theta(u,v) \Rightarrow \begin{cases} \text{either } s < u,\ t \leq v; \\ \text{or } s \leq u,\ t < v; \end{cases}$$

(B3) for each $r \in Im(\theta)$ and for each $s \in [0,r]$, there exists $t \in [0,r]$ such that $\theta(t,s) = r$;

(B4) $\theta(s,0) \leq s$, *for all* $s > 0$.

Example 1.10. *The subsequent examples illustrate the definition.*

1. $\theta_1(s,t) = \frac{\sqrt{ts}}{1+\sqrt{st}}$;
2. $\theta_2(s,t) = |t-s| + \sqrt{ts}$;
3. $\theta_3(s,t) = \frac{|t-s|}{1+|s-t|}$;
4. $\theta_4(s,t) = \frac{(t-s)^n}{1+(s-t)^n}$, *for* $n = 2, 4, 6, \ldots$.

The set of all B-action is denoted by Y.

The concept of B-action has been very much functional to formulate the notion of θ-metric spaces. We here recall the definition of the said spaces.

Definition 1.11. *Let X be a non-empty set. A mapping $d_\theta : X \times X \to [0,\infty)$ is said to be a θ-metric on X with respect to B-action $\theta \in Y$ if d_θ satisfies the following:*

(θ1) $d_\theta(x,y) = 0 \Leftrightarrow x = y$ *for all* $x,y \in X$*;*

(θ2) $d_\theta(x,y) = d_\theta(y,x)$ *for all* $x,y \in X$*;*

(θ3) $d_\theta(x,y) = \theta(d_\theta(x,z), d_\theta(z,y))$ *for all* $x,y,z \in X$.

Then the pair (X, d_θ) is said to be a θ-metric spaces.

Example 1.12. *Here we provide a non-trivial example of θ-metric space. Let $X = \{x, y, z\}$ and $d_\theta : X \times X \to [0, \infty)$ is defined as:*

$$\begin{aligned} d_\theta(x, y) &= d_\theta(y, x) = 3, \\ d_\theta(y, z) &= d_\theta(z, y) = 8, \\ d_\theta(x, z) &= d_\theta(z, x) = 10, \\ d_\theta(x, x) &= d_\theta(y, y) = d_\theta(z, z) = 0. \end{aligned}$$

Taking $\theta_2(s, t)$ in previous Example, the mapping d_θ forms a θ-metric. And therefore the pair (X, d_θ) is a θ-metric space.

Recently, Padcharoen et al. [13] defined the notion of generalized Suzuki type $\mathcal{Z}$-contraction on a metric spaces as follows.

Definition 1.13. *Let (X, d) be a metric space and $T : X \to X$ be a mapping and $\zeta \in \mathcal{Z}$. Then F is called generalized Suzuki type $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied*

$$\frac{1}{2} d\left(x, Tx\right) < d\left(x, y\right) \Rightarrow \zeta\left(d\left(Tx, Ty\right), M\left(x, y\right)\right) \geq 0, \tag{1.8}$$

for all distinct $x, y \in X$, where

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), d\left(x, Tx\right), d\left(y, Ty\right), \frac{d\left(x, Ty\right) + d\left(y, Tx\right)}{2}\right\}.$$

Motivated and inspired by Definition 1.13 and the work of Chanda et al [12], we introduce the definition of generalized Suzuki type $\mathcal{Z}$-contraction in $\theta-$metric spaces as follows.

Definition 1.14. *Let (X, d_θ) be $\theta-$metric space, T be a self-mapping on the set X and $\zeta \in \mathcal{Z}$. Then T is said to be generalized Suzuki type $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied*

$$\frac{1}{2} d_\theta\left(x, Tx\right) < d_\theta\left(x, y\right) \Rightarrow \zeta\left(d_\theta\left(Tx, Ty\right), M_\theta\left(x, y\right)\right) \geq 0, \quad \forall x, y \in X, \tag{1.9}$$

where

$$M_\theta\left(x, y\right) = \max\left\{d_\theta\left(x, y\right), d_\theta\left(x, Tx\right), d_\theta\left(y, Ty\right), \frac{d_\theta\left(x, Ty\right) + d_\theta\left(y, Tx\right)}{2}\right\}.$$

Remark 1.15. *It is clear that $\zeta\left(t, s\right) < 0$, $\forall t \geq s > 0$. Therefore T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to ζ, then*

$$\frac{1}{2} d_\theta\left(x, Tx\right) < d_\theta\left(x, y\right) \Rightarrow d_\theta\left(Tx, Ty\right) < M_\theta\left(x, y\right)$$

for all distinct $x, y \in X$.

