Fixed Point Theory for Nonlinear Mappings, Generalized Equilibrium Problems and Variational Inequality Problems by Using Hybrid Method

Bunyawee Chaloemyotphong[†] **and Atid Kangtunyakarn** [‡],[1]

[†]Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand
e-mail : rokanber4@gmail.com
[‡],Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand
e-mail : beawrock@hotmail.com

Abstract : In this article, by using hybrid method we prove strong convergence theorem for finding a common element of the set of solutions of equilibrium problems, generalized equilibrium problems and fixed points problems by using the K-mapping generated by a finite family of nonspreading mappings and a finite real numbers. Moreover, we apply our main result to obtain a strong convergence theorem for finding a solution of minimization problems, generalized equilibrium problems and fixed points problems of nonlinear mappings.

Keywords : Generalized equilibrium problem; Variational Inequality problem; Fixed Point; Nonspreading mapping

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[1]Corresponding author email: beawrock@hotmail.com (Atid Kangtunyakarn)

1   Introduction

Let C be a nonempty closed convex subset of a real Hilbert spaces H. A mapping $T : C \to C$ is called
(i) nonexpansive if
$$\|Tx - Ty\| \leq \|x - y\|,$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle,$$

(iii) quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|,$$

(iv) nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2,$$

for all $x, y \in C$ and $z \in F(T)$ where $F(T)$ denotes the set of fixed points of T (i.e., $F(T) = \{x \in C : Tx = x\}$).

A mapping $A : C \to H$ is called α-inverse strongly monotone, see [15] if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

Let $G : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for G is to determine a point $x^* \in C$ such that

$$G(x^*, y) \geq 0, \ \forall y \in C. \tag{1.1}$$

The set of all solution of (1.1) is denoted by

$$EP(G) = \{x^* \in C : G(x^*, y) \geq 0\}. \tag{1.2}$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\lim_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for all $x \in C, y \to F(x, y)$ is convex and lower semicontinuous.
Many problems in physics, optimization, and economics reduce to find a solution of $EP(G)$, see, for instance [2]-[7]. In 2007, Takahashi and Takahashi [7] proved the following theorem:

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(G) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, 1)$ satisfying
(C1) $\alpha_n \to 0$ as $n \to \infty$,
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(C3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$,
$\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(G)$, where $z = P_{F(S)\cap EP(G)}f(z)$.

Let $B : C \to H$ be a nonlinear mapping. The variational inequality problems is to find a point $u \in C$ such that

$$\langle v - u, Bu \rangle \geq 0, \text{ for all } v \in C. \tag{1.3}$$

The set of solutions of the variational inequality is denoted $VI(C, B)$. Numerous problems in physics, optimization, minimax problems are reduced to variational inequality problems, see, for instance [17]-[18].

In 2009, Kumam [17] introduced an iterative algorithm as follows:

Algorithm 1.2. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction satifying $(A1) - (A4)$ and let A be a monotone k-Lipschitz continuous mapping and let S be a nonexpansive maping. Suppose $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are guven by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.4}$$

for all $n \in \mathbb{N}$.

He proved under some control conditions on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ that the sequence $\{x_n\}$ generated by (1.4) convergence strongly to $P_{F(S)\cap VI(C,A)\cap EP(F)}u$.

The generalized equilibrium problem is to find $z \in C$ such that $G(x, y) + \langle Bz, y - z \rangle \geq 0$, $\forall y \in C$.

The set of all solutions of generalized equilibrium problem is denoted by

$$EP(G, B) = \{z \in C : G(z, y) + \langle Bz, y - z \rangle \geq 0, \ \forall y \in C\}. \tag{1.5}$$

In the case of $B = 0, EP(G, B) = EP(G)$.

Let $CB(H)$ be a family of all nonempty closed bounded subsets of H and

$\mathcal{H}(.,.)$ be the Hausdorff metric on $CB(H)$ defined as

$$\mathcal{H}(U,V) = \max\{\sup_{u \in U} d(u,V), \sup_{v \in V} d(U,v)\}, \forall U,V \in CB(H),$$

where $d(u,V) = \inf_{v \in V} d(u,v)$, $d(U,v) = \inf_{u \in U} d(u,v)$ and $d(u,v) = \|u-v\|$.

Let $\varphi : C \to H$ be a real-valued function, $T : C \to CB(H)$ be a multivalued map and $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function, that is, $\Phi(w,u,v) + \Phi(w,v,u) = 0$ for all $(w,u,v) \in H \times C \times C$ which satisfies the following conditions with respect to the multivalued $T : C \to CB(H)$;

(H1) For each fixed $v \in C$, $(w,u) \mapsto \Phi(w,u,v)$ is an upper semicontinuous function from $H \times C$ to $\mathbb{R}$, that is, for $(w,u) \in H \times C$, wherever $w_n \to w$ and $u_n \to u$ as $n \to \infty$,

$$\lim_{n \to \infty} sup(w_n, u_n, v) \to \Phi(w,u,v),$$

(H2) For each fixed $(w,v) \in H \times C \times C$, $u \mapsto \Phi(w,u,v)$ is a concave function,

(H3) For each fixed $(w,u) \in H \times C \times C$, $v \mapsto \Phi(w,u,v)$ is a convex function.

