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Approximation of Common Solutions for Proximal Split Feasibility Problems and Fixed Point Problems in Hilbert Spaces

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Abstract : In this paper, a new iterative algorithm is proposed for finding the minimum-norm solution of a proximal split minimization problem and fixed point problem of quasi-nonexpansive mappings in Hilbert spaces. Under suitable conditions, it is proved that the sequence generated by the proposed algorithm converges strongly to a common solution of the two above described problems. The iterative algorithm are proposed in such a way that the selection of the step-sizes does not need any prior information about the operator norm.

Keywords : Fixed point problem, proximal split feasibility problems, quasi-nonexpansive mapping.

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1 Introduction

Throughout this article, let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$be two proper and lower semicontinuous convex functions and $A : H_1 \to H_2$ be a bounded linear operator. Now, we will introduce one of the famous problems in many fields of pure and applied sciences, that is the split feasibility problem (SFP) was first introduced by Censor and Elfving [18] in 1994: Find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ be a bounded linear operator. Split feasibility problem can be applied to medical image reconstruction, especially intensity-modulated therapy (see, [2]). In the past decade, many researchers have increasingly stuided the split feasibility problem, see, for instance [3, 4, 5, 6, 7, 8, 9, 10], and the references therein.

In this paper, we study more general problem which is the following: find a solution $z \in H_1$ such that

$$\min_{z \in H_1} \{f(x) + g_\lambda(Ax)\}, \tag{1.2}$$

where $g_\lambda(y) := \min_{u \in H_2}\{g(u) + \frac{1}{2\lambda}\|u - y\|^2\}$ is the Moreau-Yosida approximate of the function f of parameter λ, also called proximal operator of f of order λ and below denoted by $\operatorname{prox}_{\lambda g}(x)$. If $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ ortherwise] and $g = \delta_Q$ are indicator functions of nonempty, closed, and convex sets C and Q of H_1 and H_2, respectively. Then problem (1.2) reduces to

$$\min_{x \in H_1} \{\delta_C(x) + (\delta_Q)_\lambda(Ax)\} \Leftrightarrow \min_{x \in H_1} \{\frac{1}{2\lambda}\|(I - P_Q)(Ax)\|^2\}$$

which is equivalent to **SFP** when $C \cap A^{-1}(Q)$.

In the case $\operatorname{argmin} f \cap A^{-1}(\operatorname{argmin} g) \neq \emptyset$, the split minimization problem (**SMP**) is to find a minimizer z of f such that Az minimizes g ; that is,

$$z \in \operatorname{argmin} f \text{ such that } Az \in \operatorname{argmin} g, \tag{1.3}$$

where $\operatorname{argmin} f := \{\bar{x} \in H_1 : f(\bar{x}) \leq f(x) \text{ for all } x \in H_1\}$ and $\operatorname{argmin} g := \{\bar{y} \in H_2 : g(\bar{y}) \leq g(y) \text{ for all } y \in H_2\}$. The solution set of the problem (1.3) is denote by Γ.

Recall that the proximal operator $\operatorname{prox}_{\lambda g} : H \to H$ is defined by

$$\operatorname{prox}_{\lambda g}(x) := \operatorname*{argmin}_{u \in H}\{g(u) + \frac{1}{2\lambda}\|u - x\|^2\}. \tag{1.4}$$

Moreover, the proximity operator of f is firmly nonexpansive, namely,

$$\langle \operatorname{prox}_{\lambda g}(x) - \operatorname{prox}_{\lambda g}(y), x - y\rangle \geq \|\operatorname{prox}_{\lambda g}(x) - \operatorname{prox}_{\lambda g}(y)\|^2. \tag{1.5}$$

for all $x, y \in H$, which is equivalent to

$$\| \operatorname{prox}_{\lambda g}(x) - \operatorname{prox}_{\lambda g}(y)\|^2 \leq \|x-y\|^2 - \|(I-\operatorname{prox}_{\lambda g})(x) - (I-\operatorname{prox}_{\lambda g})(y)\|^2. \quad (1.6)$$

for all $x, y \in H$. For general information on proximal operator, see the research paper by Combettes and Pesquet [23].
In 2014, Moudafi and Thakur [24] introduced the split proximal algorithm for estimating the stepsizes which do not need prior knowledge of the operator norms for solving **SMP** (1.3) as follows.

$$x_{n+1} = \operatorname{prox}_{\lambda\gamma_n f}(x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n)\forall n \geq 1, \quad (1.7)$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$, $h(x) := \frac{1}{2}\|(I-\operatorname{prox}_{\lambda g})Ax\|^2$, $l(x) := \frac{1}{2}\|(I - \operatorname{prox}_{\lambda\gamma_n f})x\|^2$ and $\theta(x) := \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$. Thay also proved the weak convergence theorem of the sequence generated by algorithm (1.7) to a solution of **SMP** (1.3).

