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Applications to Integral Equations with Coupled Fixed Point Theorems in A_b-Metric Space

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Abstract : In this paper, we establish some results on the existence and uniqueness of coupled common fixed point theorems in partially ordered A_b-metric spaces. Examples have been provided to justify the relevance of the results obtained through the analysis of extant theorem. Further, we also find application to integral equations via fixed point theorems in A_b-metric spaces. Our results generalize and extend the results of Deepak Singh et al.[6].

Keywords : Coupled fixed point, Mixed weakly monotone property, A_b-metric space, Integral equation.
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1 Introduction

The study of fixed point theory is an offshoot of non-linear function analysis. However its study began almost a century ago in the field of algebraic topology.

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Fixed point theorems find applications in proving the existence and uniqueness of the solutions of certain differential and integral equations that arise in physical, engineering and other optimization problems. In the study of fixed point theory, some of the generalizations of metric space are 2-metric space, D-metric space, D^*-metric space, G-metric space, S-metric space, Rectangular metric or metric-like space, Partial metric space, Cone metric space. In 1989, I.A.Bakhtin [3] introduced the concept of b-metric space. Consequent upon the introduction of b-metric space, many generalizations of metric spaces came into existence. In 2015, M.Abbas et al. [1] introduced the concept of n-tuple metric space and studied its topological properties. M.Ughade et al. [17] introduced the notion of A_b-metric spaces as a generalized form of n-tuple metric space. Subsequently N.Mlaiki et al. [13] obtained unique coupled common fixed point theorems in partially ordered A_b-metric spaces. The Coupled fixed point theorems in various metric spaces developed by many mathematicians (see [[2], [10], [11], [12], [15], [16]] and others). Our results extend some of these results for two self maps in A_b-metric space.

The aim of this paper is to extend the results of Deepak Singh et al. [6] for a unique Coupled fixed point theorem and to generalize the notion of mixed weakly monotone property.

In this paper, we use the notion of a mixed weakly monotone pair of maps to state a coupled common fixed point theorem on partially ordered A_b-metric space. We prove some unique coupled common fixed point theorems in partially ordered A_b-metric space and also provide example to support our results.

First we recall some notions, lemmas and examples which will be useful to prove our results.

2 Preliminaries

Definition 2.1. *[1] Let X be a non empty set and $n(\geq 2)$ be a positive integer. A function $A : X^n \to [0, \infty)$ is called an A-metric on X, if for any x_i, $a \in X.i = 1, 2, ...n$, the following conditions hold.*
(i) $A(x_1, x_2, ..., x_{n-1}, x_n) \geq 0$,
(ii) $A(x_1, x_2, ..., x_{n-1}, x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_{n-1} = x_n$,

$$(iii)\ A(x_1, x_2, ...x_{n-1}, x_n) \leq [A(x_1, x_1, ..., x_{1_{(n-1)}}, a) + A(x_2, x_2, ..., x_{2_{(n-1)}}, a) + ... + A(x_{n-1}, x_{n-1}, ..., x_{n-1_{(n-1)}}, a) + A(x_n, x_n, ..., x_{n_{(n-1)}}, a)]$$

The pair (X,A) is called an A-metric space.

Definition 2.2. *[8] Let X be a non empty set. A b-metric on X is a function $d : X^2 \to [0, \infty)$ such that the following conditions hold for all $x, y, z \in X$.*
(i) $d(x, y) = 0 \iff x = y$,
(ii) $d(x, y) = d(y, x)$,

(iii) there exists $s \geq 1$, such that $d(x,z) \leq s[d(x,y)+d(y,z)]$.
The pair (X,d) is called a b-metric space.

Definition 2.3. *[17] Let X be a non empty set and $n \geq 2$. Suppose $b \geq 1$ is a real number. A function $A_b : X^n \to [0,\infty)$ is called an A_b -metric on X, if for any $x_i, a \in X$, $i = 1, 2......n$, the following conditions hold.*
(i) $A_b(x_1, x_2, ..., x_{n-1}, x_n) \geq 0$,
(ii) $A_b(x_1, x_2, ..., x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = ... = x_{n-1} = x_n$,

$$
\begin{aligned}
(iii)\ A_b(x_1, x_2, ..., x_{n-1}, x_n) \leq b[&A_b(x_1, x_1, ..., x_{1_{(n-1)}}, a)\\
&+ A_b(x_2, x_2, ..., x_{2_{(n-1)}}, a)\\
&+ .... + A_b(x_{n-1}, x_{n-1}, ..., x_{n-1_{(n-1)}}, a)\\
&+ A_b(x_n, x_n, ..., x_{n_{(n-1)}}, a)].
\end{aligned}
$$

The pair (X, A_b) is called an A_b-metric space.

Note: In practice we write A for A_b when there is no confusion.

Example 2.4. *[17] Let $X = [1,\infty)$ and $n \geq 2$. Define $A_b : X^n \to [1,\infty)$ by $A_b(x_1, x_2, ..., x_{n-1}, x_n) = \sum\limits_{i=1}^{n} \sum\limits_{i<j} |x_i - x_j|^2$, for all $x_i \in X$, $i = 1, 2...........n$.*
Then (X, A_b) is an A_b-metric space with b=2.

Lemma 2.1. *[17] Let (X,A) be A_b metric space, so that $A : X^n \to [0,\infty)$ for some $n \geq 2$. Then $A(\underbrace{x, x, ...x}_{(n-1)times}, y) \leq bA(\underbrace{y, y, ...y}_{(n-1)times}, x)$, for all x,y $\in X$*

Lemma 2.2. *[17] Let (X,A) be A_b metric space, so that $A : X^n \to [0,\infty)$ for some $n \geq 2$.*
Then $A(\underbrace{x, x, ...x}_{(n-1)times}, z) \leq (n-1)bA(\underbrace{x, x, ...x}_{(n-1)times}, y) + b^2A(\underbrace{y, y, ...y}_{(n-1)times}, z)$, for all x,y,z $\in X$.

Lemma 2.3. *[17] Let (X,A) be A_b metric space. Then (X^2, D_A) is A_b-metric space on $X \times X$ with D_A defined by*
$D_A((x_1, y_1), (x_2, y_2), ...(x_n, y_n)) = A(x_1, x_2, ...x_n) + A(y_1, y_2, ...y_n)$, for all $x_i, y_i \in X$, $i, j = 1, 2, ...n$.

Note: We write D for D_A, when there is no confusion.

