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Stability of the New Generalized Linear Functional Equation in Normed Spaces via the Fixed Point Method in Generalized Metric Spaces

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Abstract : The aim of this paper is to apply the metric fixed point method for proving the Hyers-Ulam stability of the generalized linear functional equation of the form

$$2f(x+y)+f(x-y)+f(y-x)=2f(x)+2f(y),$$

for all $x, y \in X$, where f maps from a Banach space X into a Banach space Y.

Keywords : fixed point method; generalized linear functional equation; stability
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1 Introduction

Nowadays, the study of Hyers-Ulam stability problem is consistently becoming one of the most popular topics in the theory of functional equations. This study originated from a question of Ulam [19] concerning the stability of a group homomorphism. Afterwards, Hyers [13] gave a partial answer to Ulam's question

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and established the stability result. The result of Hyers in [13] was generalized by many mathematicians as Aiemsomboon and Sintunavarat [1, 2], Aoki [3], Bourgin [4], Forti [9], Gajda [10] and Rassias [18] and references therein.

The main objective of this paper is to introduce the new type of the generalized linear functional equations in normed spaces and establish Hyers-Ulam stability results of such functional equations via the metric fixed point method.

2 Preliminaries

The goal of this section is to present some related concepts and results which are mainly derived from fixed point theory.

Definition 2.1. *Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a* generalized metric *on X if d satisfies the following conditions for all $x, y, z \in X$:*

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$;*
3. *$d(x, z) \leq d(x, y) + d(y, z)$.*

Also, (X, d) is called a generalized metric space.

Definition 2.2. *A generalized metric space (X, d) is called* complete *if every d-Cauchy sequence in X is d-convergent, that is, if $\{x_n\}$ is a sequence in X such that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, then there is a point $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.*

We now recall the classical fundamental result from the metric fixed point theory which is the main tool of the proof used in the main result. We refer to [7] for the proof of this result.

Theorem 2.3. (Diaz and Margolis [7, Theorem]) *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a contraction mapping with the contractive constant $L < 1$, that is, $d(Jx, Jy) \leq Ld(x, y)$ for all $x, y \in X$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 satisfying the following conditions:

1. *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
2. *J has a unique fixed point y^* in the set*
$$Y := \{y \in X : d(J^{n_0} x, y) < \infty\};$$
3. *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
4. *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy),\ \forall y \in Y$.*

This result was used for the first time in [14] which was applied to investigate the stability result of some functional equations and was studied by many authors ([5, 6, 8, 11, 15, 16, 17] and references therein).

3 Main Results

First, we introduce the new functional equation as follows:

Definition 3.1. *Let X and Y be two normed spaces. A mapping $f : X \to Y$ is said to be the* generalized linear function *if it satisfies the generalized linear functional equation*

$$2f(x+y) + f(x-y) + f(y-x) = 2f(x) + 2f(y) \tag{3.1}$$

for all $x, y \in X$.

Remark 3.2. *If f is a mapping from a normed space X into a normed space Y over the same field with X such that it satisfies the linear functional equation, that is,*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta y$$

for all $x, y \in X$ and for all scalars α, β, then it also satisfies the generalized linear functional equation but the converse is not true.

Next, we give an example satisfying the generalized linear functional equation.

Example 3.3. *Let $X = Y = \mathbb{R}$ be a usual normed space over the same field and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = ax + b$ for all $x \in \mathbb{R}$, where a and b are fixed real numbers. Then f satisfies the generalized linear functional equation but f dose not satisfies linear functional equation.*

Throughout this work, if X, Y are two normed spaces, $f : X \to Y$ is a given mapping and $x, y \in X$, we will use the following symbol:

$$Df(x,y) := 2f(x+y) + f(x-y) + f(y-x) - 2f(x) - 2f(y).$$

Next, we will prove the stability result of the generalized linear functional equation which split into even and odd mappings as follows.