In 2009, Ceng et al.[8] introduced the following generalized equilibrium problem (GEP) as follows:

$$(GEP) \begin{cases} Find\ u \in C\ and\ w \in T(u)\ such\ that \\ \Phi(w,u,v) + \varphi(v) - \varphi(u) \geq 0, \forall v \in C. \end{cases} \tag{1.6}$$

The set of solutions of (GEP) is denoted by $(GEP)_s(\Phi, \varphi)$, see, for instance [4]. In the case of $\varphi \equiv 0$ and $\Phi(w,u,v) \equiv G(u,v)$, then $(GEP)_s(\Phi, \varphi)$ is denoted by $EP(G)$.

In 2012, Kangtunyakarn [14] introduced an iterative algorithm as follows:

Algorithm 1.3. *Let T_i, $i = 1, 2, ..., N$, be κ_i-pseudo-contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$ and let S_n be the S-mappings generated by $\{T_i\}_{i=1}^N$ and $\lambda_1^{(n)}, ..., \lambda_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{(n),j}, \alpha_2^{(n),j}, \alpha_3^{(n)}, j) \in I \times I \times I, I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, ..., N-1$, $\kappa \leq \alpha_1^{n,N} \leq 1$, $\kappa \leq \alpha_3^{n,N} \leq d < 1$, $\kappa \leq \alpha_2^{n,N} \leq e < 1$ for all $j = 1, 2, ..., N$. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exist sequences $\{w_n^1\}, \{w_n^2\} \in H$ and $\{x_n\}, \{u_n\}, \{v_n\} \subseteq C$ such that*

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n \rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n \rangle \geq 0, \forall v \in C, s_n > 0, \\ z_n = \delta_n P_C(I - \lambda A)u_n + (1 - \delta_n)P_C(I - \eta B)v_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n)S_n z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \forall n \geq 1, \end{cases}$$

$$\tag{1.7}$$

where $D, T : C \to CB(H)$ *are* $\mathcal{H}$-*Lipschitz continuous with constant* μ_1, μ_2, *re-spectively,* $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ *are equilibrium-like functions satisfying* $(H1) - (H3)$, $\varphi_1, \varphi_2 : C \to \mathbb{R}$ *be a lower semicontinuous and convex functional,* $A : C \to H$ *is* α-*inverse strongly monotone mapping and* $B : C \to H$ *is* β-*inverse strongly monotone mapping.*

He proved under some control conditions on $\{\delta_n\}$, $\{\alpha_n\}$, $\{r_n\}$ *and* $\{s_n\}$ *that the sequence* $\{x_n\}$ *generated by (1.7) convergence strongly to* $P_{\mathbb{F}} x_1$, *where* $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \cap F(G_1) \cap F(G_2)$, $G_1, G_2 : C \times C$ *are defined by* $G_1(x) = P_C(x - \lambda Ax)$, $G_2(x) = P_C(x - \eta Bx)$, $\forall x \in C$ *and* $P_{\mathbb{F}} x_1$ *is a solution of the following system of variational inequalities:*

$$\begin{cases} \langle Ax^*, x - x^* \rangle \geq 0, \\ \langle Bx^*, x - x^* \rangle \geq 0. \end{cases}$$

By motived of Algorithm 1.2 and Algorithm 1.3, we define the following algorithm as follows:

Algorithm 1.4. *Let* $T_i, i = 1, 2, ..., N$, *be nonspreading mappings of* C *into itself and let* K *be the* $K-$*mappings generated by* $T_1, T_2, ..., T_N$ *and* $\lambda_1, \lambda_2, ..., \lambda_N$. *Let* $x_1 \in C = C_1$ *and* $w_1 \in T(x_1)$, *there exist sequences* $\{x_n\}, \{u_n\}, \{\overline{u}_n\} \subseteq C$ *and* $\{w_n.\}\} \in H$ *such that*

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n, u_n, u) + \varphi(u) - \varphi(u_n) + \frac{1}{r_n} \langle u_n - x_n, u - u_n \rangle \geq 0, \forall u \in C, r_n > 0, \\ F(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n \rangle + \frac{1}{r} \langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n \rangle \geq 0, \forall \overline{u} \in C, \\ y_n = \alpha_n u_n + \gamma_n \overline{u}_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases}$$

(1.8)

where $T : C \to CB(H)$ *is* $\mathcal{H}$-*Lipschitz continuous with constant* μ, *respectively,* $\Phi : H \times C \times C \to \mathbb{R}$ *is equilibrium-like function satisfying* $(H1) - (H3)$, $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex function,* $F : C \times C \to \mathbb{R}$ *is a bifunction and* $A : C \to H$ *is an* α-*inverse strongly monotone mapping.*

In this article, we prove a sequence $\{x_n\}$ generated by (1.8) converges strongly to an element of the set of solutions of equilibrium problems, generalized equilibrium problems and fixed points problems by using the K-mapping generated by a finite family of nonspreading mappings and a finite real number introduced by Kangtunyakarn and Suantai [10]. Furthermore, we apply our main result to obtain a strong convergence theorem for finding a solutions of minimization problems, generalized equilibrium problems and fixed points problems of nonlinear mappings.