The purpose of this paper is to prove fixed point theorems for generalized Suzuki type $\mathcal{Z}$-contraction in $\theta-$metric spaces. Furthermore, applications of some kind of nonlinear Hammerstein integral equations.

2 Main Results

In this section, we prove fixed point theorems for generalized Suzuki type $\mathcal{Z}$-contraction in $\theta-$metric spaces as follows.

Lemma 2.1. *Every generalized Suzuki type $\mathcal{Z}$-contraction with respect to ζ in $\theta-$metric space has a unique fixed point.*

Proof. Let T be a generalized Suzuki type $\mathcal{Z}$-contraction which satisfy the condition in Definition 1.14. Now, we want to prove that the mapping T has a unique fixed point. Assume that p and q be two different fixed point for the mapping T. Since

$$0 = \frac{1}{2}d_\theta(p, Tp) < d(p, q),$$

then by using (1.9), we get that

$$0 \leq \zeta\left(d_\theta\left(Tp, Tq\right), M_\theta\left(p, q\right)\right) \tag{2.1}$$

where

$$\begin{aligned} M_\theta\left(p, q\right) &= \max\left\{d_\theta\left(p, q\right), d_\theta\left(p, Tp\right), d_\theta\left(q, Tq\right), \frac{d_\theta\left(p, Tq\right) + d_\theta\left(q, Tp\right)}{2}\right\} \\ &= d_\theta\left(p, q\right). \end{aligned}$$

From (2.1), we obtain that

$$\begin{aligned} 0 &\leq \zeta\left(d_\theta\left(Tp, Tq\right), M_\theta\left(p, q\right)\right) \\ &= \zeta\left(d_\theta\left(p, q\right), d_\theta\left(p, q\right)\right). \end{aligned}$$

This is a contradiction. So, we have $p = q$. This complete the proof. □

Theorem 2.2. *Let (X, d_θ) be $\theta-$metric space, T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Let $\{x_n\}$ be a sequence of Picard of initial point at $x_0 \in X$. Then*

$$\lim_{n\to+\infty} d_\theta\left(x_n, x_{n+1}\right) = 0.$$

Proof. Give $x_0 = x \in X$ and let $\{x_n\}$ be the Picard sequence, that is $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. If $d_\theta(x_n, Tx_n) = 0$ for some $n \in \mathbb{N}$, then $x = x_n$ becomes a fixed point of T, which completes the proof. Thus, in the rest of the proof, we assume that

$$0 < d_\theta(x_n, Tx_n), \text{ for all } n \in \mathbb{N}.$$

Therefore, we have

$$\frac{1}{2}d_\theta(x_n, Tx_n) < d_\theta(x_n, Tx_n) = d_\theta(x_n, x_{n+1}).$$

Since T is a generalized Suzuki type $\mathcal{Z}$-contraction, we have

$$\begin{aligned} 0 &\leq \zeta(d_\theta(Tx_n, T^2x_n), M_\theta(x_n, x_{n+1})) \\ &= \zeta(d_\theta(Tx_n, Tx_{n+1}), M_\theta(x_n, x_{n+1})). \end{aligned}$$

Thus

$$\begin{aligned} & M_\theta(x_n, x_{n+1}) \\ = \ & \max\{d_\theta(x_n, x_{n+1}), d_\theta(x_n, Tx_n), d_\theta(x_{n+1}, Tx_{n+1}), \\ & \frac{d_\theta(x_n, Tx_{n+1}) + d_\theta(x_{n+1}, Tx_n)}{2}\} \\ = \ & \max\{d_\theta(x_n, x_{n+1}), d_\theta(x_n, x_{n+1}), d_\theta(x_{n+1}, x_{n+2}), \\ & \frac{d_\theta(x_n, x_{n+2}) + d_\theta(x_{n+1}, x_{n+1})}{2}\} \\ = \ & \max\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n+1}, x_{n+2}), \frac{d_\theta(x_n, x_{n+2})}{2}\}. \end{aligned}$$