In 2014, Yao *et al.* [25] introduced the regularized algorithm for solving the split proximal algorithm as follows:

$$x_{n+1} = \operatorname{prox}_{\lambda\gamma_n f}(\alpha_n u + (1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n), \forall n \geq 1, \quad (1.8)$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. Then, they proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter α_n and γ_n.

Recently, Shehu and Ogbuisi [12]introduced the following algorithm for solving split proximal algorithms and fixed point problems for k-strictly pseudocontractive mappings in Hilbert spaces:

$$\begin{cases} u_n = (1-\alpha_n)x_n, \\ y_n = \operatorname{prox}_{\lambda\gamma_n f}(u_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Au_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \forall n \in \mathbb{N}, \end{cases} \quad (1.9)$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. They also showed that, under certain assumptions imposed on the parameters, the sequence $\{x_n\}$ generated by (1.9) converges strongly to $x^* \in Fix(S) \cap \Gamma$.

Very recently, Abbas *et al.* [16] studied the following algorithm for finding the minimum-norm solution of split proximal algorithm, that is,

$$x_{n+1} = \operatorname{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n)\forall n \geq 1, \quad (1.10)$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. Using the split proximal algorithm 1.10, they also proved a strong convergence theorem of the sequences generated by the proposed algorithms under some appropriate conditions.

After we have studied research related to split proximal algorithm and fixed point problem, we obtain the following question.
Question Is it possible to obtain a strong convergence theorem for finding the minimum-norm solution of a proximal split minimization problem and the set of common fixed points of a family of mappings in Hilbert spaces ? Such as a countable family of quasi-nonexpansive mappings.

In this paper, we give the answer for the mentioned questions and introduce a new iterative algorithm for finding the minimum-norm solution of a proximal split minimization problem and fixed point problem of quasi-nonexpansive mappings in Hilbert spaces. Under suitable conditions, it is proved that the sequence generated by the proposed algorithm converges strongly to a common solution of the two above described problems. The iterative algorithm are proposed in such a way that the selection of the step-sizes does not need any prior information about the operator norm.

2 Preliminaries

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $Fix(T) := \{x \in C : Tx = x\}$. A point $z \in H$ is called a *mimimum norm* fixed point of T if and only if $z \in Fix(T)$ and $\|z\| = \min\{\|x\| : x \in Fix(T)\}$.

Definition 2.1. *Let $T : C \to C$ be a nonlinear mapping, then*

(i) T is said to be nonexpansive *if*

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C,$$

(ii) T is said to be quasi-nonexpansive *if*

$$\|Tx - p\| \leq \|x - p\|, \forall x \in C \text{ and } \forall p \in Fix(T),$$

Lemma 2.2. *[28] Let C be a nonempty closed convex subset of a real Hilbert space H.For every $i = 1, 2, 3, .., N$, let $T_i : H_1 \to H_1$ be a finte fammily of quasi-nonexpansive mapping such that $\bigcap_{i=1}^{N} Fix(T_i) \neq 0$ and $I - T_i$ are demiclosed at zero. Put $T = \sum_{i=1}^{N} a_i T_i$, where $0 < a_i \leq 1$, for every $i = 1, 2, ..., N$ with $\sum_{i=1}^{N} a_i = 1$. Then the following hold:*

1. *$Fix(T) = \bigcap_{i=1}^{N} Fix(T_i)$;*
2. *T is a quasi-nonexpansive mapping;*

3. T is demiclosed at zero.

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.3 ([21]). *Given $x \in H_1$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Lemma 2.4 ([19]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n) s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\displaystyle\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\displaystyle\lim_{n\to\infty} s_n = 0$.

Lemma 2.5. ([22]) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq ...$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3 Main Theorem

In this section, we prove a strong convergence theorem for for finding the minimum-norm solution of a proximal split minimization problem and fixed point problem of quasi-nonexpansive mappings in Hilbert spaces. Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$be two proper and lower semicontinuous convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, 3, .., N$, let $T_i : H_1 \to H_1$ be a finite family of quasi-nonexpansive mapping such that $\bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ and $I - T_i$ are demiclosed at zero.