2   Preliminaries

In this section, we need the following lemmas and definitions to prove our main result.

Let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \ \forall y \in C.$$

The following lemma is a property of P_C.

Lemma 2.1. *(See [9].)Given $x \in H$ and $y \in C$, then $P_C x = y$ if and only if the following inequality holds*

$$\langle x - z, z - y \rangle \ge 0, \ \forall z \in C.$$

Lemma 2.2. *(See [1].)Let H be a Hilbert space and C a nonempty closed convex subset of H. Let T be a nonspreading mapping of C into itself. Then $F(T)$ is closed and convex.*

In 2009, Kangtunyakarn and Suantai [10] introduced K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ as follows:

Definition 2.3. *Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into itself, and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i = 1, ..., N$. Define a mapping $K : C \to C$ as follows:*

$$U_1 = \lambda_1 T_1 + (1 - \lambda_1)I$$
$$U_2 = \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1$$
$$U_3 = \lambda_3 T_3 U_2 + (1 - \lambda_3)U_2$$
$$\vdots$$
$$U_{N-1} = \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1})U_{N-2}$$
$$K = U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}.$$

Such a mapping K is called the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$.

Lemma 2.4. *(See [16].)Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \ne \emptyset$ and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \le 1$. Let K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$ and K is a quasi-nonexpansive mapping.*

Lemma 2.5. *(See [11].)Let H be a Hilbert space, C a closed convex subset of H, and $S : C \to C$ a nonspreading mapping with $F(S) \ne \emptyset$.Then S is demiclosed, that is, $x_n \rightharpoonup u$ and $x_n - Sx_n \to 0$ imply $u \in F(S)$.*

Let $\omega(x_n)$ be the set of all weakly ω-limit of $\{x_n\}$, i.e., $\omega(x_n) = \{c \mid x_{n_k} \rightharpoonup c$ as $k \to \infty\}$ where $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Lemma 2.6. *(See [12].) Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$, if $\{x_n\}$ is such that $\omega(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \forall n \in \mathbb{N}.$$

Then $x_n \to q$ as $n \to \infty$.

Definition 2.7. *A multivalued mapping $T : C \to CB(H)$ is said to be $\mathcal{H}$-Lipschitz continuous if there exists a constant $\mu > 0$ such that*

$$\mathcal{H}(T(u), T(v)) \leq \mu\|u - v\|, \forall u, v \in C,$$

where $\mathcal{H}(.,.)$ is the Hausdorff metric on $CB(H)$.

Lemma 2.8. *(Nadler's theorem, See [13].) Let $(X, \|\cdot\|)$ be a normed vector space and $\mathcal{H}(.,.)$ be the Hausdorff metric on $CB(H)$. If $U, V \in CB(X)$, then for any given $\varepsilon > 0$ and $u \in U$, there exists $v \in V$ such that*

$$\|u - v\| \leq (1 + \varepsilon)\mathcal{H}(U, V).$$

Theorem 2.9. *(See [8].) Let C be a nonempty, bounded, closed and convex subset of a real Hilbert space H, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with a constant μ, and $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying (H1)-(H3). Let $r > 0$ be a constant. For each $x \in C$, take $w_x \in T(x)$ arbitrarity and define a mapping $T_r : C \to C$ as follows:*

$$T_r(x) = \{u \in C : \Phi(\omega_x, u, v) + \varphi(v) - \varphi(u) + \frac{1}{r}\langle u - x, v - u \rangle \geq 0, \forall v \in C\}.$$

Then, the following hold:
  (a) T_r is a single-valued;
  *(b) T_r is a firmly nonexpansive (that is, for any $u, v \in C$,
$\|T_r u - T_r v\|^2 \leq \langle T_r u - T_r v, u - v \rangle$) if*

$$\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq 0,$$

for all $(x_1, x_2) \in C \times C$ and all $w_i \in T(x_i)$, $i = 1, 2$;
  (c) $F(T_r) = (GEP)_s(\Phi, \varphi)$;
  (d) $(GEP)_s(\Phi, \varphi)$ is closed and convex.

Lemma 2.10. *(See [6].) Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \text{ for all } x \in C.$$

Lemma 2.11. *(See [5].) Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $\overline{T}_r : H \times C$ as follows:*

$$\overline{T}_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C\},$$

for all $z \in H$. Then the following hold:
  (a) $\overline{T}_r$ is a single-valued;
  (b) $\overline{T}_r$ is a firmly nonexpansive, that is,

$$\|\overline{T}_r u - \overline{T}_r v\|^2 \leq \langle \overline{T}_r u - \overline{T}_r v, u - v \rangle, \forall u, v \in H;$$

  (c) $F(\overline{T}_r) = EP(F)$;
  (d) $EP(F)$ is closed and convex.