Next, we get

$$\frac{d_\theta(x_n, x_{n+2})}{2} \leq \max\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n+1}, x_{n+2})\}.$$

Thus,

$$M_\theta(x_n, x_{n+1}) = \max\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n+1}, x_{n+2})\}$$

by (1.9), we get that

$$\begin{aligned} 0 &\leq \zeta(d_\theta(Tx_n, Tx_{n+1}), M_\theta(x_n, x_{n+1})) \\ &= \zeta(d_\theta(x_{n+1}, x_{n+2}), \max\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n+1}, x_{n+2})\}) \\ &< \max\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n+1}, x_{n+2})\} - d_\theta(x_{n+1}, x_{n+2}). \end{aligned} \tag{2.2}$$

From (2.2), we obtain that

$$M_\theta(x_n, x_{n+1}) = d_\theta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Therefore, the sequence $\{d_\theta(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative reals. So there is some $r \geq 0$ such that

$$\lim_{n\to+\infty} d_\theta(x_n, x_{n+1}) = r.$$

As T is a genralized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ therefore by the condition (Δ_3) in Definition 1.3, we obtain

$$0 \leq \lim_{n\to+\infty} \zeta(d_\theta(x_{n+1}, x_{n+2}), d_\theta(x_n, x_{n+1})) < 0.$$

This contradiction proves that $r = 0$ and hence

$$\lim_{n\to+\infty} d_\theta(x_n, x_{n+1}) = r = 0.$$

□

Theorem 2.3. *Let (X, d_θ) be $\theta-$metric space, T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then the sequence $\{x_n\}$ is bounded.*

Proof. Let $x_0 \in X$ and $\{x_n\}$ be the corresponding Picard sequence, i.e., $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. We claim that the sequence $\{x_n\}$ is bounded. Reasoning by contradiction, we suppose that $\{x_n\}$ is not bounded. So, we can construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the least integer such that

$$d_\theta(x_{n_{k+1}}, x_{n_k}) > 1$$

and

$$d_\theta(x_m, x_{n_k}) \leq 1 \tag{2.4}$$

for $n_k \leq m < n_{k+1} - 1$. Now, using the triangle inequality (θ_3) and (2.4), we get

$$\begin{aligned} 1 &< d_\theta(x_{n_{k+1}}, x_{n_k}) \\ &\leq \theta(d_\theta(x_{n_{k+1}}, x_{n_{k+1}-1}), d_\theta(x_{n_{k+1}-1}, x_{n_k})) \\ &\leq \theta(d_\theta(x_{n_{k+1}}, d_\theta(x_{n_{k+1}-1}), 1). \end{aligned} \tag{2.5}$$

Letting $k \to \infty$ on both sides of (2.5) and using Theorem 2.2 and the condition (B4) in Definition 1.9, we deduce that

$$d_\theta(x_{n_{k+1}}, x_{n_k}) \to 1.$$

On the other hand, using the condition (θ_3) in Definition 1.11 and (2.4), we obtain that

$$\begin{aligned} 1 &< d_\theta(x_{n_{k+1}}, x_{n_k}) \\ &\leq d_\theta(x_{n_{k+1}-1}, x_{n_k-1}) \\ &\leq \theta(d_\theta(x_{n_{k+1}-1}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1})) \\ &\leq \theta(1, d_\theta(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Thus, we get

$$d_\theta(x_{n_{k+1}-1}, x_{n_k-1}) \to 1.$$

Since

$$\begin{aligned} \frac{1}{2} d_\theta(x_{n_{k+1}-1}, Tx_{n_k-1}) &= \frac{1}{2} d_\theta(x_{n_{k+1}-1}, x_{n_k}) \\ &< d_\theta(x_{n_{k+1}-1}, x_{n_k}), \end{aligned}$$

by T be a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we obtain that

$$\zeta(d_\theta(x_{n_{k+1}}, x_{n_k}), M_\theta(x_{n_{k+1}-1}, x_{n_k})) \geq 0,$$

