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Best Proximity Point Theorems for Suzuki Type Proximal Contractive Multimaps

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Abstract : The purpose of this manuscript is to establish new best proximity point results for Suzuki type proximal contractive multimaps. Our results extend some recent known results by Hussain *et al.* [11] as well as other results in the literature. An illustrative example is provided to highlight our main results.

Keywords : proximal contractive multivalued mapping; best proximity point; fixed point; Suzuki type proximal contraction
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1 Introduction and Preliminaries

The background concept on best proximity point results and related fixed point theory in (ordered) metric spaces, Banach spaces and fuzzy metric spaces

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is very abundant in the literature; (see, for instance, [12],[13], [14], [16], [18], [19]) and references therein.

Let A and B be two nonempty subsets of a metric space (X, d). An element $x \in A$ is said to be a fixed point of a given map $T : A \to B$ if $Tx = x$. Clearly, $T(A) \cap A \neq \emptyset$ is a necessary(but not sufficient) condition for the existence of a fixed point of T. If $T(A) \cap A = \emptyset$, then $d(x, Tx) > 0$ for all $x \in A$, that is, the set of fixed point of T is empty. In a such situation, one often attempts to find an element x which is in some sense closed to Tx. Best approximation theory and best proximity point analysis have been developed in this direction. Recently, Jleli and Samet [14] introduced the notion of α-ψ-proximal contractive type mappings and established some best proximity point theorems. Many authors obtained best proximity point theorems in different settings; (see [1],[2],[8],[9],[10], [15],[20], [21],[23], for examples). Abkar and Gbeleh [4], Al-Thagafi and Shazad ([5],[6]), Ali *et al.* [7], Xu and Fan [24] investigated best proximity points for multivalued mappings.

Later, Hussain *et al.* [11] introduced new Suzuki and convex type contractions and established new best proximity results for those contractions in the setting of a metric space.

Let (X, d) be a metric space. For $A, B \subset X$, we use the following notations subsequently:

$$\begin{aligned} d(A, B) &= \inf \{d(a, b) : a \in A, b \in B\}, \\ D(x, B) &= \inf \{d(x, b) : b \in B\}, \\ A_0 &= \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}, \\ B_0 &= \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}, \end{aligned}$$

$2^X \backslash \emptyset$ is the set of all nonempty subsets of X, $\mathrm{CL}(X)$ is the set of all nonempty closed subsets of X, and $\mathrm{K}(X)$ is the set of all nonempty compact subsets of X. For every $A, B \in \mathrm{CL}(X)$, let

$$H(A, B) = \begin{cases} \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{1.1}$$

Such a map H is called the generalized Hausdorff metric induced by d. A point $x^* \in X$ is said to be the best proximity point of a mapping $T : A \to B$ if $d(x^*, Tx^*) = d(A, B)$. When $A = B$, the best proximity point is essentially the fixed point of the mapping T. We review the following essential definitions.

Definition 1.1 (see [22]). *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if, for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,*

$$\left. \begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned} \right\} \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2). \tag{1.2}$$

Let Ψ denote the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following properties:

(a) ψ is monotone nondecreasing;

(b) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$.

It is well-known that $\psi(t) < t$ for all $t > 0$.

Let Θ denote the set of all functions $\theta : (0,\infty) \to [1,\infty)$ with the following conditions:

(a) θ is increasing;

(b) for all sequences $\{\alpha_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} \theta(\alpha_n) = 1$;

(c) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \dfrac{\theta(t)-1}{t^r} = l$.

Definition 1.2 (see [11])**.** *Let A and B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is called α^+-proximal admissible if there exists a mapping $\alpha : A \times A \to [-\infty,\infty)$ such that*

$$\left.\begin{array}{l} \alpha(x_1,x_2) \geq 0 \\ d(u_1,Tx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1,u_2) \geq 0 \tag{1.3}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 1.3 (see [11])**.** *The mapping $T : A \to B$ is called a Suzuki type α^+ ψ-proximal contraction, if there exists a mapping $\alpha : A \times A \to [-\infty,\infty)$ such that*

$$\frac{1}{2}d^*(x,Tx) \leq d(x,y) \Rightarrow \alpha(x,y) + d(Tx,Ty) \leq \psi(\mathcal{M}(x,y)) \tag{1.4}$$

for all $x, y \in A$, where $d^*(x,Tx) = d(x,Tx) - d(A,B), \psi \in \Psi$, and

$$\mathcal{M}(x,y) = \max\Big\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2} - d(A,B), \frac{d(x,Ty)+d(y,Tx)}{2} - d(A,B)\Big\}.$$