3 Main Results

Theorem 3.1. *Let C be a nonempty bounded, closed, and convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ, respectively, $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $(H1) - (H3)$. Let $A : C \to H$ be an α-inverse strongly monotone mapping, let $F : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$, let T_i be nonspreading mappings of C into itself for all $i = 1, 2, ..., N$ with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap EP(F, A) \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1 \in T(x_1)$. Support that there exist sequences $\{x_n\}, \{u_n\}, \{\overline{u}_n\} \subseteq C$ and $\{w_n\} \in H$ be sequences generated by (1.8) where $\{\alpha_n\}, \{\gamma_n\}$ and $\{\eta_n\}$ are sequences in $[0, 1]$ for all $n \in \mathbb{N}$, $r, r_n \subset (0, 2\alpha)$ and suppose the following conditions hold:*

(i) $\alpha_n + \gamma_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \gamma_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there is $\lambda > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \quad (3.1)$$

for all $(r_1, r_2) \in \Theta \times \Theta$, $w_i \in T(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$.

Proof. From (3.1) for every $r \in \Theta$, we have

$$\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq -\lambda_1\|T_r(x_1) - T_r(x_2)\|^2 \leq 0, \quad (3.2)$$

for all $(x_1, x_2) \in C \times C$ and $w_i \in T(x_i), i = 1, 2$.

It is easy to see that $I - rA$ is a nonexpansive mapping. Indeed, since A is an α-inverse strongly monotone mapping with $r \in (0, 2\alpha)$, we have

$$
\begin{aligned}
\|(I - rA)x - (I - rA)y\|^2 &= \|x - y - r(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2r\langle x - y, Ax - Ay\rangle + r^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\alpha r\|Ax - Ay\|^2 + r^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2. \quad\quad\quad\quad\quad\quad\quad\quad\quad (3.3)
\end{aligned}
$$

Thus $I - rA$ is a nonexpansive mapping.

From (1.8) and Theorem 2.9, we have $u_n = T_{r_n}x_n$.

From (1.8) and Lemma 2.11, we have $\bar{u}_n = \overline{T}_r(I - rA)x_n$.

Let $z \in \mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap EP(F, A) \cap (GEP)_s(\Phi, \varphi)$.

From Theorem 2.9 and Lemma 2.11, we have $z = T_{r_n}z = \overline{T}_r(I - rA)z$.

From nonexpansiveness of $\{T_{r_n}\}$, we have

$$\|u_n - z\| \leq \|x_n - z\|,$$

From nonexpansiveness of $\{\overline{T}_r\}, \{I - rA\}$, we have

$$
\begin{aligned}
\|\bar{u}_n - z\| &= \|\overline{T}_r(I - rA)x_n - z\| \\
&\leq \|x_n - z\|,
\end{aligned}
$$

From the definition of y_n, we have

$$
\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n u_n + \gamma_n \bar{u}_n + \eta_n K x_n - z\|^2 \\
&\leq \alpha_n \|u_n - z\|^2 + \gamma_n \|\bar{u}_n - z\|^2 + \eta_n \|K x_n - z\|^2 \quad\quad (3.4) \\
&\leq \|x_n - z\|^2.
\end{aligned}
$$

Next, we show that C_n is closed and convex for every $n \in \mathbb{N}$. It is obvious that C_n is closed. In fact, we know that, for $z \in C_n$,

$$\|y_n - z\| \leq \|x_n - z\| \text{ is equivalent to } \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0. \quad (3.5)$$

Let $z_1, z_2 \in C_n$ and $t \in (0, 1)$, it follows that

$$
\begin{aligned}
\|y_n - x_n\|^2 &+ 2\langle y_n - x_n, x_n - (tz_1 + (1-t)z_2)\rangle \\
&= t(2\langle y_n - x_n, x_n - z_1\rangle + \|y_n - x_n\|^2) \\
&\quad + (1-t)(2\langle y_n - x_n, x_n - z_2\rangle + \|y_n - x_n\|^2) \\
&\leq 0.
\end{aligned}
$$

From (3.5), we have $tz_1 + (1-t)z_2 \in C_n$. Then, we have C_n is convex. By Theorem 2.9, Lemma 2.2 and 2.11, we conclude that $\mathbb{F}$ is closed and convex. Then

$P_{\mathbb{F}}$ is well defined.

Next, we show that $\mathbb{F} \subset C_n$ for every $n \in \mathbb{N}$.

Putting $q \in \mathbb{F}$, by (3.4), it is easy to see that $q \in C_n$, then we have $\mathbb{F} \subset C_n$ for all $n \in \mathbb{N}$. Since $x_n = P_{C_n} x_1$, for every $w \in C_n$, we have

$$\|x_n - x_1\| \leq \|w - x_1\|, \forall n \in \mathbb{N}.$$

Since $P_{\mathbb{F}} x_1 \in \mathbb{F} \subset C_n$ and $x_n = P_{C_n} x_1$, we have

$$\|x_n - x_1\| \leq \|P_{\mathbb{F}} x_1 - x_1\|. \tag{3.6}$$

We will show that $\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0$

Since C is bounded, we have $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{\bar{u}_n\}$, $\{y_n\}$. Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and $x_n = P_{C_n} x_1$, we have

$$\begin{aligned}
0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\
&= -\langle x_n - x_1, x_n - x_1 \rangle + \langle x_1 - x_n, x_1 - x_{n+1} \rangle \\
&\leq -\|x_n - x_1\|^2 + \|x_n - x_1\| \|x_1 - x_{n+1}\|.
\end{aligned}$$

It implies that

$$\|x_n - x_1\| \leq \|x_1 - x_{n+1}\|.$$

It follows that $\lim_{n \to \infty} \|x_n - x_1\|$ exists.