Theorem 3.4. *Let X and Y be two normed spaces and $\rho : X \times X \to [0, \infty)$ be a function such that there exists a real number L with $L < 1$ and*

$$\rho(x,y) \leq 2L\rho\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.2}$$

for all $x, y \in X$. Suppose that $f_e : X \to Y$ is an even mapping satisfying $f_e(0) = 0$ and

$$\|Df_e(x,y)\| \leq \rho(x,y) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique mapping $A_e : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_e(x) - A_e(x)\| \leq \frac{1}{4-4L}\rho(x,x) \tag{3.4}$$

for all $x \in X$.

Proof. Let V be the class of all mappings from X into Y and a function $d : V \to [0, \infty]$ be defined by

$$d(u, v) = \inf\{\beta \in \mathbb{R}^+ : \|u(x) - v(x)\| \leq \beta\rho(x, x),\ \forall x \in X\} \tag{3.5}$$

for all $u, v \in V$, where $\inf \emptyset = \infty$. Then (V, d) is a complete generalized metric space(see [**?**]).

Next, we define the linear mapping $J : V \to V$ by

$$(Ju)(x) = \frac{1}{2}u(2x) \text{ for all } x \in X \tag{3.6}$$

and for all $u \in V$. First, we claim that J is a contraction mapping. Let $u, v \in V$. If $d(u, v) = \infty$, we get

$$d(Ju, Jv) \leq Ld(u, v). \tag{3.7}$$

If $d(u, v) < \infty$, we will assume that

$$\Omega := \{\beta > 0 : \|u(x) - v(x)\| \leq \beta\rho(x, x) \text{ for all } x \in X\}. \tag{3.8}$$

Since $d(u, v) < \infty$, we obtain $\Omega \neq \emptyset$. Suppose that $\beta \in \Omega$. For each $x \in X$, we have

$$\|(Ju)(x) - (Jv)(x)\| = \left\|\frac{1}{2}u(2x) - \frac{1}{2}v(2x)\right\| \leq \frac{1}{2}\beta\rho(2x, 2x) \leq L\beta\rho(x, x). \tag{3.9}$$

It follows that

$$d(Ju, Jv) \leq L\beta. \tag{3.10}$$

By taking the infimum on β, we obtain

$$d(Ju, Jv) \leq Ld(x, y). \tag{3.11}$$

From (3.7) and (3.11), we conclude that

$$d(Ju, Jv) \leq Ld(u, v) \tag{3.12}$$

for all $u, v \in V$. This means that J is a contraction mapping.

Putting $y = x$ in (3.3), we obtain

$$\|2f_e(2x) - 4f_e(x)\| \leq \rho(x, x) \tag{3.13}$$

and hence

$$\left\|f_e(x) - \frac{1}{2}f_e(2x)\right\| \leq \frac{1}{4}\rho(x, x) \tag{3.14}$$

for all $x \in X$. It follows from (3.14) that

$$d(f_e, Jf_e) \leq \frac{1}{4} < \infty.$$

By Theorem 2.3, there exist a positive integer n_0 and a mapping $A_e : X \to Y$ satisfying the following conditions.

1. A_e is a unique fixed point of J, that is,

$$A_e(2x) = 2A_e(x) \tag{3.15}$$

for all $x \in X$. Also, the mapping A_e belongs to the set

$$P := \{v \in V \mid d(f_e, v) < \infty\}.$$

This implies that A_e is a unique mapping satisfying (3.15). Furthermore, there exists a $\beta \in (0, \infty)$ such that

$$\|f_e(x) - A_e(x)\| \leq \beta\rho(x, x)$$

for all $x \in X$.

2. $d(J^n f_e, A_e) \to 0$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \frac{1}{2^n} f_e(2^n x) = A_e(x) \tag{3.16}$$

for all $x \in X$.