Definition 1.4 (see [11])**.** *A mapping $T : A \to B$ is called a Suzuki type $\alpha^+\theta$-proximal contraction, if for all $x, y \in A$ with $\frac{1}{2}d^*(x,Tx) \leq d(x,y)$ and $d(Tx,Ty) > 0$,*

$$\Rightarrow \alpha(x,y) + \theta\big(d(Tx,Ty)\big) \leq \big[\theta\big(\mathcal{M}(x,y)\big)\big]^\beta \tag{1.5}$$

where $\alpha : A \times A \to [-\infty,\infty), 0 \leq \beta < 1$, and $\theta \in \Theta$.

The main results of Hussain *et al.* in [11] are the following.

Theorem 1.5 (see [11]). *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ satisfy* (1.4) *together with the following assertions:*

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;*

(ii) *T is an α^+-proximal admissible map;*

(iii) *there exist elements x_0, x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A,B)\ , \quad \alpha(x_0, x_1) \geq 0 \tag{1.6}$$

(iv) *T is continuous, or*

(v) *A is α-regular, that is, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 0$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \geq 0$ for all $n \in N$.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A,B)$.

Theorem 1.6 (see [11]). *Let X, A, A_0, and B be as Theorem 1.5. Assume that $T : A \to B$ satisfies* (1.4) *and the assertions (i)-(v) in Theorem 1.5 and*

$$\alpha(p,q) + d(Tp, Tq) \leq \psi(\mathcal{M}(p,q)) \tag{1.7}$$

holds for all $p, q \in A$. Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A,B)$.

Theorem 1.7 (see [11]). *Let X, A, A_0, and B be as Theorem 1.5. Assume that $T : A \to B$ satisfies* (1.5) *and the assertions (i)-(v) in Theorem 1.5. Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A,B)$.*

Definition 1.8 (see [3]). *An element $x^* \in A$ is said to be the best proximity point of a multivalued nonself mapping T, if $D(x^*, Tx^*) = d(A,B)$.*

Inspired and motivated by the results of Hussain *et al.* in [11] and by those of Ali *et al.* in [7], we establish the best proximity point results for Suzuki type proximal contractive multimaps. Our results extend the recent results of Hussain *et al.* [11] to the best proximity point results for nonself multivalued mappings. We also give an illustrative example to support our main results.

2 Main Results

We begin this section by introducing the following definitions.

Definition 2.1. *Let A and B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to 2^B \backslash \emptyset$ is called an α^+-proximal admissible multimap, if there exists a mapping $\alpha : A \times A \to [-\infty, \infty)$ such that*

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 0 \\ D(u_1, Tx_1) = d(A,B) \\ D(u_2, Tx_2) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq 0 \tag{2.1}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 2.2. *Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to CL(B)$ is called a Suzuki type $\alpha^+\psi$-proximal contractive multimap, if there exists a mapping $\alpha : A \times A \to [-\infty, \infty)$ such that*

$$\frac{1}{2}D^*(x, Tx) \leq d(x, y) \Rightarrow \alpha(x, y) + H(Tx, Ty) \leq \psi(\mathcal{M}(x, y)) \tag{2.2}$$

for all $x, y \in A$, where $D^*(x, Tx) = D(x, Tx) - d(A, B), \psi \in \Psi$, and

$$\mathcal{M}(x, y) = \max\Big\{d(x, y), \frac{D(x, Tx) + D(y, Ty)}{2} - d(A, B), \frac{D(x, Ty) + D(y, Tx)}{2} - d(A, B)\Big\}.$$

Definition 2.3. *A mapping $T : A \to CL(B)$ is called a Suzuki type $\alpha^+\theta$-proximal contractive multimap, if for all $x, y \in A$ with $\frac{1}{2}D^*(x, Tx) \leq d(x, y)$ and $H(Tx, Ty) > 0$,*

$$\Rightarrow \alpha(x, y) + \theta\big(H(Tx, Ty)\big) \leq \big[\theta\big(\mathcal{M}(x, y)\big)\big]^\beta \tag{2.3}$$

where $\alpha : A \times A \to [-\infty, \infty), 0 \leq \beta < 1, \theta \in \Theta$, and

$$\mathcal{M}(x, y) = \max\Big\{d(x, y), \frac{D(x, Tx) + D(y, Ty)}{2} - d(A, B), \frac{D(x, Ty) + D(y, Tx)}{2} - d(A, B)\Big\}.$$

The following are our main results.