Definition 2.5. *Let (X,A) be A_b-metric space. A sequence $\{x_k\}$ in X is said to converge to a point $x \in X$, if $A(\underbrace{x_k, x_k, ...x_k}_{(n-1)times}, x) \to 0$ as $k \to \infty$. That is, to each $\varepsilon \geq 0$ there exist $N \in \mathbb{N}$ such that for all $k \geq N$, we have $A(\underbrace{x_k, x_k, ...x_k}_{(n-1)times}, x) \leq \varepsilon$ and we write $\lim\limits_{k\to\infty} x_k = x$.*

Note: x is called the limit of the sequence $\{x_k\}$

Lemma 2.4. *[13] Let (X,A) be A_b-metric space. If the sequence $\{x_k\}$ in X converges to a point x, then the limit x is unique.*

Definition 2.6. *Let (X,A) be A_b-metric space. A sequence $\{x_k\}$ in X is called a Cauchy sequence, if $A(\underbrace{x_k, x_k, ...x_k}_{(n-1)times}, x_m) \to 0$ as $k, m \to \infty$.*
That is, to each $\varepsilon \geq 0$, there exists $N \in \mathbb{N}$ such that for all $k, m \geq N$, we have $A(\underbrace{x_k, x_k, ...x_k}_{(n-1)times}, x_m) \leq \varepsilon$.

Lemma 2.5. *[13] Every convergent sequence in a A_b-metric space is a Cauchy sequence.*

Definition 2.7. *A A_b-metric space (X,A) is said to be complete, if every Cauchy sequence in X is convergent.*

Definition 2.8. *[9] Let $(X, \leq)$ be a partially ordered set and $f, g : X \times X \to X$ be mappings. We say that (f, g) has the mixed weakly monotone property on X, if for any $x, y \in X$,*
$x \leq f(x, y),\ y \geq f(y, x)$
$\implies f(x, y) \leq g((f(x, y), f(y, x)),\ f(y, x) \geq g((f(y, x), f(x, y))$
and $x \leq g(x, y),\ y \geq g(y, x)$
$\implies g(x, y) \leq f((g(x, y), g(y, x)),\ g(y, x) \geq f((g(y, x), g(x, y))$

Definition 2.9. *Let X be a non-empty set and $f, g : X \times X \to X$ be maps on $X \times X$.*
(i) A point $(x, y) \in X \times X$ is called a coupled fixed pint of f, if $x = f(x, y)$ and $y = f(y, x)$
(ii) A point $(x, y) \in X \times X$ is said to be a common coupled fixed pint of f and g, if $x = f(x, y) = g(x, y)$ and $y = f(y, x) = g(y, x)$.

Note: (x, y) is said to be a Coupled coincidence point of f *and* g, if $f(x, y) = g(x, y)$ and $f(y, x) = g(y, x)$.

We observe that a common coupled fixed pint of f and g is necessarily a Coupled coincidence point of f *and* g.

3 Main Results

Now we prove our first main result.

Theorem 3.1. *Let $(X, \leq, A)$ be a partially ordered, complete A_b-metric space and let $f, g : X \times X \to X$ be the mappings such that*
(i) the pair (f, g) has mixed weakly monotone property on X and there exists

$x_0, y_0 \in X$ such that $x_0 \le f(x_0, y_0), f(y_0, x_0) \le y_0$ or $x_0 \le g(x_0, y_0), g(y_0, x_0) \le y_0$,
(ii) there is an $a_i > 0$, $i = 1, .., 4$. Such that
$b^2(a_1 + a_2) + a_3(1 + b^2) + b^2 a_4((n-1)b + 1) < 1$ and

$$\begin{aligned} & A(f(x,y), f(x,y), ...f(x,y), g(u,v)) + A(f(y,x), f(y,x), ...f(y,x), g(v,u)) \\ & \le a_1 \Bigg[(1 + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))) \\ & \qquad \frac{(D((u,v),(u,v),...,(u,v),(g(u,v),g(v,u))))}{(1 + D((x,y),(x,y),...(x,y),(u,v)))} \Bigg] \\ & \quad + a_2[D((x,y),(x,y),...(x,y),(u,v))] \\ & \quad + a_3[D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))) \\ & \qquad + D((u,v),(u,v),...,(u,v),(g(u,v),g(v,u)))] \\ & \quad + a_4[D((u,v),(u,v),...,(u,v),(f(x,y),f(y,x))) \\ & \qquad + D((x,y),(x,y),...(x,y),(g(u,v),g(v,u)))] \end{aligned} \tag{3.1}$$

for all $x, y, u, v \in X$ with $x \le u$ and $y \ge v$,
(iii) if f or g is continuous.
Then f and g have a coupled common fixed point in X.

Proof. Let (x_0, y_0) be a given point in $X \times X$, satisfying (i).
Write $x_1 = f(x_0, y_0), y_1 = f(y_0, x_0), x_2 = g(x_1, y_1), y_2 = g(y_1, x_1)$
Define the sequences $\{x_n\}$ and $\{y_n\}$ inductively

$$\begin{aligned} x_{2n+1} = f(x_{2n}, y_{2n}),\ y_{2n+1} = f(y_{2n}, x_{2n}) \\ x_{2n+2} = g(x_{2n+1}, y_{2n+1}),\ y_{2n+2} = g(y_{2n+1}, x_{2n+1}) \\ for\ all\ n \in \mathbb{N} \end{aligned} \tag{3.2}$$

Since $x_0 \le f(x_0, y_0)$ *and* $y_0 \ge f(y_0, x_0)$
and since f has mixed weakly monotone property, we get
$x_1 = f(x_0, y_0) \le f(f(x_0, y_0), f(y_0, x_0)) = f(x_1, y_1) = x_2 \implies x_1 \le x_2$
and $x_2 = f(x_1, y_1) \le f(f(x_1, y_1), f(y_1, x_1)) = f(x_2, y_2) = x_3 \implies x_2 \le x_3$ also
$y_1 = f(y_0, x_0) \ge f(f(y_0, x_0), f(x_0, y_0)) = f(y_1, x_1) = y_2 \implies y_1 \ge y_2$
and $y_2 = f(y_1, x_1) \ge f(f(y_1, x_1), f(x_1, y_1)) = f(y_2, x_2) = y_3 \implies y_2 \ge y_3$
By induction,

$$\begin{aligned} x_0 \le x_1 \le x_2 \le ...... \le x_n \le x_{n+1} \le .... \\ y_0 \ge y_1 \ge y_2 ...... \ge y_n \ge y_{n+1} \ge .... \\ for\ all\ n \in \mathbb{N} \end{aligned} \tag{3.3}$$