3. $d(f_e, A_e) \leq \frac{1}{1-L} d(f_e, Jf_e)$, which implies the inequality

$$d(f_e, A_e) \leq \frac{1}{4 - 4L}.$$

This implies that the inequality (3.4) holds.
It follows from (3.2), (3.3) and (3.16) that

$$\begin{aligned} \|DA_e(x, y)\| &= \lim_{n\to\infty} \frac{1}{2^n} \|Df_e(2^n x, 2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \rho(2^n x, 2^n y) \\ &\leq \lim_{n\to\infty} L^n \rho(x, y) \\ &= 0 \end{aligned}$$

for all $x, y \in X$. This implies that

$$DA_e(x, y) = 0$$

for all $x, y \in X$, that is,

$$2A_e(x + y) + A_e(x - y) + A_e(y - x) = 2A_e(x) + 2A_e(y)$$

for all $x, y \in X$. Therefore, $A_e : X \to Y$ is a unique mapping satisfying the generalized linear functional equation and (3.4). □

Corollary 3.5. *Let X and Y be two normed spaces, $c > 0$ and $0 \leq p < 1$. Suppose that $f_e : X \to Y$ is an even mapping satisfying $f_e(0) = 0$ and*

$$\|Df_e(x, y)\| \leq c(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique mapping $A_e : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_e(x) - A_e(x)\| \leq \frac{c}{2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. Letting $\rho(x, y) = c(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and choosing $L = 2^{p-1}$ in Theorem 3.4, we obtain this result. □

Corollary 3.6. *Let X and Y be two normed spaces, $c > 0, p > 0$ and $q > 0$ with $p + q < 1$. Suppose that $f_e : X \to Y$ is an even mapping satisfying $f_e(0) = 0$ and*

$$\|Df_e(x, y)\| \leq c\|x\|^p\|y\|^p$$

for all $x, y \in X$. Then there exists a unique mapping $A_e : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_e(x) - A_e(x)\| \leq \frac{c}{4 - 2^{p+q+1}}\|x\|^{p+q}$$

for all $x \in X$.

Proof. Letting $\rho(x, y) = c\|x\|^p\|y\|^q$ for all $x, y \in X$ and choosing $L = 2^{p+q-1}$ in Theorem 3.4, we obtain this result. □

Corollary 3.7. *Let X and Y be two normed spaces, $c_1 > 0, c_2 > 0$ and $0 \leq p < 1$. Suppose that $f_e : X \to Y$ is an even mapping satisfying $f_e(0) = 0$ and*

$$\|Df_e(x, y)\| \leq c_1(\|x\|^p + \|y\|^p) + c_2\|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}$$

for all $x, y \in X$. Then there exists a unique mapping $A_e : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_e(x) - A_e(x)\| \leq \frac{(2c_1 + c_2)}{4 - 2^{p+1}}\|x\|^p$$

for all $x \in X$.

Proof. Letting $\rho(x, y) = c_1(\|x\|^p + \|y\|^p) + c_2\|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}$ for all $x, y \in X$ and choosing $L = 2^{p-1}$ in Theorem 3.4, we obtain this result. □

In the same way of the proof in Theorem 3.4, we can prove the following theorem.

Theorem 3.8. *Let X and Y be two normed spaces and $\rho : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\rho(x, y) \leq \frac{L}{2} \rho(2x, 2y)$$

for all $x, y \in X$. Suppose that $f_e : X \to Y$ is an even mapping satisfying $f_e(0) = 0$ and (3.3). Then there exists a unique mapping $A_e : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_e(x) - A_e(x)\| \leq \frac{L}{4 - 4L} \rho(x, x) \tag{3.17}$$

for all $x \in X$.

Proof. Let (V, d) be the generalized metric space defined in the proof of Theorem 3.4. Now we consider the linear mapping $J : V \to V$ such that

$$(Ju)(x) := 2u\left(\frac{x}{2}\right) \text{ for all } x \in X$$

and for all $u \in V$. The remain of the proof is similar to the proof of Theorem 3.4. □

Corollary 3.9. *Let X and Y be two normed spaces, $c > 0$ and $p > 1$. Suppose that $f_e : X \to Y$ is an even mapping satisfying $f_e(0) = 0$ and*

$$\|Df_e(x, y)\| \leq c(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique mapping $A_e : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_e(x) - A_e(x)\| \leq \frac{c}{2^p - 2} \|x\|^p$$

for all $x \in X$.