Since

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\
&\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2,
\end{aligned}$$

and $\lim_{n \to \infty} \|x_n - x_1\|$ exists, we have

$$\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0. \tag{3.7}$$

Since $x_{n+1} = P_{Cn+1} x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|.$$

From (3.7), we have

$$\lim_{n \to \infty} \|y_n - x_{n+1}\| = 0. \tag{3.8}$$

Since

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

by (3.7) and (3.8), we have

$$\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{3.9}$$

Next, we show that

$$\lim_{n \to \infty} \|Kx_n - x_n\| = 0.$$

By definition y_n, we have

$$y_n - x_n = \alpha_n(u_n - x_n) + \gamma_n(\bar{u}_n - x_n) + \eta_n(Kx_n - x_n)$$

It implies that

$$\eta_n(Kx_n - x_n) = (y_n - x_n) + \alpha_n(x_n - u_n) + \gamma_n(x_n - \bar{u}_n) \\ + \delta_n(x_n - \bar{v}_n). \tag{3.10}$$

Since T_{r_n} is a firmly nonexpansive mapping and $T_{r_n}x_n = u_n$, we have

$$\begin{aligned} \|u_n - z\|^2 &= \|T_{r_n}x_n - T_{r_n}z\|^2 \\ &\leq \langle T_{r_n}x_n - T_{r_n}z, x_n - z \rangle \\ &= \frac{1}{2}(\|u_n - z\|^2 + \|x_n - z\|^2 - \|u_n - x_n\|^2), \end{aligned} \tag{3.11}$$

it implies that

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.12}$$

From (3.4) and (3.12), we have

$$\begin{aligned} \|y_n - z\|^2 &= \|\alpha_n u_n + \gamma_n \bar{u}_n + \eta_n Kx_n - z\|^2 \\ &\leq \alpha_n\|u_n - z\|^2 + \gamma_n\|\bar{u}_n - z\|^2 + \eta_n\|Kx_n - z\|^2 \\ &\leq \alpha_n(\|x_n - z\|^2 - \|u_n - x_n\|^2) + \gamma_n\|\bar{u}_n - z\|^2 + \eta_n\|Kx_n - z\|^2 \\ &\leq \|x_n - z\|^2 - \alpha_n\|u_n - x_n\|^2. \end{aligned} \tag{3.13}$$

it implies that

$$\begin{aligned} \alpha_n\|u_n - x_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\ &\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|. \end{aligned} \tag{3.14}$$

By (3.9) and condition (ii), we have

$$\lim_{n \to \infty} \|u_n - x_n\| = 0. \tag{3.15}$$

Since $\overline{T}_r$ is a nonexpansive mapping, $\overline{T}_r(I - rA)x_n = \bar{u}_n$ and A is an α−inverse strongly monotone mapping with $r \in (0, 2\alpha)$, we have

$$\begin{aligned} \|\bar{u}_n - z\|^2 &= \|\overline{T}_r(I - rA)x_n - \overline{T}_r(I - rA)z\|^2 \\ &\leq \|(x_n - z) - r(Ax_n - Az)\|^2 \\ &\leq \|x_n - z\|^2 - 2r\langle x_n - z, Ax_n - Az \rangle + r^2\|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 - r(2\alpha - r)\|Ax_n - Az\|^2. \end{aligned} \tag{3.16}$$

By (3.13) and (3.16), we have

$$\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n u_n + \gamma_n \bar{u}_n + \eta_n K x_n - z\|^2 \\
&\leq \alpha_n \|u_n - z\|^2 + \gamma_n \|\bar{u}_n - z\|^2 + \eta_n \|K x_n - z\|^2 \\
&\leq \alpha_n \|u_n - z\|^2 + \gamma_n(\|x_n - z\|^2 - r(2\alpha - r)\|A x_n - A z\|^2) + \eta_n \|K x_n - z\|^2 \\
&\leq \|x_n - z\|^2 - \gamma_n r(2\alpha - r)\|A x_n - A z\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
\gamma_n r(2\alpha - r)\|A x_n - A z\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\
&\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|. \qquad (3.17)
\end{aligned}$$

By condition (ii), $r \in (0, 2\alpha)$ and (3.9), we have

$$\lim_{n \to \infty} \|A x_n - A z\| = 0. \qquad (3.18)$$

By $\overline{T}_r$ is a firmly nonexpansive mapping and $\overline{T}_r(I - rA)x_n = \bar{u}_n$, we have

$$\begin{aligned}
\|\bar{u}_n - z\|^2 &= \|\overline{T}_r(I - rA)x_n - \overline{T}_r(I - rA)z\|^2 \\
&\leq \langle \bar{u}_n - z, (I - rA)x_n - (I - rA)z \rangle \\
&= \frac{1}{2}\Big(\|\bar{u}_n - z\|^2 + \|(I - rA)x_n - (I - rA)z\|^2 \\
&\quad - \|((I - rA)x_n - (I - rA)z) - (\bar{u}_n - z)\|^2 \Big) \\
&\leq \frac{1}{2}\Big(\|\bar{u}_n - z\|^2 + \|x_n - z\|^2 - \|((x_n - \bar{u}_n) - r(A x_n - A z)\|^2 \Big) \\
&= \frac{1}{2}\Big(\|\bar{u}_n - z\|^2 + \|x_n - z\|^2 - \|x_n - \bar{u}_n\|^2 \\
&\quad + 2r\langle x_n - \bar{u}_n, A x_n - A z \rangle - r^2\|A x_n - A z\|^2 \Big), \qquad (3.19)
\end{aligned}$$

it implies that

$$\|\bar{u}_n - z\| \leq \|x_n - z\|^2 - \|x_n - \bar{u}_n\|^2 + 2r\langle x_n - \bar{u}_n, A x_n - A z \rangle. \qquad (3.20)$$