Now we show that these sequences are Cauchy
Define$D_n : X \times X \to X$ by

$$\begin{aligned} D_n &= D((x_n, y_n), (x_n, y_n), ..., (x_n, y_n), (x_{n+1}, y_{n+1})) \\ &= A(x_n, x_n, ...x_n, x_{n+1}) + A(y_n, y_n, ...y_n, y_{n+1}) \\ & \qquad for\ all\ x_i, y_i \in X, i, j = 1, 2, ...n. \end{aligned}$$

Now

$$\begin{aligned} D_{2n+1} &= A(x_{2n+1}, x_{2n+1}, ...x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, ...y_{2n+1}, y_{2n+2}) \\ &= A(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n})...f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1})) \\ &\quad + A(f(y_{2n}, x_{2n}), f(y_{2n}, x_{2n})...f(y_{2n}, x_{2n}), g(y_{2n+1}, x_{2n+1})) \end{aligned}$$

$$\begin{aligned} &\le a_1 \Bigg[(1 + D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})))) \\ &\quad \frac{(D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))}{(1 + D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})))} \Bigg] \\ &\quad + a_2[D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))] \\ &\quad + a_3[D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))) \\ &\qquad + (D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))] \\ &\quad + a_4[(D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})))) \\ &\qquad + (D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ..., (x_{2n}, y_{2n}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))] \end{aligned}$$

From 3.2

$$\begin{aligned} &\le a_1 \Bigg[(1 + D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))) \\ &\quad \frac{(D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})))}{(1 + D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})))} \Bigg] \\ &\quad + a_2[D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))] \\ &\quad + a_3[D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\ &\qquad + (D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})))] \\ &\quad + a_4[D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1})) \\ &\qquad + (D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ..., (x_{2n}, y_{2n}), (x_{2n+2}, y_{2n+2})))] \end{aligned}$$

$$\begin{aligned} &= a_1 \ [D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))] \\ &\quad + a_2 \ [D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))] \\ &\quad + a_3 \ [D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\ &\qquad + (D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})))] \\ &\quad + a_4 \ [D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ..., (x_{2n}, y_{2n}), (x_{2n+2}, y_{2n+2}))] \end{aligned}$$

From lemma 2.2,

$$\begin{aligned} D_{2n+1} &\le a_1 \ [D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))] \\ &\quad + a_2 \ [D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))] \\ &\quad + a_3 \ [D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\ &\qquad + (D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})))] \\ &\quad + a_4 \ [(n-1)b \ D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ..., (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\ &\qquad + b^2 \ D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ..., (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))] \end{aligned}$$

$$\begin{aligned}&= (a_1 + a_3 + a_4 b^2)\ [D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), ...,\\ &\qquad (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})]\\ &+ (a_2 + a_3 + a_4(n-1)b)\ [D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), ...,\\ &\qquad (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))]\end{aligned}$$

$$\begin{aligned}&= (a_1 + a_3 + a_4 b^2)\ [A(x_{2n+1}, x_{2n+1}, ...x_{2n+1}, x_{2n+2})\\ &\qquad + A(y_{2n+1}, y_{2n+1}, ...y_{2n+1}, y_{2n+2})]\\ &+ (a_2 + a_3 + a_4(n-1)b)\ [A(x_{2n}, x_{2n}, ...x_{2n}, x_{2n+1})\\ &\qquad + A(y_{2n}, y_{2n}, ...y_{2n}, y_{2n+1})]\end{aligned} \tag{3.4}$$

Similarly we get,

$$\begin{aligned}&= (a_1 + a_3 + a_4 b^2)\ [A(y_{2n+1}, y_{2n+1}, ...y_{2n+1}, y_{2n+2}\\ &\qquad + A(x_{2n+1}, x_{2n+1}, ...x_{2n+1}, x_{2n+2})]\\ &+ (a_2 + a_3 + a_4(n-1)b)\ [A(y_{2n}, y_{2n}, ...y_{2n}, y_{2n+1})\\ &\qquad + A(x_{2n}, x_{2n}, ...x_{2n}, x_{2n+1})]\end{aligned} \tag{3.5}$$

From 3.4 and 3.5 we have,

$$\begin{aligned}2D_{2n+1} &= 2[A(x_{2n+1}, x_{2n+1}, ...x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, ...y_{2n+1}, y_{2n+2})]\\ &\leq 2\{(a_1 + a_3 + a_4 b^2)\ [A(x_{2n+1}, x_{2n+1}, ...x_{2n+1}, x_{2n+2})\\ &\qquad + A(y_{2n+1}, y_{2n+1}, ...y_{2n+1}, y_{2n+2})]\\ &\quad + (a_2 + a_3 + a_4(n-1)b)\ [A(x_{2n}, x_{2n}, ...x_{2n}, x_{2n+1})\\ &\qquad + A(y_{2n}, y_{2n}, ...y_{2n}, y_{2n+1})]\}\end{aligned}$$

Therefore

$$\begin{aligned}D_{2n+1} &\leq \{(a_1 + a_3 + a_4 b^2)\ [A(x_{2n+1}, x_{2n+1}, ...x_{2n+1}, x_{2n+2})\\ &\qquad + A(y_{2n+1}, y_{2n+1}, ...y_{2n+1}, y_{2n+2})]\\ &\quad + (a_2 + a_3 + a_4(n-1)b)\ [A(x_{2n}, x_{2n}, ...x_{2n}, x_{2n+1})\\ &\qquad + A(y_{2n}, y_{2n}, ...y_{2n}, y_{2n+1})]\}\end{aligned} \tag{3.6}$$

$$\Longrightarrow\ (1 - (a_1 + a_3 + a_4 b^2))\ D_{2n+1} \leq (a_2 + a_3 + a_4(n-1)b)\ D_{2n}$$

$$\Longrightarrow\ D_{2n+1} \leq \frac{a_2 + a_3 + a_4(n-1)b}{1 - (a_1 + a_3 + a_4 b^2)}\ D_{2n} \tag{3.7}$$

Put $\gamma = \frac{a_2+a_3+a_4(n-1)b}{1-(a_1+a_3+a_4b^2)}$, then $0 \leq \gamma < 1$.
From 3.7,

$$D_{2n+1} \leq \gamma\ D_{2n}$$

Similarly we can show that

$$D_{2n+2} \leq \gamma\ D_{2n+1}\ for\ n = 0, 1, 2, ...$$

Hence

$$D_{n+1} \leq \gamma\ D_n$$

Therefore

$$D_{n+1} \leq \gamma^{n+1} D_0 \tag{3.8}$$

Define

$$\begin{aligned} D_{n,m} &= D(\underbrace{(x_n, y_n), (x_n, y_n), ..., (x_n, y_n)}_{(n-1)-times}, (x_m, y_m)) \\ &= A(\underbrace{x_n, x_n, ...x_n}_{(n-1)-times}, x_m) + A(\underbrace{y_n, y_n, ...y_n}_{(n-1)-times}, y_m) \end{aligned}$$