Now, we introduce the following algorithm for finding the solution set of $\Gamma \cap \bigcap_{i=1}^{N} Fix(T_i)$.

Algorithm 3.1

Step 1: Choose an initial point $x_1 \in H_1$.

Step 2: Assume that x_n has been constructed.
Set $\theta(x_n) := \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$ where $h(x_n) := \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g})Ax_n\|^2$ and $l(x_n) := \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f})x_n\|^2$ with $\theta(x_n) \neq 0$.
We compute x_{n+1} in the following iterative scheme:

$$\begin{cases} y_n = \mathrm{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n) \\ x_{n+1} = \beta_n y_n + (1-\beta_n)\sum_{i=1}^{N} a_i T_i y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, and $0 \le a_i \le 1$, for every $i = 1, 2, ..., N$ with $\displaystyle\sum_{i=1}^{N} a_i = 1$.

Using algorithm (3.1), we prove a strong convergence theorem for approximation of solutions of problem (1.3) and the set of fixed points of quasi-nonexpansive mappings as follows:

Theorem 3.1. *Suppose that $\Omega := \Gamma \cap \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. If the parameters satisfy the following conditions:*

(C1) $\displaystyle\lim_{n\to\infty} \alpha_n = 0$ *and* $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*;*

(C3) $\varepsilon \le \rho_n \le \dfrac{4(1-\alpha_n)h(x_n)}{h(x_n) + l(x_n)} - \varepsilon$ *for some* $\varepsilon > 0$ *and for any* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to a solution z which is also a minimum norm solution of Ω. In other words, $z = P_\Omega(0)$.

Proof. Let $z = P_\Omega(0)$. Then $z = \mathrm{prox}_{\lambda\gamma_n f} z$ and $Az = \mathrm{prox}_{\lambda g} z$. Note that $\nabla h(x_n) = A^*(I - \mathrm{prox}_{\lambda g})Ax_n$, $\nabla l(x_n) = (I - \mathrm{prox}_{\lambda\gamma_n f})x_n$

Since $\mathrm{prox}_{\lambda g}$ is firmly nonexpansive, we have that $I - \mathrm{prox}_{\lambda g}$ is also firmly nonexpansive. Hence

$$\begin{aligned} \langle A^*(I - \mathrm{prox}_{\lambda g})Ax_n, x_n - z\rangle &= \langle (I - \mathrm{prox}_{\lambda g})Ax_n, Ax_n - Az\rangle \\ &= \langle (I - \mathrm{prox}_{\lambda g})Ax_n, Ax_n - Az\rangle \\ &= \langle (I - \mathrm{prox}_{\lambda g})Ax_n - (I - \mathrm{prox}_{\lambda g})Az, Ax_n - Az\rangle \\ &\ge \|(I - \mathrm{prox}_{\lambda g})Ax_n\|^2 = 2h(x_n). \end{aligned} \tag{3.2}$$

From the deffinition of y_n and the nonexpansivity of $\mathrm{prox}_{\lambda\gamma_n f}$, we have

$$\begin{aligned}
\|y_n - z\| &= \| \mathrm{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I-\mathrm{prox}_{\lambda g})Ax_n) - z\| \\
&\leq \|(1-\alpha_n)x_n - \gamma_n A^*(I-\mathrm{prox}_{\lambda g})Ax_n - z\| \\
&= \|\alpha_n(z) + (1-\alpha_n)\left(x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I-\mathrm{prox}_{\lambda g})Ax_n - z\right)\| \\
&\leq \alpha_n\|z\| + (1-\alpha_n)\left\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I-\mathrm{prox}_{\lambda g})Ax_n - z\right\|. \qquad (3.3)
\end{aligned}$$