From (3.13) and (3.20) , we have

$$\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n u_n + \gamma_n \bar{u}_n + \eta_n K x_n - z\|^2 \\
&\leq \alpha_n \|u_n - z\|^2 + \gamma_n \|\bar{u}_n - z\|^2 + \eta_n \|K x_n - z\|^2 \\
&\leq \|x_n - z\|^2 - \gamma_n \|x_n - \bar{u}_n\|^2 + 2r\gamma_n\langle x_n - \bar{u}_n, A x_n - A z \rangle.
\end{aligned}$$

It implies that

$$\begin{aligned}
\gamma_n \|x_n - \bar{u}_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 + 2r\gamma_n\langle x_n - \bar{u}_n, A x_n - A z \rangle \\
&\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\| \\
&\quad + 2r\gamma_n \|x_n - \bar{u}_n\|\|A x_n - A z\|. \qquad (3.21)
\end{aligned}$$

From (3.9), (3.18), condition (ii), $r \in (0, 2\alpha)$, we have

$$\lim_{n \to \infty} \|x_n - \bar{u}_n\| = 0. \tag{3.22}$$

Since

$$\eta_n \|Kx_n - x_n\| \le \|y_n - x_n\| + \alpha_n \|x_n - u_n\| + \gamma_n \|x_n - \overline{u}_n\|,$$

from (3.10), (3.15) and (3.22) and condition (ii), we have

$$\lim_{n \to \infty} \|Kx_n - x_n\| = 0.$$

Next, we will show that $\{x_n\}$, $\{w_n\}$ are Cauchy sequences.

Let $a \in (0, 1)$, by (3.7), there exists $N_0 \in \mathbb{N}$ such that

$$\|x_{n+1} - x_n\| < a^n, \forall n \ge N_0. \tag{3.23}$$

Thus, for any number $n, p \in \mathbb{N}$, we have

$$\|x_{n+p} - x_n\| \le \sum_{k=n}^{n+p-1} \|x_{k+1} - x_k\| \le \sum_{k=n}^{n+p-1} a^k \le \frac{a^n}{1-a}. \tag{3.24}$$

Since $a \in (0, 1)$, we have $\lim_{n \to \infty} a^n = 0$. By (3.24), we have $\{x_n\}$ is a Cauchy sequence in Hilbert space. Then, there exists $x^* \in C$ such that $\lim_{n \to \infty} x_n = x^*$.

Since $T : C \to CB(H)$ is $\mathcal{H}$-Lipschitz continuous with constant μ and (1.8), we have

$$\|w_n - w_{n+1}\| \le \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})) \le \left(1 + \frac{1}{n}\right)\mu\|x_{n+1} - x_n\|. \tag{3.25}$$

By (3.23), (3.25) and for any number $n, p \in \mathbb{N}$, $p > 0$, we have

$$\begin{aligned}
\|w_{n+p} - w_n\| &\le \Sigma_{k=n}^{n+p-1}\|w_{k+1} - w_k\| \\
&\le \Sigma_{k=n}^{n+p-1}\left(1 + \frac{1}{k}\right)\mu\|x_{k+1} - x_k\| \\
&\le \Sigma_{k=n}^{n+p-1}2\mu a^k \\
&\le 2\mu\frac{a^n}{1-a}. \tag{3.26}
\end{aligned}$$

Since $a \in (0, 1)$, we have $\lim_{n \to \infty} a^n = 0$. By (3.26), we have $\{w_n\}$ is a Cauchy sequence in Hilbert space. Then, there exists $w^* \in C$ such that $\lim_{n \to \infty} w_n = w^*$. Next, we show that $w^* \in T(x^*)$.

Since $w_n \in T(x_n)$, we have

$$\begin{aligned}
d(w_n, T(x^*)) &\le \max\left\{d(w_n, T(x^*)), \sup_{w \in T(x^*)} d(T(x_n), w)\right\} \\
&\le \max\left\{\sup_{z \in T(x_n)} d(z, T(x^*)), \sup_{w \in T(x^*)} d(T(x_n), w)\right\} \\
&= \mathcal{H}(T(x_n), T(x^*)). \tag{3.27}
\end{aligned}$$

It implies that

$$
\begin{aligned}
d(w^*, T(x^*)) &\leq \|w^* - w_n\| + d(w_n, T(x^*)) \\
&\leq \|w^* - w_n\| + \mathcal{H}(T(x_n), T(x^*)) \\
&\leq \|w^* - w_n\| + \mu \|x_n - x^*\|.
\end{aligned}
$$

By $\lim_{n \to \infty} x_n = x^*$ and $\lim_{n \to \infty} w_n = w^*$, we have $d(w^*, T(x^*)) = 0$. Since $T(x^*)$ is a closed set, we have $w^* \in T(x^*)$.