Now we have to show that $D_{n,m}$ is a Cauchy sequence
By lemma 2.2, for all $k, m \in \mathbb{N},\ k \leq m$
we have

$$\begin{aligned} D_{n+1,m+1} &= A(x_{n+1}, x_{n+1}, ...x_{n+1}, x_{m+1}) + A(y_{n+1}, y_{n+1}, ...y_{n+1}, y_{m+1}) \\ &\leq b(n-1)[A(x_{n+1}, x_{n+1}, ...x_{n+1}, x_{n+2}) + A(y_{n+1}, y_{n+1}, ...y_{n+1}, y_{n+2})] \\ &\quad + b^2[A(x_{n+2}, x_{n+2}, ...x_{n+2}, x_{m+1}) + A(y_{n+2}, y_{n+2}, ...y_{n+2}, y_{m+1})] \\ &= b(n-1)D_{n+1} + b^2 b(n-1)[A(x_{n+2}, x_{n+2}, ...x_{n+2}, x_{n+3}) \\ &\quad + A(y_{n+2}, y_{n+2}, ...y_{n+2}, y_{n+3})] \\ &\quad + b^2 b^2[A(x_{n+3}, x_{n+3}, ...x_{n+3}, x_{m+1}) + A(y_{n+3}, y_{n+3}, ...y_{n+3}, y_{m+1})] \\ &\leq b(n-1)D_{n+1} + b^3(n-1)D_{n+2} + b^5(n-1)D_{n+3} \\ &\quad \vdots \\ &\quad + b^{2(m-n)-3}(n-1)[A(x_{m-1}, x_{m-1}, ...x_{m-1}, x_m) \\ &\quad + A(y_{m-1}, y_{m-1}, ...y_{m-1}, y_m)] \\ &\quad + b^{2(m-n)-1}(n-1)[A(x_m, x_m, ...x_m, x_{m+1}) \\ &\quad + A(y_m, y_m, ...y_m, y_{m+1})] \end{aligned}$$

From 3.8

$$\begin{aligned} D_{n+1,m+1} &\leq b(n-1)[\gamma^{n+1} + b^2\gamma^{n+2} + b^4\gamma^{n+3} ... + b^{2(m-n)-2}\gamma^m]D_0 \\ \Longrightarrow D_{n+1,m+1} &\leq b(n-1)\gamma^{n+1}[1 + b^2\gamma + (b^2\gamma)^2 + ... + (b^2\gamma)^{(m-n-1)}]D_0 \\ &= b(n-1)\gamma^{n+1}[1 + \delta + \delta^2 + ... + \delta^{(m-n-1)}]D_0 \\ &= b(n-1)\gamma^{n+1}\left(\frac{1}{1-\delta}\right)D_0 \end{aligned}$$

Where $\delta = b^2\gamma$
Hence for all $n, m \in \mathbb{N}$, with $n \leq m$, we have

$$D_{n,m} = A(x_n, x_n, ...x_n, x_m) + A(y_n, y_n, ...y_n, y_m) \leq b(n-1)\gamma^n\left(\frac{1}{1-\delta}\right)D_0$$

Since $0 \leq \delta = b^2(a_1 + a_2) + a_3(1 + b^2) + b^2a_4((n-1)b + 1) < 1$, we have

$$\lim_{n,m\to\infty} A(x_n, x_n, ...x_n, x_m) + A(y_n, y_n, ...y_n, y_m) = 0$$

$$That\ is,\ \lim_{n,m\to\infty} A(x_n, x_n, ...x_n, x_m) = \lim_{n,m\to\infty} A(y_n, y_n, ...y_n, y_m) = 0$$

Therefore $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in X.
By the completeness of X, there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
Therefore $D_{n,m}$ is a Cauchy sequence.
Now we show that (x, y) is a coupled fixed point of f and g.
Without loss of generality, we may suppose that f is continuous, we have

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n}, y_{2n}) = f\left(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}\right) = f(x, y)$$

and

$$y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} f(y_{2n}, x_{2n}) = f\left(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}\right) = f(y, x)$$

Thus (x, y) is a coupled fixed point of f.
From 3.1, taking $u = x\ and\ v = y$, we have,

$$\begin{aligned}
&A(x, x, ...x, g(x, y)) + A(y, y, ...y, g(y, x)) \\
&= A(f(x, y), f(x, y), ...f(x, y), g(x, y)) + A(f(y, x), f(y, x), ...f(y, x), g(y, x)) \\
&\leq a_1 \Bigg[(1 + D((x, y), (x, y), ...(x, y), (f(x, y), f(y, x)))) \\
&\qquad \frac{(D((x, y), (x, y), ..., (x, y), (g(x, y), g(y, x))))}{(1 + D((x, y), (x, y), ...(x, y), (x, y)))}\Bigg] \\
&\quad + a_2\, [D((x, y), (x, y), ...(x, y), (x, y))] \\
&\quad + a_3\, [D((x, y), (x, y), ...(x, y), (f(x, y), f(y, x)) \\
&\qquad + D((x, y), (x, y), ..., (x, y), (g(x, y), g(y, x))))] \\
&\quad + a_4\, [D((x, y), (x, y), ..., (x, y), (f(x, y), f(y, x))) \\
&\qquad + D((x, y), (x, y), ...(x, y), (g(x, y), g(y, x)))] \\
&\leq a_1 \Bigg[(1 + D((x, y), (x, y), ...(x, y), (x, y))) \\
&\qquad \frac{(D((x, y), (x, y), ..., (x, y), (g(x, y), g(y, x))))}{(1 + D((x, y), (x, y), ...(x, y), (x, y)))}\Bigg] \\
&\quad + a_2\, [D((x, y), (x, y), ...(x, y), (x, y))] \\
&\quad + a_3\, [D((x, y), (x, y), ...(x, y), (x, y)) \\
&\qquad + D((x, y), (x, y), ..., (x, y), (g(x, y), g(y, x)))] \\
&\quad + a_4\, [D((x, y), (x, y), ..., (x, y), (x, y)) \\
&\qquad + D((x, y), (x, y), ...(x, y), (g(x, y), g(y, x)))]
\end{aligned}$$

$=(a_1 + a_3 + a_4)\ D((x, y), (x, y), ..., (x, y), (g(x, y), g(y, x)))$
$\leq b(a_1 + a_3 + a_4)\ ((g(x, y), g(y, x)), (g(x, y), g(y, x)), ...(g(x, y), g(y, x)), (x, y))$