Since $\nabla h(x_n) = A^*(I-\mathrm{prox}_{\lambda g})Ax_n$, $\nabla l(x_n) = (I-\mathrm{prox}_{\lambda\gamma_n f})x_n$ and (3.2), we have

$$\begin{aligned}
&\left\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I-\mathrm{prox}_{\lambda g})Ax_n - z\right\|^2 \\
&= \|x_n - z\|^2 + \frac{\gamma_n^2}{(1-\alpha_n)^2}\|A^*(I-\mathrm{prox}_{\lambda g})Ax_n - z\|^2 \\
&\quad - 2\frac{\gamma_n}{(1-\alpha_n)}\langle A^*(I-\mathrm{prox}_{\lambda g})Ax_n, x_n - z\rangle \\
&= \|x_n - z\|^2 + \frac{\gamma_n^2}{(1-\alpha_n)^2}\|\nabla h(x_n)\|^2 - 2\frac{\gamma_n}{(1-\alpha_n)}\langle \nabla h(x_n), x_n - z\rangle \\
&\leq \|x_n - z\|^2 + \frac{\gamma_n^2}{(1-\alpha_n)^2}\|\nabla h(x_n)\|^2 - 4\frac{\gamma_n}{(1-\alpha_n)}h(x_n) \\
&= \|x_n - z\|^2 + \rho_n^2\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)^2\theta^4(x_n)}\|\nabla h(x_n)\|^2 - 4\rho_n\frac{(h(x_n)+l(x_n))}{(1-\alpha_n)\theta^2(x_n)}h(x_n) \\
&\leq \|x_n - z\|^2 + \rho_n^2\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)^2\theta^4(x_n)} - 4\rho_n\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\frac{h(x_n)}{(h(x_n)+l(x_n))} \\
&= \|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\right). \qquad (3.4)
\end{aligned}$$

Without loss of generality, by condition (C3), we can assume that $\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n} \geq 0$ for all $n \geq 1$. From (3.3), (3.4), we have

$$\begin{aligned}
\|y_n - z\| &\leq \alpha_n\|z\| + (1-\alpha_n)\left\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I-\mathrm{prox}_{\lambda g})Ax_n - z\right\| \\
&\leq \alpha_n\|z\| + (1-\alpha_n)\|x_n - z\|. \qquad (3.5)
\end{aligned}$$

Put $T = \sum_{i=1}^N a_iT_i$, where $0 \leq a_i \leq 1$, for every $i = 1, 2, ..., N$ with $\sum_{i=1}^N a_i = 1$.

From Lemma 2.2, we have T is a quasi-nonexpansive mapping. It follows that

$$\begin{aligned}\|x_{n+1}-z\| &= \|\beta_n y_n + (1-\beta_n)Ty_n - z\| \\ &\le \beta_n\|y_n - z\| + (1-\beta_n)\|Ty_n - z\| \\ &\le \beta_n\|y_n - z\| + (1-\beta_n)\|y_n - z\| \\ &= \|y_n - z\| \\ &\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|z\| \\ &\le \max\{\|x_n - z\|, \|z\|\}.\end{aligned}$$

By mathematical induction, we have

$$\|x_n - z\| \le \max\{\|x_1 - z\|, \|z\|\}, \forall n \in \mathbb{N}.$$

It implies that $\{x_n\}$ is bounded and so are , $\{T(y_n)\}$.

From the definition of y_n, we have

$$\begin{aligned}\|y_n - z\|^2 &= \|\operatorname{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n) - z\|^2 \\ &\le \|(1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\|^2, \\ &= \|\alpha_n(z) + (1-\alpha_n)\left(x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\right)\|^2 \\ &\le \alpha_n\|z\|^2 + (1-\alpha_n)\left\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\right\|^2 \\ &\le \alpha_n\|z\|^2 + (1-\alpha_n)\left(\|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\right)\right) \\ &= \alpha_n\|z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right).\end{aligned} \tag{3.6}$$

It follows from (3.6), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\beta_n y_n + (1-\beta_n)Ty_n - z\|^2 \\ &\le \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \\ &\le \|y_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \\ &\le \alpha_n\|z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \\ &\le \alpha_n\|z\|^2 + \|x_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2.\end{aligned}$$

It implies that

$$\beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \le \alpha_n\|z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2. \tag{3.7}$$

From the definition of x_n and (3.6), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\beta_n y_n + (1-\beta_n)Ty_n - z\|^2 \\
&\le \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 \\
&\le \|y_n - z\|^2 \\
&\le \alpha_n\|z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right) \\
&\le \alpha_n\|z\|^2 + \|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right).
\end{aligned}$$

It implies that

$$\rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right) \le \alpha_n\|z\|^2 + \|x_n - z\|^2 - \|x_{n+1}-z\|^2. \tag{3.8}$$

Now we divide the rest of the proof into two cases.