which implies that

$$d_\theta(x_{n_{k+1}}, x_{n_k}) \leq M_\theta(x_{n_{k+1}-1}, x_{n_k}).$$

Now,

$$\begin{aligned}
1 &< d_\theta(x_{n_{k+1}}, x_{n_k}) \\
&\le M_\theta(x_{n_{k+1}-1}, x_{n_k}) \\
&= \max\{d_\theta(x_{n_{k+1}-1}, x_{n_k-1}), d_\theta(x_{n_{k+1}-1}, x_{n_{k+1}}), \\
&\quad d_\theta(x_{n_k-1}, x_{n_k}), \frac{d_\theta(x_{n_{k+1}-1}, x_{n_k}) + d_\theta(x_{n_k-1}, x_{n_{k+1}})}{2}\} \\
&\le \max\{\theta(d_\theta(x_{n_{k+1}-1}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1})), d_\theta(x_{n_{k+1}-1}, x_{n_{k+1}}), \\
&\quad d_\theta(x_{n_k-1}, x_{n_k}), \frac{1 + d_\theta(x_{n_k-1}, x_{n_{k+1}})}{2}\} \\
&\le \max\{\theta(1, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1})), d_\theta(x_{n_{k+1}-1}, x_{n_{k+1}}), \\
&\quad d_\theta(x_{n_k-1}, x_{n_k}), \frac{1 + \theta(d_\theta(x_{n_k-1}, x_{n_k}) + d_\theta(x_{n_k}, x_{n_{k+1}}))}{2}\}.
\end{aligned}$$

Letting $k \to +\infty$, we obtain

$$1 \le \lim_{k\to+\infty} M_\theta(x_{n_{k+1}-1}, x_{n_k}) \le 1,$$

that is,

$$\lim_{k\to+\infty} M_\theta(x_{n_{k+1}-1}, x_{n_k}) = 1.$$

Furthermore, since $\frac{1}{2}d_\theta(x_{n_{k+1}-1}, x_{n_k-1}) < d_\theta(x_{n_{k+1}-1}, x_{n_k-1})$. Therefore, T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we get

$$\begin{aligned}
0 &\le \zeta(d_\theta(Tx_{n_{k+1}-1}, Tx_{n_k-1}), M_\theta(x_{n_{k+1}-1}, x_{n_k-1})) \\
&\le \limsup_{n\to\infty} \zeta(d_\theta(x_{n_{k+1}}, x_{n_k}), M_\theta(x_{n_{k+1}-1}, x_{n_k-1})) \\
&< 0.
\end{aligned}$$

This is a contradiction. Hence, $\{x_n\}$ is bounded. □

Theorem 2.4. *Let (X, d_θ) be $\theta-$metric space, T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then the sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Now, we will prove that $\{x_n\}$ is Cauchy sequence. Let $C_n = \sup\{d_\theta(x_i, x_j) : i, j \ge n\}$, $n \in \mathbb{N}$. From Theorem 2.3, we know that $C_n < +\infty$ for each $n \in \mathbb{N}$. Note that $\{C_n\}$ is a decreasing sequence of non-negative reals. Thus there exists $C \ge 0$ such that

$$\lim_{n\to+\infty} C_n = C.$$

Our claim is that $C = 0$. Let us suppose that $C > 0$. Considering C_n, for any $y \in \mathbb{N}$, there exists n_k, m_k such that $n_k > m_k \ge k$ and

$$C_k - \frac{1}{k} < d_\theta(x_{m_k}, x_{n_k}) \le C_k.$$

Letting $k \to \infty$ in the above inequality, we obtain

$$\lim_{k\to+\infty} d_\theta(x_{m_k}, x_{n_k}) = C. \tag{2.6}$$

Now,

$$\begin{aligned} & d_\theta(x_{m_k}, x_{n_k}) \\ \leq\ & d_\theta(x_{m_k-1}, x_{n_k-1}) \\ \leq\ & \theta(d_\theta(x_{m_k-1}, x_{m_k}), d_\theta(x_{m_k}, x_{n_k-1})) \\ \leq\ & \theta(d_\theta(x_{m_k-1}, x_{m_k}), \theta(d_\theta(x_{m_k}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1}))). \end{aligned}$$

Letting $k \to +\infty$ in the previous inequality and using the condition (B4) in Definition 1.9, we derive

$$\begin{aligned} C &\leq \lim_{k\to+\infty} d_\theta(x_{m_k-1}, x_{n_k-1}) \\ &\leq \theta(0, \theta(d_\theta(x_{m_k}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1}))) \\ &\leq \theta(d_\theta(x_{m_k}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1})). \end{aligned} \tag{2.7}$$