Proof. Letting $\rho(x, y) = c(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and choosing $L = 2^{1-p}$ in Theorem 3.8, we obtain this result. □

Corollary 3.10. *Let X and Y be two normed spaces, $c > 0, p > 0$ and $q > 0$ with $p + q > 1$. Suppose that $f_e : X \to Y$ is an even mapping satisfying $f_e(0) = 0$ and*

$$\|Df_e(x, y)\| \leq c\|x\|^p \|y\|^q$$

for all $x, y \in X$. Then there exists a unique mapping $A_e : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_e(x) - A_e(x)\| \leq \frac{c}{2^{p+q+1} - 4} \|x\|^{p+q}$$

for all $x \in X$.

Proof. Letting $\rho(x,y) = c\|x\|^p\|y\|^q$ for all $x, y \in X$ and choosing $L = 2^{1-(p+q)}$ in Theorem 3.8, we obtain this result. □

Corollary 3.11. *Let X and Y be two normed spaces, $c_1 > 0, c_2 > 0$ and $p > 1$. Suppose that $f_e : X \to Y$ is an even mapping satisfying $f_e(0) = 0$ and*

$$\|Df_e(x,y)\| \le c_1(\|x\|^p + \|y\|^p) + c_2\|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}$$

for all $x, y \in X$. Then there exists a unique mapping $A_e : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_e(x) - A_e(x)\| \le \frac{(2c_1 + c_2)}{2^{p+1} - 4}\|x\|^p$$

for all $x \in X$.

Proof. Letting $\rho(x,y) = c_1(\|x\|^p + \|y\|^p) + c_2\|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}$ for all $x, y \in X$ and choosing $L = 2^{1-p}$ in Theorem 3.8, we obtain this result. □

The condition of $f_e(0) = 0$ of the even function f_e in Theorem 3.4 is very important in the proof of such theorem but we can omit this condition in the case of the odd function. Next, we give the mentioned results.

Theorem 3.12. *Let X and Y be two normed spaces and $\rho : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\rho(x,y) \le 2L\rho\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f_o : X \to Y$ is an odd mapping satisfying

$$\|Df_o(x,y)\| \le \rho(x,y) \tag{3.18}$$

for all $x, y \in X$. Then there exists a unique mapping $A_o : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_o(x) - A_o(x)\| \le \frac{1}{4 - 4L}\rho(x,x)$$

for all $x \in X$.

Remark 3.13. *For letting $\rho(x,y) = c(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $c > 0$ and $p > 1$ or $\rho(x,y) = c\|x\|^p\|y\|^q$ for all $x, y \in X$ with $c, p, q > 0$ and $p + q > 1$ or $\rho(x,y) = c_1(\|x\|^p + \|y\|^p) + c_2\|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}$ for all $x, y \in X$ with $c_1, c_2 > 0$ and $p > 1$ in Theorem 3.4, we have results which are similar to Corollaries 3.5, 3.6 and 3.7.*

Theorem 3.14. *Let X and Y be two normed spaces and $\rho : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\rho(x,y) \le \frac{L}{2}\rho(2x, 2y)$$

for all $x, y \in X$. Suppose that $f_o : X \to Y$ is an odd mapping satisfying (3.18). Then there exists a unique mapping $A_o : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f_o(x) - A_o(x)\| \leq \frac{L}{4-4L}\rho(x,x)$$

for all $x \in X$.