Since $b(a_1 + a_3 + a_4) < 1$, we have $(g(x, y), g(y, x)) = (x, y)$
$\implies g(x, y) = x$ and $g(y, x) = y$
Therefore (x, y) is a coupled fixed point of g.
Thus (x, y) is a coupled common fixed point of f and g. □

Note: (i) It may be observed that putting $g = f$ in Theorem 3.1 is an extension of (Theorem 37 of W.sintunawarat et al.[16])
(ii) It may be observed that putting $g = f$ in Theorem 3.1, we extend (Corollary 3.6 of M.Abbas et al.[2]) in A_b metric space.
(iii) In Theorem 3.1, putting $g = f,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = k$ and $n = 2$, we obtain (Theorem 3.2 of E.Carpinar et al.[10]).
(iv) Our Theorem 3.1 is generalization of (Corollary 3.2 of W.sintunawarat et al.[16]). Under the assumption $x = u,\ y = v$ for $n = 3$.
(v) In Theorem 3.1, putting $g = f,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = k$ and $n = 3$, we obtain (Corollary 24 of E.Carpinar et al.[11]).

Theorem 3.2. *Let $(X, \leq, A)$ be a partially ordered, complete A_b-metric space and $f, g : X \times X \to X$ be the mappings such that*
(i) the pair (f, g) has mixed weakly monotone property on X and there exists $x_0, y_0 \in X$ such that $x_0 \leq f(x_0, y_0), f(y_0, x_0) \leq y_0$ or
$x_0 \leq g(x_0, y_0), g(y_0, x_0) \leq y_0$,
(ii) there is an $a_i > 0$, $i = 1, .., 4$. Such that
$b^2(a_1 + a_2) + a_3(1 + b^2) + b^2 a_4((n-1)b + 1) < 1$ and

$$\begin{aligned}
&A(f(x,y), f(x,y), ...f(x,y), g(u,v)) + A(f(y,x), f(y,x), ...f(y,x), g(v,u)) \\
&\leq a_1 \Bigg[(1 + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))) \\
&\qquad \frac{(D((u,v),(u,v),...,(u,v),(g(u,v),g(v,u))))}{(1 + D((x,y),(x,y),...(x,y),(u,v)))} \Bigg] \\
&\quad + a_2\ [D((x,y),(x,y),...(x,y),(u,v))] \\
&\quad + a_3\ [D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))) \\
&\qquad + D((u,v),(u,v),...,(u,v),(g(u,v),g(v,u)))] \\
&\quad + a_4\ [D((u,v),(u,v),...,(u,v),(f(x,y),f(y,x))) \\
&\qquad + D((x,y),(x,y),...(x,y),(g(u,v),g(v,u)))]
\end{aligned} \tag{3.9}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$,
(iii) X has the following properties
(a). if $\{x_n\}$ is an increasing sequence with $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
(b). if $\{y_n\}$ is a decreasing sequence with $y_k \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.
Then f and g have coupled common fixed points in X.

Proof. Suppose X satisfies (a) and (b), by 3.3 we get $x_n \leq x$ and $y_n \geq y$ for all $n \in \mathbb{N}$

Applying lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))\\
&\le b(n-1)D((x,y),(x,y),...(x,y),(x_{2n+2},y_{2n+2})\\
&\quad + b^2D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),...(x_{2n+2},y_{2n+2}),(f(x,y),f(y,x)))\\
&= b(n-1)D((x,y),(x,y),...(x,y),(x_{2n+2},y_{2n+2}))\\
&\quad + b^2D((g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),\\
&\qquad ....(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(f(x,y),f(y,x)))
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\ &D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))\\
&\le b(n-1)[A(x,x,...x,x_{2n+2})+A(y,y,...y,y_{2n+2})]\\
&\quad + b^2A[g(x_{2n+1},y_{2n+1}),g(x_{2n+1},y_{2n+1}),...g(x_{2n+1},y_{2n+1}),f(x,y)]\\
&\quad + b^2A[g(y_{2n+1},x_{2n+1}),g(y_{2n+1},x_{2n+1}),...g(y_{2n+1},x_{2n+1}),f(y,x)]
\end{aligned} \tag{3.10}$$

By 3.1, we get

$$\begin{aligned}
&A((g(x_{2n+1},y_{2n+1})),(g(x_{2n+1},y_{2n+1})),...(g(x_{2n+1},y_{2n+1}),(f(x,y))\\
&\quad + A((g(y_{2n+1},x_{2n+1})),(g(y_{2n+1},x_{2n+1})),...(g(y_{2n+1},x_{2n+1}),(f(y,x))\\
&\le a_1\Bigg[(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,\\
&\qquad (x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1}))))\\
&\qquad \frac{(D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))))}{(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x,y)))}\Bigg]\\
&\quad + a_2\ [D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x,y))]\\
&\quad + a_3\ [D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\\
&\qquad ...,(x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))\\
&\qquad + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))]\\
&\quad + a_4\ [D((x,y),(x,y),...(x,y),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))\\
&\qquad + D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(f(x,y),f(y,x)))]
\end{aligned}$$

$$\begin{aligned}
&= a_1\Bigg[(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2})))\\
&\qquad \frac{(D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))))}{(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x,y)))}\Bigg]\\
&\quad + a_2\ [D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x,y))]\\
&\quad + a_3\ [D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\\
&\qquad + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))]\\
&\quad + a_4[D((x,y),(x,y),...(x,y),(x_{2n+2},y_{2n+2}))\\
&\qquad + D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(f(x,y),f(y,x)))]
\end{aligned}$$

From 3.7 and 3.8

$$
\begin{aligned}
&D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))\\
&\leq b(n-1)[(A(x,x,...x,x_{2n+2})+A(y,y,...y,y_{2n+2}))]\\
&\quad + b^2\Big\{a_1\ \Big[(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2})))\\
&\qquad\qquad \frac{(D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))))}{(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x,y)))}\Big]\\
&\quad + a_2\ [D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x,y))]\\
&\quad + a_3\ [D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\\
&\qquad + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))]\\
&\quad + a_4[D((x,y),(x,y),...(x,y),(x_{2n+2},y_{2n+2}))\\
&\qquad + D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),...,(x_{2n+1},y_{2n+1}),(f(x,y),f(y,x)))]\Big\}
\end{aligned} \tag{3.11}
$$