CASE 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=1}^{\infty}$ is non-increasing. Then $\{\|x_n - z\|\}_{n=1}^{\infty}$ coverges and $\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0$ as $n \to \infty$. From (3.8), the condition (C1) and (C3), we obtain

$$\rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right) \to 0 \text{ as } n \to \infty.$$

Then, we have

$$\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)} \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Observe that $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ is bounded (see [16]). It follows that

$$\lim_{n\to\infty}((h(x_n)+l(x_n))^2) = 0.$$

It implies that

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} l(x_n) = 0.$$

Next, we will show that $\limsup_{n\to\infty}\langle -z, x_n - z\rangle \le 0$, where $z = P_\omega(0)$. To show this,since $\{x_n\}$ is bounded, there exits a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying $x_{n_j} \rightharpoonup q$ and

$$\limsup_{n\to\infty}\langle -z, x_n - z\rangle = \lim_{j\to\infty}\langle -z, x_{n_j} - z\rangle.$$

By the lower semicontinuity of h, we have

$$0 \le h(q) \le \liminf_{j\to\infty} h(x_{n_j}) = \lim_{n\to\infty} h(x_n) = 0.$$

So, $h(q) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Aq\|^2 = 0$. Therefore, Aq is a fixed point of the proximal mapping of g or equivalently $0 \in \partial f(Aq)$. In other words, Aq is a minimizer of g . Similarly, from the lower semicontinuity of l, we obtain

$$0 \leq l(q) \leq \liminf_{j\to\infty} l(x_{n_j}) = \lim_{n\to\infty} l(x_n) = 0.$$

So, $l(q) = \frac{1}{2}\|(I - \text{prox}_{\lambda\gamma_n f})q\|^2 = 0$. Therefore, q is a fixed point of the proximal mapping of f or equivalently $0 \in \partial g(q)$. In other words, q is a minimizer of f. Hence $q \in \Gamma$.
From the definition of γ_n, we have

$$0 < \gamma_n < 4\frac{h(x_n) + l(x_n)}{\theta^2(x_n)} \to 0 \text{ as } n \to \infty$$

implies that $\gamma_n \to 0$ as $n \to \infty$.
Next, we will show that $q \in Fix(T) = \bigcap_{i=1}^N Fix(T_i)$. From (3.7) and the condition (C1) (C2), we have

$$\|y_n - Ty_n\| \to 0 \text{ as } n \to \infty. \tag{3.10}$$

For each $n \geq 1$, let $u_n := (1 - \alpha_n)x_n$. Then,

$$\begin{aligned}\|u_n - x_n\| &= \|(1 - \alpha_n)x_n - x_n\| \\ &= \alpha_n\|x_n\|.\end{aligned}$$

From the condition (C1), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.11}$$

Observe that

$$\|u_n - \text{prox}_{\lambda\gamma_n f} x_n\| \leq \|u_n - x_n\| + \|(I - \text{prox}_{\lambda\gamma_n f})x_n\|.$$

From $\lim_{n\to\infty} l(x_n) = \lim_{n\to\infty} \frac{1}{2}\|(I - \text{prox}_{\lambda\gamma_n f})x_n\|^2 = 0$ and (3.11), we have

$$\lim_{n\to\infty} \|u_n - \text{prox}_{\lambda\gamma_n f} x_n\| = 0. \tag{3.12}$$

By the nonexpansiveness of $\text{prox}_{\lambda\gamma_n f}$, we have

$$\begin{aligned}\|y_n - \text{prox}_{\lambda\gamma_n f} x_n\| &= \|\text{prox}_{\lambda\gamma_n f}(u_n - \gamma_n A^*(I - \text{prox}_{\lambda g})Ax_n) - \text{prox}_{\lambda\gamma_n f} x_n\| \\ &\leq \|u_n - \gamma_n A^*(I - \text{prox}_{\lambda g})Ax_n - x_n\| \\ &\leq \|u_n - x_n\| + \gamma_n\|A^*(I - \text{prox}_{\lambda g})Ax_n\|.\end{aligned}$$

From (3.12) and $\gamma_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|y_n - \text{prox}_{\lambda\gamma_n f} x_n\| = 0. \tag{3.13}$$

We observe that

$$\|y_n - u_n\| \le \|y_n - \operatorname{prox}_{\lambda\gamma_n f} x_n\| + \|u_n - \operatorname{prox}_{\lambda\gamma_n f} x_n\|.$$