Again taking limit as $k \to +\infty$ in (2.7), and the condition (B4) in Definition 1.9, we obtain

$$\begin{aligned} C &\leq \lim_{k\to+\infty} d_\theta(x_{m_k-1}, x_{n_k-1}) \\ &\leq \theta(0, C) \\ &\leq C. \end{aligned}$$

Thus,

$$\lim_{k\to+\infty} d_\theta(x_{m_k-1}, x_{n_k-1}) = C. \tag{2.8}$$

From Theorem 2.4, we have

$$\frac{1}{2} d_\theta(x_{m_k-1}, Tx_{m_k-1}) < \frac{1}{2} d_\theta(x_{m_k-1}, x_{n_k-1}) < d_\theta(x_{m_k-1}, x_{n_k-1}).$$

Since T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we have

$$0 \leq \zeta(d_\theta(Tx_{m_k-1}, Tx_{n_k-1}), M_\theta(x_{m_k-1}, x_{n_k-1})).$$

By the condition Δ_3 in Definition 1.7, we obtain

$$\begin{aligned}
& d_\theta(x_{m_k}, x_{n_k}) \\
= & d_\theta(Tx_{m_k-1}, Tx_{n_k-1}) \\
< & M_\theta(x_{m_k-1}, x_{n_k-1}) \\
= & \max\{d_\theta(x_{m_k-1}, x_{n_k-1}), d_\theta(x_{m_k-1}, Tx_{m_k-1}), d_\theta(x_{n_k-1}, Tx_{n_k-1}), \\
& \frac{d_\theta(x_{m_k-1}, Tx_{n_k-1}) + d_\theta(x_{n_k-1}, Tx_{m_k-1})}{2}\} \\
\leq & \max\{d_\theta(x_{m_k-1}, x_{n_k-1}), d_\theta(x_{m_k-1}, x_{m_k}), d_\theta(x_{n_k-1}, x_{n_k}), \\
& \frac{d_\theta(x_{m_k-1}, x_{n_k}) + d_\theta(x_{n_k-1}, x_{m_k})}{2}\} \\
\leq & \max\{d_\theta(x_{m_k-1}, x_{n_k-1}), d_\theta(x_{m_k-1}, x_{m_k}), d_\theta(x_{n_k-1}, x_{n_k}), \\
& \frac{\theta(d_\theta(x_{m_k-1}, x_{m_k}), d_\theta(x_{m_k}, x_{n_k})) + \theta(d_\theta(x_{n_k-1}, x_{n_k}), d_\theta(x_{n_k}, x_{m_k}))}{2}\}.
\end{aligned}$$

Letting $k \to +\infty$, by Theorem 2.2 and 2.6, we obtain

$$\lim_{k\to+\infty} M_\theta(x_{m_k-1}, x_{n_k-1}) = C. \tag{2.9}$$

By using (2.8), (2.9) and the condition Δ_3 in Definition 1.7, we obtain

$$0 \leq \limsup_{k\to+\infty} \zeta(d_\theta(x_{m_k}, x_{n_k}), M_\theta(x_{m_k-1}, x_{n_k-1})) < 0,$$

which is a contradiction. Consequently, $\{x_n\}$ is Cauchy sequence. This complete the proof. □

Theorem 2.5. *Let (X, d_θ) be a complete θ-metric space, T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then T has fixed point.*

Proof. From the previous result in Theorem 2.4, $\{x_n\}$ is Cauchy sequence and X is complete there exists $\rho \in X$ such that

$$\lim_{n\to+\infty} x_n = \rho. \tag{2.10}$$

Now, we prove that ρ is a fixed point of T. We claim that

$$\frac{1}{2}d_\theta(x_n, Tx_n) < d_\theta(x_n, \rho) \text{ or } \frac{1}{2}d_\theta(x_{n+1}, Tx_{n+1}) < d_\theta(x_{n+1}, \rho), \ \forall n \in \mathbb{N}.$$