Taking the limit as $n \to \infty$ in 3.11, we obtain

$$
\begin{aligned}
&D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))\\
&\leq b(n-1)[A(x,x,...x,x)+A(y,y,...y,y)]\\
&\quad + b^2\Big\{a_1\ \Big[(1+D((x,y),(x,y),...,(x,y),(x,y)))\\
&\qquad\qquad \frac{(D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))))}{(1+D((x,y),(x,y),...,(x,y),(x,y)))}\Big]\\
&\quad + a_2\ [D((x,y),(x,y),...,(x,y),(x,y))]\\
&\quad + a_3\ [D((x,y),(x,y),...,(x,y),(x,y))\\
&\qquad + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))]\\
&\quad + a_4[D((x,y),(x,y),...,(x,y),(x,y))\\
&\qquad + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&Therefore\ D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))\\
&\qquad \leq b^2(a_1+a_3+a_4)\ D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))
\end{aligned}
$$

Since $b^2(a_1+a_3+a_4) < 1$, we have
$D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))) = 0$
$\Longrightarrow\ (f(x,y),f(y,x)) = (x,y)$
That is, $f(x,y) = x$ and $f(y,x) = y$
Therefore (x,y) is a coupled fixed point of f.
Similarly we can show that $g(x,y) = x$ and $g(y,x) = y$
Hence $f(x,y) = x = g(x,y)$ and $f(y,x) = y = g(y,x)$
Thus (x,y) is a coupled common fixed point of f and g. □

Theorem 3.3. *Suppose Theorem 3.1 or Theorem 3.2 satisfied, if further $\{x_n\}$ is an increasing sequence with $x_n \to x$ and $x_n \leq u$ for each n, then $x \leq u$. Then f and g have a unique coupled common fixed points. Further more, any fixed point of f is a fixed point of g, and conversely.*

Proof. Suppose the given condition holds,
Let (x, y) and $(u, v) \in X \times X$, there exist $(x^*, y^*) \in X \times X$, that is, comparable to (x, y) and (u, v).

$$\begin{aligned} D((x,y),(x,y),...(x,y),(u,v)) &= A(x,x,...x,u) + A(y,y,...y,u) \\ &= A(f(x,y),f(x,y),...f(x,y),g(u,v)) \\ &\quad + A(f(y,x),f(y,x),...f(y,x),g(v,u)) \end{aligned}$$

$$\begin{aligned} \leq a_1 &\Bigg[(1 + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))) \\ &\quad \frac{(D((u,v),(u,v),...,(u,v),(g(u,v),g(v,u))))}{(1 + D((x,y),(x,y),...(x,y),(u,v)))}\Bigg] \\ &+ a_2\ [D((x,y),(x,y),...(x,y),(u,v))] \\ &+ a_3\ [D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))) \\ &\quad + D((u,v),(u,v),...,(u,v),(g(u,v),g(v,u)))] \\ &+ a_4\ [D((u,v),(u,v),...,(u,v),(f(x,y),f(y,x))) \\ &\quad + D((x,y),(x,y),...(x,y),(g(u,v),g(v,u)))] \end{aligned}$$

$$\begin{aligned} &\leq a_1 \Bigg[(1 + D((x,y),(x,y),...(x,y),(x,y))) \\ &\quad \frac{(D((u,v),(u,v),...,(u,v),(u,v)))}{(1 + D((x,y),(x,y),...(x,y),(u,v)))}\Bigg] \\ &\quad + a_2\ [D((x,y),(x,y),...(x,y),(u,v))] \\ &\quad + a_3\ [D((x,y),(x,y),...(x,y),(x,y)) \\ &\qquad + D((u,v),(u,v),...,(u,v),(u,v))] \\ &\quad + a_4\ [D((u,v),(u,v),...,(u,v),(x,y)) + D((x,y),(x,y),...(x,y),(u,v))] \\ &\leq a_2\ D((x,y),(x,y),...(x,y),(u,v)) \\ &\quad + a_4\ (D((u,v),(u,v),...(u,v),(x,y)) \\ &\qquad + D((x,y),(x,y),...(x,y),(u,v))) \\ &\leq a_2\ D((x,y),(x,y),...(x,y),(u,v)) \\ &\quad + a_4\ (b\ D((x,y),(x,y),...(x,y),(u,v)) + D((x,y),(x,y),...(x,y),(u,v))) \\ &= (a_2 + a_4(b+1))\ D((x,y),(x,y),...(x,y),(u,v)) \end{aligned}$$

Since $(a_2 + a_4(b + 1)) < 1$, so that
$D((x, y), (x, y), ...(x, y), (u, v)) = 0$
$\implies (x, y) = (u, v) \implies x = u \text{ and } y = v$
Suppose (x, y) and (x^*, y^*) are Coupled common fixed points such that $x \leq x^*$

and $y \geq y^*$, then $x = x^*$ and $y = y^*$.
Now

$$\begin{aligned} D((x,y),(x,y),...(x,y),(x^*,y^*)) &= A(x,x,...x,x^*) + A(y,y,...y,y^*) \\ &= A(f(x,y),f(x,y),...f(x,y),g(x^*,y^*)) \\ &\quad + A(f(y,x),f(y,x),...f(y,x),g(y^*,x^*)) \end{aligned}$$

$$\begin{aligned} &\leq a_1 \left[(1 + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))) \right. \\ &\qquad \left. \frac{(D((x^*,y^*),(x^*,y^*),...,(x^*,y^*),(g(x^*,y^*),g(y^*,x^*))))}{(1 + D((x,y),(x,y),...(x,y),(x^*,y^*)))} \right] \\ &\quad + a_2 \left[D((x,y),(x,y),...(x,y),(x^*,y^*)) \right] \\ &\quad + a_3 \left[D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))) \right. \\ &\qquad \left. + D((x^*,y^*),(x^*,y^*),...,(x^*,y^*),(g(x^*,y^*),g(y^*,x^*))) \right] \\ &\quad + a_4 \left[D((x^*,y^*),(x^*,y^*),...,(x^*,y^*),(f(x,y),f(y,x))) \right. \\ &\qquad \left. + D((x,y),(x,y),...(x,y),(g(x^*,y^*),g(y^*,x^*))) \right] \\ &\leq a_1 \left[(1 + D((x,y),(x,y),...(x,y),(x,y))) \right. \\ &\qquad \left. \frac{(D((x^*,y^*),(x^*,y^*),...,(x^*,y^*),(x^*,y^*)))}{(1 + D((x,y),(x,y),...(x,y),(x^*,y^*)))} \right] \\ &\quad + a_2 \left[D((x,y),(x,y),...(x,y),(x^*,y^*)) \right] \\ &\quad + a_3 \left[D((x,y),(x,y),...(x,y),(x,y)) \right. \\ &\qquad \left. + D((x^*,y^*),(x^*,y^*),...,(x^*,y^*),(x^*,y^*)) \right] \\ &\quad + a_4 \left[D((x^*,y^*),(x^*,y^*),...,(x^*,y^*),(x,y)) \right. \\ &\qquad \left. + D((x,y),(x,y),...(x,y),(x^*,y^*)) \right] \\ &= a_2 \left[D((x,y),(x,y),...(x,y),(x^*,y^*)) \right] \\ &\quad + a_4 \left[D((x^*,y^*),(x^*,y^*),...,(x^*,y^*),(x,y)) \right. \\ &\qquad \left. + D((x,y),(x,y),...(x,y),(x^*,y^*)) \right] \\ &\leq a_2 \left[D((x,y),(x,y),...(x,y),(x^*,y^*)) \right] \\ &\quad + a_4 \left[b\ D((x,y),(x,y),...,(x,y),(x^*,y^*)) \right. \\ &\qquad \left. + D((x,y),(x,y),...(x,y),(x^*,y^*)) \right] \\ &= (a_2 + a_4(b+1)) D((x,y),(x,y),...,(x,y),(x^*,y^*)) \end{aligned}$$