From (3.12) and (3.13), we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.14}$$

Also, observe that $\|y_n - x_n\| \le \|y_n - u_n\| + \|u_n - x_n\|$ and from (3.12) and (3.13), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.15}$$

Using $x_{n_j} \rightharpoonup q \in H_1$ and (3.15), we obtain $y_{n_j} \rightharpoonup q \in H_1$. Since $y_{n_j} \rightharpoonup q \in H_1$, $\|y_n - Ty_n\| \to 0$ as $n \to \infty$ and Lemma 2.2, we have $q \in Fix(T) = \bigcap_{i=1}^N Fix(T_i)$. Hence $q \in \Omega = \bigcap_{i=1}^N Fix(T_i) \cap \Gamma$. Since $x_{n_j} \rightharpoonup q$ as $j \to \infty$ and $q \in \Omega$. Lemma 2.3, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle -z, x_n - z\rangle &= \lim_{j\to\infty} \left\langle -z, x_{n_j} - z\right\rangle \\ &= \langle -z, q - z\rangle \\ &\le 0. \end{aligned} \tag{3.16}$$

Now, from (3.1)and (3.4) , we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 \\ &\le \beta_n\|y_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 \\ &\le \|y_n - z\|^2 \\ &\le \|(1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\|^2 \\ &= \|(1-\alpha_n)\left(x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\right) + \alpha_n z\|^2 \\ &= (1-\alpha_n)^2\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\|^2 + \alpha_n^2\|z\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z, -z\rangle \\ &\le (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n^2\|z\|^2 + 2\alpha_n(1-\alpha_n)\langle x_n - z, -z\rangle \\ &\quad - 2\alpha_n\gamma_n\langle A^*(I - \operatorname{prox}_{\lambda g})Ax, -z\rangle \\ &= (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n^2\|z\|^2 + 2\alpha_n(1-\alpha_n)\langle x_n - z, -z\rangle \\ &\quad + 2\alpha_n\gamma_n\langle \nabla h(x_n), z\rangle \\ &\le (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\big(\alpha_n\|z\|^2 + 2(1-\alpha_n)\langle x_n - z, -z\rangle \\ &\quad + 2\gamma_n\|\nabla h(x_n)\|\|z\|\big). \end{aligned} \tag{3.17}$$

Since $\nabla h(x_n)$ is Lipschitz continuous with Lipschitzian constant $\|A\|^2$and $\nabla l(x_n)$ is nonexpansive, $\nabla h(x_n)$, $\nabla l(x_n)$, and $\theta^2(x_n)$ are bounded. From the condition

(C1), (3.16), (3.17) and Lemma 2.4, we can conclude that the sequence $\{x_n\}$ converges strongly to z.

CASE 2. Assume that $\{\|x_n - z\|\}$ is not monotonically decreasing sequence. Then there exists a subsequence n_k of n such that $\|x_{n_k} - \bar{x}\| < \|x_{n_k+1} - \bar{x}\|$ for all $k \in \mathbb{N}$. Now we define a positive interger sequence $\tau(n)$ by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \|x_{n_k} - \bar{x}\| < \|x_{n_k+1} - \bar{x}\|\}.$$

for all $n \geq n_0$ (for some n_0 large enough). By lemma 2.5, we have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\tau(n)} - \bar{x}\|^2 - \|x_{\tau(n)+1} - \bar{x}\|^2 \leq 0, \forall n \geq n_0.$$

By continuing in the same direction as in CASE 1, we can show that

$$\rho_{\tau(n)} \left(\frac{4h(x_{\tau(n)})}{(h(x_{\tau(n)}) + l(x_{\tau(n)}))} - \frac{\rho_{\tau(n)}}{1 - \alpha_{\tau(n)}} \right) \left(\frac{(h(x_{\tau(n)}) + l(x_{\tau(n)}))^2}{\theta^2(x_{\tau(n)})} \right) \to 0 \text{ as } n \to \infty.$$