Next, we show that $\omega(x_n) \subset \mathbb{F}$.

Since $\{x_n\}$ is bounded, then $\omega(x_n) \neq \emptyset$. Let $q \in \omega(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to q. Since $\{x_n\}$ is a Cauchy sequence in Hilbert space, we have $x_n \to q$ as $n \to \infty$. Since $\lim_{n \to \infty} x_n = x^*$, we have $x^* = q$, it follows that $w^* \in T(q)$.

From (3.15) and $x_n \to q$ as $n \to \infty$, we have $u_n \to q$ as $n \to \infty$.

By $u_n = T_{r_n} x_n$, we have

$$
\Phi(w_n, u_n, u) + \varphi(u) - \varphi(u_n) + \frac{1}{r_n} \langle u_n - x_n, u - u_n \rangle \geq 0, \forall u \in C, r_n > 0.
$$

By (3.15), $(H1)$ and lower semicontinuity of φ, we have

$$
\Phi(w^*, q, u) + \varphi(u) - \varphi(q) \geq 0, \forall u \in C.
$$

Then, we have

$$
q \in (GEP)_s(\Phi_1, \varphi_1). \tag{3.28}
$$

From (3.22) and $x_n \to q$ as $n \to \infty$, we have $\bar{u}_n \to q$ as $n \to \infty$.

By $\bar{u}_n = \overline{T}_r(I - rA)x_n$, we have

$$
F(\bar{u}_n, \bar{u}) + \langle Ax_n, \bar{u} - \bar{u}_n \rangle + \frac{1}{r} \langle \bar{u} - \bar{u}_n, \bar{u}_n - x_n \rangle \geq 0, \forall \bar{u} \in C.
$$

From $(A2)$, we have

$$
\begin{aligned}
F(\bar{u}, \bar{u}_n) &\leq F(\bar{u}_n, \bar{u}) + F(\bar{u}, \bar{u}_n) + \langle Ax_n, \bar{u} - \bar{u}_n \rangle + \frac{1}{r} \langle \bar{u} - \bar{u}_n, \bar{u}_n - x_n \rangle \\
&\leq \langle Ax_n, \bar{u} - \bar{u}_n \rangle + \frac{1}{r} \langle \bar{u} - \bar{u}_n, \bar{u}_n - x_n \rangle \\
&= \langle Ax_n, \bar{u} - \bar{u}_n \rangle + \langle \bar{u} - \bar{u}_n, \frac{\bar{u}_n - x_n}{r} \rangle, \tag{3.29}
\end{aligned}
$$

Put $z_t = ty + (1-t)q$ for all $t \in (0, 1]$ and $y \in C$. Then we have $z_t \in C$.

So, from (3.29), we have

$$
F(z_t, \bar{u}_n) - \langle z_t - \bar{u}_n, Ax_n \rangle - \langle z_t - \bar{u}_n, \frac{\bar{u}_n - x_n}{r} \rangle \leq 0. \tag{3.30}
$$

By (3.30), we have

$$
\begin{aligned}
\langle z_t - \bar{u}_n, Az_t \rangle &\geq \langle z_t - \bar{u}_n, Az_t \rangle - \langle z_t - \bar{u}_n, Ax_n \rangle - \langle z_t - \bar{u}_n, \frac{\bar{u}_n - x_n}{r} \rangle \\
&\quad + F(z_t, \bar{u}_n) \\
&= \langle z_t - \bar{u}_n, Az_t - A\bar{u}_n \rangle + \langle z_t - \bar{u}_n, A\bar{u}_n \rangle - \langle z_t - \bar{u}_n, Ax_n \rangle \\
&\quad - \langle z_t - \bar{u}_n, \frac{\bar{u}_n - x_n}{r} \rangle + F(z_t, \bar{u}_n) \\
&= \langle z_t - \bar{u}_n, Az_t - A\bar{u}_n \rangle + \langle z_t - \bar{u}_n, A\bar{u}_n - Ax_n \rangle \\
&\quad - \langle z_t - \bar{u}_n, \frac{\bar{u}_n - x_n}{r} \rangle + F(z_t, \bar{u}_n).
\end{aligned} \tag{3.31}
$$

Since $\lim_{n \to \infty} \|\bar{u}_n - x_n\| = 0$, we have $\lim_{n \to \infty} \|A\bar{u}_n - Ax_n\| = 0$.