This is,

$$\frac{1}{2}d_\theta(x_n, Tx_n) < d_\theta(x_n, \rho) \text{ or } \frac{1}{2}d_\theta(Tx_n, T^2x_n) < d_\theta(Tx_n, \rho), \ \forall n \in \mathbb{N}. \tag{2.11}$$

It follow from (2.11). Let

$$(A) := \frac{1}{2}d_\theta(x_n, Tx_n) < d_\theta(x_n, \rho)$$

and

$$(B) := \frac{1}{2}d_\theta(Tx_n, T^2x_n) < d_\theta(Tx_n, \rho).$$

Suppose that there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} &\frac{1}{2}d_\theta(x_m, Tx_m) \geq d_\theta(x_m, \rho) \text{ and} \\ &\frac{1}{2}d_\theta(Tx_m, T^2x_m) \geq d_\theta(Tx_m, \rho). \end{aligned} \tag{2.12}$$

Therefore,

$$2d_\theta(x_m, \rho) \leq d_\theta(x_m, Tx_m) \leq \theta(d_\theta(x_m, \rho), d_\theta(\rho, Tx_m)).$$

Next, we have

$$\begin{aligned} d_\theta(Tx_m, T^2x_m) &< d_\theta(x_m, Tx_m) \\ &\leq \lim_{m\to\infty} d_\theta(x_m, Tx_m) \\ &\leq \lim_{m\to\infty} \theta(d_\theta(x_m, \rho), d_\theta(\rho, Tx_m)) \\ &\leq \theta(0, d_\theta(\rho, Tx_m)) \\ &\leq d_\theta(\rho, Tx_m) \\ &\leq 2d_\theta(\rho, Tx_m) \\ &\leq d_\theta(Tx_m, T^2x_m). \end{aligned} \tag{2.13}$$

Thus, we get $d_\theta(Tx_m, T^2x_m) < d_\theta(Tx_m, T^2x_m)$. This is a contradiction. Hence, (2.11) holds.

If the part (A) of is true, by T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we have

$$0 \leq \zeta(d_\theta(Tx_n, T\rho), M_\theta(x_n, \rho)). \tag{2.14}$$

By the condition Δ_2 in Definition 1.3, we obtain

$$\begin{aligned} &d_\theta(Tx_n, T\rho) \\ < \;& M_\theta(x_n, \rho) \\ = \;& \max\{d_\theta(x_n, \rho), d_\theta(x_n, Tx_n), d_\theta(\rho, T\rho), \frac{d_\theta(x_n, T\rho) + d_\theta(\rho, Tx_n)}{2}\}. \end{aligned}$$

Letting $n \to +\infty$ and by using (2.10), we get

$$\lim_{n\to\infty} M_\theta(x_n, \rho) = d_\theta(\rho, T\rho). \tag{2.15}$$

It follows, from (2.14), (2.15) and the condition Δ_3 in Definition 1.7, we have for every $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \zeta(d_\theta(Tx_n, T\rho), M_\theta(x_n, \rho)) \\ &\leq \limsup_{n\to+\infty} \zeta(d_\theta(Tx_n, T\rho), M_\theta(x_n, \rho)) \\ &\leq d_\theta(\rho, T\rho) - d_\theta(\rho, T\rho). \end{aligned}$$

From the condition Δ_3 in Definition 1.7, since the both sequences $d_\theta(Tx_n, T\rho)$, $M_\theta(x_n, \rho)$ converge to $d_\theta(\rho, T\rho) > 0$ it is clear that

$$\limsup_{n\to+\infty} \zeta(d_\theta(Tx_n, T\rho), M_\theta(x_n, \rho)) < 0.$$

This is a contradiction. Therefore, ρ is a fixed point of T.

Also, if the part (B) of is true, by T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we have

$$0 \leq \zeta(d_\theta(T^2x_n, T\rho), M_\theta(Tx_n, \rho)). \tag{2.16}$$

By the condition Δ_2 in Definition 1.3, we obtain

$$\begin{aligned} & d_\theta(T^2x_n, T\rho) \\ < \; & M_\theta(Tx_n, \rho) \\ = \; & \max\{d_\theta(Tx_n, \rho), d_\theta(Tx_n, T^2x_n), d_\theta(\rho, T\rho), \\ & \frac{d_\theta(Tx_n, T\rho) + d_\theta(\rho, T^2x_n)}{2}\} \\ = \; & \max\{d_\theta(Tx_n, \rho), d_\theta(Tx_n, T^2x_n), d_\theta(\rho, T\rho), \\ & \frac{d_\theta(Tx_n, T\rho) + \theta(d_\theta(\rho, Tx_n), d_\theta(Tx_n, T^2x_n))}{2}\}. \end{aligned}$$