Hence, we have

$$\frac{(h(x_{\tau(n)}) + l(x_{\tau(n)}))^2}{\theta^2(x_{\tau(n)})} \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Consequently, we have

$$\lim_{n\to\infty} ((h(x_{\tau(n)}) + l(x_{\tau(n)}))^2) = 0.$$

It implies that

$$\lim_{n\to\infty} h(x_{\tau(n)}) = \lim_{n\to\infty} l(x_{\tau(n)}) = 0.$$

Moreover, By continuing in the same direction as in Case 1, we can prove that

$$\limsup_{n\to\infty} \left\langle -z, x_{\tau(n)} - z \right\rangle \leq 0.$$

From (3.17), we have

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - z\|^2 - \|x_{\tau(n)} - z\|^2 \\ &\leq (1 - \alpha_{\tau(n)})\|x_{\tau(n)} - z\|^2 + \alpha_{\tau(n)}\rho_{\tau(n)} - \|x_{\tau(n)} - z\|^2 \\ &= \alpha_{\tau(n)}(\rho_{\tau(n)} - \|x_{\tau(n)} - z\|^2). \end{aligned}$$

It follows that

$$\|x_{\tau(n)} - z\|^2 \leq \rho_{\tau(n)},$$

where $\rho_{\tau(n)} = \alpha_{\tau(n)}\|z\|^2 + 2(1 - \alpha_{\tau(n)})\langle x_{\tau(n)} - z, -z\rangle + 2\gamma_{\tau(n)}\|\nabla h(x_{\tau(n)})\|\|z\|$. By using Lemma 2.4, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0.$$

It follows from Lemma 2.5 that

$$0 \leq \|x_{\tau(n)} - \bar{x}\| \leq \|x_{\tau(n)+1} - \bar{x}\| \to 0$$

as $n \to \infty$. Hence $\{x_n\}$ converges strongly to z. This completes the proof.

□

As a direct proof of Theorem 3.1, we obtain the following results.

When $f = \delta_C$ and $g = \delta_Q$ are indicator functions of nonempty, closed, and convex sets C and Q of H_1 and H_2, respectively, then SMP (1.3) reduces to the split feasibility problem (1.1). In this case, we obtain the following results.

Algorithm 3.2

Step 1: Choose an initial point $x_1 \in H_1$.

Step 2: Assume that x_n has been constructed.
Set $h(x_n) := \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ with $\|\nabla h(x_n\| \neq 0$. We compute x_{n+1} in the following iterative scheme:

$$\begin{cases} y_n = P_C((1-\alpha_n)x_n - \gamma_n A^*(I - P_Q)Ax_n) \\ x_{n+1} = \beta_n y_n + (1-\beta_n)\displaystyle\sum_{i=1}^{N} a_i T_i y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.19}$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n)}{\|\nabla h(x_n\|^2}$ with $0 < \rho_n < 4$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, and $0 \leq a_i \leq 1$, for every $i = 1, 2, ..., N$ with $\displaystyle\sum_{i=1}^{N} a_i = 1$.

Using algorithm 3.2, we prove a strong convergence theorem for approximation of solutions of problem (1.1) and the set of fixed points of quasi-nonexpansive mappings as follows:

Corollary 3.1. *Suppose that $\Omega := \Psi \cap \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. If the parameters satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(C3) $\varepsilon \leq \rho_n \leq 4(1-\alpha_n) - \varepsilon$ *for some* $\varepsilon > 0$.

Then the sequence $\{x_n\}$ converges strongly to a solution z which is also a minimum norm solution of Ω. In other words, $z = P_\Omega(0)$.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semicontinuous convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $T : H_1 \to H_1$ be a quasi-nonexpansive mapping such that $Fix(T) \neq \emptyset$ and $I - T$ are demiclosed at zero. Suppose that $\Omega := \Gamma \cap Fix(T) \neq \emptyset$. Set $\theta(x) := \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$ where $h(x) := \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g})Ax\|^2$ and $l(x) := \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f})x\|^2$ with $\theta(x) \neq 0$ for each $n \geq 1$. For given $x_1 \in H_1$ and let $\{x_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{cases} y_n = \mathrm{prox}_{\lambda \gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n) \\ x_{n+1} = \beta_n y_n + (1-\beta_n)Ty_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.20}$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$, and $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$. If the parameters satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(C3) $\varepsilon \leq \rho_n \leq \dfrac{4(1-\alpha_n)h(x_n)}{h(x_n) + l(x_n)} - \varepsilon$ *for some* $\varepsilon > 0$*.*

Then the sequence $\{x_n\}$ converges strongly to a solution z which is also a minimum norm solution of Ω. In other words, $z = P_\Omega(0)$.

Proof. Take $T = T_i$ for all $i = 1, 2, 3, ..., N$ in Theorem 3.1. So, from Theorem 3.1, we obtain the desired result. □

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