From monotone of A, we have $\langle z_t - \bar{u}_n, Az_t - A\bar{u}_n \rangle \geq 0$. From $\bar{u}_n \to q$ as $n \to \infty$ and $(A4)$, we have

$$
\langle z_t - q, Az_t \rangle \geq F(z_t, q). \tag{3.32}
$$

From $(A1), (A4)$ and (3.32), we have

$$
\begin{aligned}
0 = F(z_t, z_t) = F(z_t, ty - (1-t)q) &\leq tF(z_t, y) + (1-t)F(z_t, q) \\
&\leq tF(z_t, y) + (1-t)\langle z_t - q, Az_t \rangle \\
&= tF(z_t, y) + (1-t)t\langle y - q, Az_t \rangle.
\end{aligned}
$$

It implies that

$$
F(z_t, y) + (1-t)\langle y - q, Az_t \rangle \geq 0. \tag{3.33}
$$

Letting $t \to 0^+$ and (3.33), we have $0 \leq F(q, y) + \langle y - q, Aq \rangle$, for all $y \in C$. Then

$$
q \in EP(F, A). \tag{3.34}
$$

By Lemma 2.4, we have K is a quasi-noneapansive mapping and $F(K) = \bigcap_{i=1}^{N} F(T_i)$.

Since $x_{n_i} \rightharpoonup q$ as $i \to \infty$ and $\lim_{n \to \infty} \|Kx_n - x_n\| = 0$ and Lemma 2.5, we have

$$
q \in F(K) = \bigcap_{i=1}^{N} F(T_i) \tag{3.35}
$$

From (3.28), (3.34) and (3.35), we have $q \in \mathbb{F}$.

Hence $\omega(x_n) \subset \mathbb{F}$. By Lemma 2.6 and (3.6), it implies that $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$. This completes the proof. $\qquad \square$

The following corollary is a consequence which is applied by Theorem (3.1). Therefore, we omit the proof. In the case of $F \equiv 0$, then $EP(F, A)$ is reduced to $VI(C, A)$. So, we prove the next result as follows:

Corollary 3.2. *Let C be a nonempty bounded, closed and convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be lower semicontinuous and convex function. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ, respectively, $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $(H1) - (H3)$. Let $A : C \to H$ be an $\alpha-$inverse strongly monotone mapping, let $T_i, i = 1, 2, .., N$, be nonspreading mappings of C into itself with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N - 1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1 \in T(x_1)$, there exist sequences $\{x_n\}, \{u_n\}, \{\overline{u}_n\} \subseteq C$ and $\{w_n\} \in H$ generated by*

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n, u_n, u) + \varphi(u) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n \rangle \geq 0, \forall u \in C, r_n > 0, \\ y_n = \alpha_n u_n + \gamma_n P_C(I - rA)x_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \forall n \geq 1, \end{cases}$$

$$(3.36)$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{\eta_n\}$ are sequences in $[0, 1]$, $r, r_n \subset (0, 2\alpha)$, for every $n \in \mathbb{N}$ and suppose the following conditions hold:

(i) $\alpha_n + \gamma_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \gamma_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there exists $\lambda > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2,$$

$$(3.37)$$

for all $(r_1, r_2) \in \Theta \times \Theta$, $w_i \in T(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$.

4 Application

In this section, by using our main result, we obtain Theorem 4.1. Before we prove strong convergence theorem in this section, we consider the following standard constrained convex optimization problem as follows:

$$\text{find } x^* \in C, \text{ such that } f(x^*) = \min_{x \in C} f(x), \tag{4.1}$$

where $f : C \to \mathbb{R}$ is a convex, Fréchet differentiable function, C is a closed convex subset of H.

It is known that the optimization problem (4.1) is equivalent to the following variational inequality problem

$$\text{find } x^* \in C, \text{ such that } \langle v - x^*, \nabla f(x^*) \rangle \geq 0, \forall v \in C, \tag{4.2}$$

where $\nabla f : C \to C$ is the gradient of f.

It is also known that the optimality condition (4.2) is equivalent to the following fixed point equation

$$x^* = P_C(x^* - \mu \nabla f(x^*)), \tag{4.3}$$

where P_C is the metric projection onto C and $\mu > 0$ is a positive constant. The set of all solutions of (4.1) is denoted by Ω_f.

Next, we prove a result involving optimization problem as follows:

Theorem 4.1. *Let C be a nonempty bounded,closed and convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ, respectively, $\Phi : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying $(H1) - (H3)$. Let $f : C \to \mathbb{R}$ be a convex function with ∇f be an $\frac{1}{L_f}$-inverse strongly monotone mapping, where $L_f > 0$, let $T_i, i = 1, 2, .., N$, be nonspreading mappings of C into itself with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap \Omega_f \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N - 1$ and $0 < \lambda_N \leq 1$. Let K be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1 \in T(x_1)$, there exist sequences $\{x_n\}, \{u_n\}, \{\overline{u}_n\} \subseteq C$ and $\{w_n\} \in H$ generated by*

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n, u_n, u) + \varphi(u) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n \rangle \geq 0, \forall u \in C, r_n > 0, \\ y_n = \alpha_n u_n + \gamma_n P_C(I - r\nabla f)x_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases}$$

$$(4.4)$$

where $\{\alpha_n\}, \{\gamma_n\}$ and $\{\eta_n\}$ are sequences in $[0,1]$, $r, r_n \subset (0, \frac{2}{L_f})$, for every $n \in \mathbb{N}$ and suppose the following conditions hold:

(i) $\alpha_n + \gamma_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \gamma_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there exists $\lambda > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \tag{4.5}$$

for all $(r_1, r_2) \in \Theta \times \Theta$, $w_i \in T(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

Proof. Putting $A \equiv \nabla f$ and Corollary 3.2, we can conclude the desired conclusion. $\square$

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