Letting $n \to +\infty$ and by using (2.10), we get

$$\lim_{n\to\infty} M_\theta(Tx_n, \rho) = d_\theta(\rho, T\rho). \tag{2.17}$$

It follows, from (2.16), (2.17) and the condition Δ_3 in Definition 1.7, we have for every $n \in \mathbb{N}$,

$$\begin{aligned} 0 \;\leq\; & \zeta(d_\theta(T^2x_n, T\rho), M_\theta(Tx_n, \rho)) \\ \leq\; & \limsup_{n\to+\infty} \zeta(d_\theta(T^2x_n, T\rho), M_\theta(Tx_n, \rho)) \\ \leq\; & d_\theta(\rho, T\rho) - d_\theta(\rho, T\rho). \end{aligned}$$

From the condition Δ_3 in Definition 1.7, since the both sequences $d_\theta(T^2x_n, T\rho)$, $M_\theta(Tx_n, \rho)$ converge to $d_\theta(\rho, T\rho)$ it is clear that

$$\limsup_{n\to+\infty} \zeta(d_\theta(T^2x_n, T\rho), M_\theta(Tx_n, \rho)) < 0.$$

This is a contradiction. Therefore, ρ is a fixed point of T. Uniqueness is guaranteed from Lemma 2.1. $\square$

Next, it will demonstrate an example that corresponds to Theorem 2.5 as follows.

Example 2.6. *Let $X = [0,1]$ be equipped with the Euclidean metric. We define a mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{2}{20}, & x \in S_1 = [1, \frac{1}{2}); \\ \frac{1}{20}, & x \in S_2 = [\frac{1}{2}, 1]; \end{cases}$$

where $y - x > \frac{1}{5}$ for all $x \in S_1$, $y \in S_2$. We show that T is a generalized Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Now, we have $0 \leq d_\theta(Tx, Ty) \leq \frac{1}{20}$ for all $x, y \in X$. Next, if both $x, y \in S_1$ or S_2, then $d_\theta(Tx, Ty) = 0$. On the other hand, let $x \in S_1$ and $y \in S_2$. Now, we obtain $\frac{1}{5} < d_\theta(x, y) \leq 1$. Also, $0 \leq d_\theta(x, Tx) \leq \frac{2}{5}$, $\frac{9}{20} \leq d_\theta(y, Ty) \leq \frac{19}{20}$, $0 \leq d_\theta(x, Ty) \leq \frac{9}{20}$, $\frac{4}{10} \leq d_\theta(y, Tx) \leq \frac{9}{10}$ and $\frac{4}{20} \leq \frac{d_\theta(x,Ty)+d_\theta(y,Tx)}{2} \leq \frac{27}{40}$. Thus, $M_\theta(x, y) \geq \frac{9}{20}$. From the calculation, it is clear that

$$\frac{1}{2} d_\theta(x, Tx) < d_\theta(x, y)$$

for all $x, y \in X$, then

$$d_\theta(Tx, Ty) < M_\theta(x, y).$$

Therefore, we obtain

$$\zeta(d_\theta(Tx, Ty), M_\theta(x, y)) \geq 0,$$

for all $x, y \in X$. Hence, T is the generalized Suzuki type $\mathcal{Z}-$contraction with respect to $\zeta \in \mathcal{Z}$. Taking into account Theorem 2.5, we can say that T has a unique fixed point. Here $\rho = \frac{2}{20}$ is that required fixed point.

Corollary 2.7. *Let (X, d_θ) be a complete θ-metric space, $T : X \to X$ be a mapping such that there exists $k \in (0, 1)$ satisfying*

$$d_\theta(Tx, Ty) < kM_\theta(x, y), \quad \forall x, y \in X,$$

where

$$M_\theta(x, y) = \{d_\theta(x, y), d_\theta(x, Tx), d_\theta(y, Ty), \frac{d_\theta(x, Ty) + d_\theta(y, Tx)}{2}\}.$$

Then T has fixed point.