Since $(a_2 + a_4(b+1)) < 1$, so that
$D((x,y),(x,y),...,(x,y),(x^*,y^*)) = 0$
$\implies (x,y) = (x^*,y^*)$
$\implies x = x^* \textit{ and } y = y^*$
we show that any fixed point of f is a fixed point of g, and conversely.
That is, to show that (x,y) is a fixed point of $f \iff (x,y)$ is a fixed point of g.

Suppose that (x, y) is a coupled fixed point of f.

$$\begin{aligned}
&D((x,y),(x,y),...(x,y),(g(x,y),g(y,x)))\\
&= A(f(x,y),f(x,y),...f(x,y),g(x,y)) + A(f(y,x),f(y,x),...f(y,x),g(y,x))\\
&\leq a_1 \Bigg[(1 + D((x,y),(x,y),...(x,y),(f(x,y),f(y,x))))\\
&\qquad \frac{(D((x,y),(x,y),...,(x,y),(g(x,y),g(y,x))))}{(1 + D((x,y),(x,y),...(x,y),(x,y)))}\Bigg]\\
&\quad + a_2\ [D((x,y),(x,y),...(x,y),(x,y))]\\
&\quad + a_3\ [D((x,y),(x,y),...(x,y),(f(x,y),f(y,x)))\\
&\qquad + D((x,y),(x,y),...,(x,y),(g(x,y),g(y,x)))]\\
&\quad + a_4\ [D((x,y),(x,y),...,(x,y),(f(x,y),f(y,x)))\\
&\qquad + D((x,y),(x,y),...(x,y),(g(x,y),g(y,x)))]\\
&= a_1 \Bigg[(1 + D((x,y),(x,y),...(x,y),(x,y)))\\
&\qquad \frac{(D((x,y),(x,y),...,(x,y),(g(x,y),g(y,x))))}{(1 + D((x,y),(x,y),...(x,y),(x,y)))}\Bigg]\\
&\quad + a_3\ [D((x,y),(x,y),...(x,y),(x,y))\\
&\qquad + D((x,y),(x,y),...,(x,y),(g(x,y),g(y,x)))]\\
&\quad + a_4\ [D((x,y),(x,y),...,(x,y),(x,y))\\
&\qquad + D((x,y),(x,y),...(x,y),(g(x,y),g(y,x)))]\\
&= a_1\ [D((x,y),(x,y),...,(x,y),(g(x,y),g(y,x)))]\\
&\quad + a_3\ [D((x,y),(x,y),...,(x,y),(g(x,y),g(y,x)))]\\
&\quad + a_4\ [D((x,y),(x,y),...(x,y),(g(x,y),g(y,x)))]\\
&= (a_1 + a_3 + a_4)\ D((x,y),(x,y),...(x,y),(g(x,y),g(y,x)))\\
&\leq b(a_1 + a_3 + a_4)\ D((g(x,y),g(y,x)),(g(x,y),g(y,x)),\\
&\qquad .....(g(x,y),g(y,x)),(x,y))
\end{aligned}$$

Since $b(a_1 + a_3 + a_4) < 1$, we have
$D((g(x,y),g(y,x)),(g(x,y),g(y,x)),...(g(x,y),g(y,x)),(x,y)) = 0$
$\implies (g(x,y),g(y,x)) = (x,y)$
$\implies x = g(x,y)$ *and* $y = g(y,x)$
Therefore (x, y) is a coupled fixed point of g, and conversely. □

Taking $g = f$ and $a_1 = a_3 = a_4 = 0$ in Theorem 3.1, we get the following

Corollary 3.4. *Let $(X, \leq, A)$ be a partially ordered, complete A_b-metric space and let $f : X \times X \to X$ be the mapping such that*
(i) f has mixed weakly monotone property on X and there exists $x_0, y_0 \in X$ such that $x_0 \leq f(x_0, y_0), f(y_0, x_0) \leq y_0$,

(ii) there is an a_2 such that $a_2 < 1$ and

$$A(f(x,y), f(x,y), ...f(x,y), f(u,v)) + A(f(y,x), f(y,x), ...f(y,x), f(v,u)) \\ \leq a_2\ D((x,y),(x,y),...(x,y),(u,v)) \tag{3.12}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$,
(iii) if f is continuous.
Then f has a coupled fixed point in X.

we give an example to illustrate the Theorem 3.1 as follows.

Example 3.1. *Let $(\mathbb{R}, \leq, A)$ be a partially ordered complete A_b-metric space with A_b-metric, with index n, defined on $X = [-\infty, +\infty]$ as $A_b : X^n \to [-\infty, +\infty]$ by $A_b(x_1, x_2, ...x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2$, for all $x_i \in X$, $i = 1, 2............n$. Then (X, A_b) is an A_b-metric space with b=2.*
Let $f, g : \mathbb{R} \to \mathbb{R}$ be two maps defined by $f(x,y) = \frac{4x-2y+32n-2}{32n}$ and $g(x,y) = \frac{6x-3y+48n-3}{48n}$. Then the pair (f, g) has mixed weakly monotone property on $\mathbb{R}$

$$\begin{aligned}
&A(f(x,y), f(x,y), ...f(x,y), g(u,v)) + A(f(y,x), f(y,x), ...f(y,x), g(v,u)) \\
&= (n-1)(|f(x,y) - g(u,v)|) + (n-1)(|f(y,x) - g(v,u)|) \\
&= (n-1)\left(\left|\frac{4x-2y+32n-2}{32n} - \frac{6u-3v+48n-3}{48n}\right|\right) \\
&\quad + (n-1)\left(\left|\frac{4y-2x+32n-2}{32n} - \frac{6v-3u+48n-3}{48n}\right|\right) \\
&= \frac{(n-1)}{16n}(|2(x-u) - (y-v)| + |2(y-v) - (x-u)|) \\
&\leq \frac{(n-1)}{16n}(3\,|x-u| + 3\,|y-v|) \\
&\leq \frac{3(n-1)}{16n}(|x-u| + |y-v|) \\
&= \frac{3(n-1)}{16n}\ D((x,y),(x,y), ..., (x,y),(u,v))
\end{aligned}$$

For $n = 2$ and b=2, since $b^2 a_2 < 1 \implies a_2 < \frac{1}{4}$.
Then the contractive condition 3.1 is satisfied with $a_1 = a_3 = a_4 = 0$ and $a_2 < \frac{3}{32} < \frac{1}{4}$ and also $(1, 1)$ is the unique coupled common fixed point of f and g.