Remark 3.15. *For letting $\rho(x,y) = c(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $c > 0$ and $p > 1$ or $\rho(x,y) = c\|x\|^p\|y\|^q$ for all $x, y \in X$ with $c, p, q > 0$ and $p + q > 1$ or $\rho(x,y) = c_1(\|x\|^p + \|y\|^p) + c_2\|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}$ for all $x, y \in X$ with $c_1, c_2 > 0$ and $p > 1$ in Theorem 3.8, we have results which are similar to Corollaries 3.9, 3.10 and 3.11.*

Let X, Y be two normed spaces and $f : X \to Y$ be a mapping. Define $f_e, f_o : X \to Y$ by

$$f_e(x) = \frac{f(x) + f(-x)}{2}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. Then f_e is and even mapping and f_o is an odd mapping such that $f = f_e + f_o$.

In order to obtain the stability results of the generalized linear functional equation concerning an even mapping and an odd mapping, we also need the next lemma.

Lemma 3.16. *If f is a mapping from a normed space X into a normed space Y and f satisfies*

$$\|Df(x,y)\| \leq \rho(x,y)$$

for all $x, y \in X$, then an even mapping $f_e : X \to Y$ satisfies

$$\|Df_e(x,y)\| \leq \rho(x,y)$$

for all $x, y \in X$, where $f_e(x) = \frac{f(x)+f(-x)}{2}$.

Proof. For each $x, y \in X$, we have $\|2f_e(x+y) + f_e(x-y) + f_e(y-x) - 2f_e(x) -$

$2f_e(y)\|$

$$\begin{aligned}
&= \left\|2\left[\frac{f(x+y)-f(-(x+y))}{2}\right]+\frac{f(x-y)-f(-(x-y))}{2}+\frac{f(y-x)-f(-(y-x))}{2}\right.\\
&\quad \left.-2\left[\frac{f(x)-f(-x)}{2}\right]-2\left[\frac{f(y)-f(-y)}{2}\right]\right\|\\
&\leq \left\|f(x+y)+\frac{f(x-y)}{2}+\frac{f(y-x)}{2}-f(x)-f(y)\right\|\\
&\quad +\left\|f(-x-y)+\frac{f(-x+y)}{2}+\frac{f(-y+x)}{2}-f(-x)-f(-y)\right\|\\
&\leq \frac{\rho(x,y)}{2}+\frac{\rho(x,y)}{2}\\
&\leq \rho(x,y).
\end{aligned}$$

□

By using the same technique in the previous lemma, we obtain the next result.

Lemma 3.17. *If f is a mapping from a normed space X into a normed space Y and f satisfies*

$$\|Df(x,y)\| \leq \rho(x,y)$$

for all $x, y \in X$, then an odd mapping $f_o : X \to Y$ satisfies

$$\|Df_o(x,y)\| \leq \rho(x,y)$$

for all $x, y \in X$, where $f_o(x) = \frac{f(x)-f(-x)}{2}$.

From Theorems 3.4, 3.12 and Lemma 3.16 with the fact that $f = f_o + f_e$, we get the following result.

Theorem 3.18. *Let X and Y be two normed spaces and $\rho : X \times X \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\rho(x,y) \leq 2L\rho\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|Df(x,y)\| \leq \rho(x,y)$$

for all $x, y \in X$. Then there exist mappings $A_e, A_o : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f(x) - A_e(x) - A_o(x)\| \leq \frac{1}{2-2L}\rho(x,x)$$

for all $x \in X$.

From Theorems 3.8, 3.14 and Lemma 3.17 with the fact that $f = f_o + f_e$, we get the following result.

Theorem 3.19. *Let X and Y be two normed spaces and $\rho : X \times X \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\rho(x,y) \leq \frac{L}{2}\rho(2x,2y)$$

for all $x, y \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|Df(x,y)\| \leq \rho(x,y)$$

for all $x, y \in X$. Then there exist mappings $A_e, A_o : X \to Y$ satisfying the generalized linear functional equation (3.1) and

$$\|f(x) - A_e(x) - A_o(x)\| \leq \frac{L}{2-2L}\rho(x,x)$$

for all $x \in X$.

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