3 Application to nonlinear operator equations

In this section, we give an application of Theorem 2.5 to show the existence and uniqueness problem of solutions for some kind of nonlinear Hammerstein integral equations. We consider this integral equations as follows.

$$x(t) = f(t) + \int_0^t K(t, s)\, h(s, x(s))\, ds, \tag{3.1}$$

where the unknown function $x(t)$ takes real values.

Let $X = C([0,1])$ be the space of all real continuous functions defined on $[0,1]$. It is well known that $C([0,1])$ endowed with the θ-metric

$$d_\theta(x,y) = \|x-y\| = \max_{t\in[0,1]} |x(t) - y(t)|, \tag{3.2}$$

with respect to B-action $\theta \in Y$. Thus, (X, d_θ) is a complete θ-metric space. Define a mapping $T : X \to X$ by

$$T(x)(t) = f(t) + \int_0^t K(t,s)\, h(s, x(s))\, ds, \quad \forall t \in (0,1). \tag{3.3}$$

Assumption 3.1

(I) $f \in C([0,1] \times (-\infty, +\infty))$, $f \in X$ and $K \in C([0,1] \times [0,1])$ such that $K(t,s) \geq 0$;

(II) $h(t,\cdot) : (-\infty, +\infty) \to (-\infty, +\infty)$ is increasing for all $t \in (0,1)$ such that

$$\frac{1}{2} d_\theta(x, T(x)) < d_\theta(x,y) \Rightarrow \zeta(|h(t,x) - h(t,y)|, \mathcal{M}_\theta(x,y)) \geq 0$$

implies

$$|h(t,x) - h(t,y)| < \mathcal{M}_\theta(x,y) \quad \forall x, y \in X, t \in (0,1),$$

where $\mathcal{M}_\theta(x,y) = \max\left\{|x-y|, |x-Tx|, |y-Ty|, \dfrac{|x-Ty| + |y-Tx|}{2}\right\}$;

(III) $\max_{t,s\in[0,1]} |Kv(t,s)| \leq 1.$

Theorem 3.1. *Let $X = C([0,1])$, (X, d_θ), T, h, $K(t,s)$ are satisfied in Assumption 3.1, then the nonlinear Hammerstein integral equation* (3.1) *has a unique solution $w \in C([0,1])$ and for each $x \in C([0,1])$ the iterative sequence $\{x_n = T^n x\}$ converges to the unique solution $w \in X$ of equation* (3.1).

Proof. Now, we show that the mapping $T : X \to X$ define by (3.2) is a Suzuki type $\mathcal{Z}$-contraction. By assumption (II) and (III), $\forall x, y \in C([0,1])$, $t \in (0,1)$, we have

$$\begin{aligned}
& |Tx(t) - Ty(t)| \\
< \;& M_\theta(Tx_n, \rho) \\
= \;& |\int_0^t K(t,s)\left(h(s,x(s)) - h(s,y(s))\right) ds| \\
\le \;& \int_0^t |K(t,s)||h(s,x(s)) - h(s,y(s))| ds \\
\le \;& \int_0^t |h(s,x(s)) - h(s,y(s))| ds \\
< \;& \int_0^t \mathcal{M}_\theta(x(s), y(s)) ds \\
= \;& \int_0^t \max\left\{ \begin{array}{c} |x(s) - y(s)|, |x(s) - Tx(s)|, |y(s) - Ty(s)|, \\ \dfrac{|x(s) - Ty(s)| + |y(s) - Tx(s)|}{2} \end{array} \right\} ds \\
\le \;& \int_0^t \max\left\{ d_\theta(x,y), d_\theta(x,Tx), d_\theta(y,Ty), \frac{d_\theta(x,Ty) + d_\theta(y,Tx)}{2} \right\} ds \\
= \;& M_\theta(x,y) \int_0^t ds \\
= \;& t M_\theta(x,y) \\
\le \;& M_\theta(x,y).
\end{aligned}$$

Therefore, T is a Suzuki type $\mathcal{Z}$-contraction and Theorem 2.5 applies to T, which guarantee the existence of a unique fixed point $w \in X$. That is, w is the unique solution of the integral equations (3.1). For each $x \in X$, the sequence $\{x_n = T^n x\}$ converges to w. This complete the proof. □

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