4 Application

The following system of Volterra type integral equations:

$$\begin{aligned} u(t) &= q(t) + \int_0^T \lambda(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds \\ v(t) &= q(t) + \int_0^T \lambda(t,s)(f_1(s,v(s)) + f_2(s,u(s)))ds. \end{aligned} \tag{4.1}$$

where the space $X = C([0,T],\mathbb{R})$ of continuous functions defined in $[0,T]$. Obviously, the space with the metric is given by

$$A(u,v) = \max_{t\in[0,T]} |u(t) - v(t)|, \; u,v \in C([0,T],\mathbb{R})$$

is a complete metric space.
Let $X = C([0,T],\mathbb{R})$ the natural partial order relation,
that is, $u,v \in C([0,T],\mathbb{R})$, $u \le v \iff u(t) \le v(t), \; t \in [0,T]$.

Theorem 4.1. *Consider the corollary 3.4 and assume that the following conditions are hold:*
(i) $f_1, f_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ *are continuous;*
(ii) $q : [0,T] \to \mathbb{R}$ *is continuous;*
(iii) $\lambda : [0,T] \times \mathbb{R} \to [0,\infty)$ *is continuous;*
(iv) there exist $c > 0$ *and* $a_2 < 1$, *such that for all* $u,v \in \mathbb{R}, \; v \ge u$,
$0 \le f_1(s,v) - f_1(s,u) \le ca_2 \; (v-u)$
$0 \le f_2(s,v) - f_2(s,u) \le ca_2 \; (v-u)$;
(v) assume that $c \max_{t\in[0,T]} \int_0^T \lambda(t,s)ds \le 1$;

(vi) there exist $x_0, y_0 \in X$ *such that*

$$\begin{aligned} x_0(t) &\ge q(t) + \int_0^T \lambda(t,s)(f_1(s,x_0(s)) + f_2(s,y_0(s)))ds \\ y_0(t) &\le q(t) + \int_0^T \lambda(t,s)(f_1(s,y_0(s)) + f_2(s,x_0(s)))ds. \end{aligned}$$

Then the system of Volterra type integral equation 4.1 has a unique solution in $X \times X$ *with* $X = C([0,T],\mathbb{R})$.

Proof. Define the mapping $F : X \times X \to X$ by

$$F(u,v)(t) = q(t) + \int_0^T \lambda(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds \tag{4.2}$$

for all $u,v \in X$ *and* $t \in [0,T]$.
Now we have to show that all the conditions of Corollary 3.4 are satisfied.

From (iv) of the Theorem 3.1, clearly F has mixed monotone property.
For $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, we have

$$
\begin{aligned}
&A(F(x,y),F(x,y),.....,F(x,y),F(u,v)) + A(F(y,x),F(y,x),.....,F(y,x),F(v,u)) \\
&= (n-1) \max_{t\in[0,T]} (|F(x,y)(t) - F(u,v)(t)| + |F(y,x)(t) - F(v,u)(t)|) \\
&= (n-1) \max_{t\in[0,T]} \left| \int_0^T \lambda(t,s)(f_1(s,x(s)) + f_2(s,y(s)))ds - \int_0^T \lambda(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds \right| \\
&+ (n-1) \max_{t\in[0,T]} \left| \int_0^T \lambda(t,s)(f_1(s,y(s)) + f_2(s,x(s)))ds - \int_0^T \lambda(t,s)(f_1(s,v(s)) + f_2(s,u(s)))ds \right| \\
&\leq (n-1) \max_{t\in[0,T]} \bigg(\int_0^T |f_1(s,x(s)) - f_1(s,u(s))| \, |\lambda(t,s)| \, ds \\
&\qquad + \int_0^T |f_2(s,y(s)) - f_2(s,v(s))| \, |\lambda(t,s)| \, ds \\
&\qquad + \int_0^T |f_1(s,y(s)) - f_1(s,v(s))| \, |\lambda(t,s)| \, ds + \int_0^T |f_2(s,x(s)) - f_2(s,u(s))| \, |\lambda(t,s)| \, ds \bigg) \\
&\leq (n-1) \max_{t\in[0,T]} ca_2 \left(\int_0^T |x(s) - u(s)| \, |\lambda(t,s)| \, ds + \int_0^T |y(s) - v(s)| \, |\lambda(t,s)| \, ds \right. \\
&\qquad \left. + \int_0^T |y(s) - v(s)| \, |\lambda(t,s)| \, ds + \int_0^T |x(s) - u(s)| \, |\lambda(t,s)| \, ds \right) \\
&\leq (n-1) \left(\max_{t\in[0,T]} |x(t) - u(t)| + \max_{t\in[0,T]} |y(t) - v(t)| \right. \\
&\qquad \left. + \max_{t\in[0,T]} |y(t) - v(t)| + \max_{t\in[0,T]} |x(t) - u(t)| \right) ca_2 \int_0^T |\lambda(t,s)| \, ds \\
&\leq 2(n-1) \left(\max_{t\in[0,T]} |x(t) - u(t)| + \max_{t\in[0,T]} |y(t) - v(t)| \right) ca_2 \int_0^T |\lambda(t,s)| \, ds \\
&\leq 2(n-1) \; a_2 \; (A(x,x,...,x,u) + A(y,y,...,y,v)) \\
&= 2(n-1) \; a_2 \; D((x,y),(x,y),......(x,y),(u,v))
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&A(F(x,y),F(x,y),.....,F(x,y),F(u,v)) + A(F(y,x),F(y,x),.....,F(y,x),F(v,u)) \\
&\leq 2(n-1) \; a_2 \; D((x,y),(x,y),......(x,y),(u,v))
\end{aligned}
$$

For n=2, $a_2 < \frac{1}{2} < 1$. Which is the contractive condition in Corollary 3.4.
Thus, F has a coupled fixed point in X.
That is, the system of Volterra type integral equations has a solution. □

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