Theorem 2.4. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [-\infty, \infty)$ and let $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : A \to \mathrm{CL}(B)$ is a mapping satisfying(2.2) and the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*

(ii) *T is an α^+-proximal admissible multimap;*

(iii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$D(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 0; \tag{2.4}$$

(iv) *T is continuous, or*

(v) *A is α-regular, that is, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 0$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \geq 0$ for all n.*

Then there exists an element $x^* \in A_0$ such that

$$D\big(x^*, Tx^*\big) = d(A, B).$$

Proof. Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$D(x_2, Tx_1) = d(A, B). \tag{2.5}$$

As T satisfies (iii) and is α^+-proximal admissible, we obtain $\alpha(x_1, x_2) \geq 0$. That is

$$D(x_2, Tx_1) = d(A, B), \quad \alpha(x_1, x_2) \geq 0. \tag{2.6}$$

Again, since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$D(x_3, Tx_2) = d(A, B). \tag{2.7}$$

Therefore we have

$$D(x_2, Tx_1) = d(A, B),\ D(x_3, Tx_2) = d(A, B),\ \alpha(x_1, x_2) \geq 0. \tag{2.8}$$

Again, since T is α^+-proximal admissible, we obtain $\alpha(x_2, x_3) \geq 0$. Hence, we have

$$D(x_3, Tx_2) = d(A, B), \quad \alpha(x_2, x_3) \geq 0. \tag{2.9}$$

Continuing this method, we get

$$D(x_{n+1}, Tx_n) = d(A, B),\ \alpha(x_n, x_{n+1}) \geq 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.10}$$

From (2.10), definition of D^* and triangle inequality, we can write

$$\begin{aligned} \frac{1}{2} D^*\big(x_{n-1}, Tx_{n-1}\big) &= \frac{1}{2}\Big(D(x_{n-1}, Tx_{n-1}) - d(A, B)\Big) \\ &\leq \frac{1}{2}\Big(d(x_{n-1}, x_n) + D(x_n, Tx_{n-1}) - d(A, B)\Big) \\ &= \frac{1}{2} d(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n). \end{aligned} \tag{2.11}$$

That is

$$\frac{1}{2} D^*\big(x_{n-1}, Tx_{n-1}\big) \leq d(x_{n-1}, x_n). \tag{2.12}$$

From (2.2), we get

$$H(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n) + H(Tx_{n-1}, Tx_n) \leq \psi(\mathcal{M}(x_{n-1}, x_n)). \tag{2.13}$$

By using (2.10), triangle inequality and the P-property, we obtain

$$\begin{aligned}
\mathcal{M}(x_{n-1},x_n) &= \max\Big\{d(x_{n-1},x_n), \frac{D(x_{n-1},Tx_{n-1})+D(x_n,Tx_n)}{2}-d(A,B),\\
&\qquad \frac{D(x_{n-1},Tx_n)+D(x_n,Tx_{n-1})}{2}-d(A,B)\Big\}\\
&\le \max\Big\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+D(x_n,Tx_{n-1})+d(x_n,x_{n+1})+D(x_{n+1},Tx_n)}{2}-d(A,B),\\
&\qquad \frac{d(x_{n-1},x_n)+D(x_n,Tx_{n-1})+d(Tx_{n-1},Tx_n)+D(x_n,Tx_{n-1})}{2}-d(A,B)\Big\}\\
&= \max\Big\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(A,B)+d(x_n,x_{n+1})+d(A,B)}{2}-d(A,B),\\
&\qquad \frac{d(x_{n-1},x_n)+d(A,B)+d(Tx_{n-1},Tx_n)+d(A,B)}{2}-d(A,B)\Big\}\\
&= \max\Big\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(A,B)+d(x_n,x_{n+1})+d(A,B)}{2}-d(A,B),\\
&\qquad \frac{d(x_{n-1},x_n)+d(A,B)+d(x_n,x_{n+1})+d(A,B)}{2}-d(A,B)\Big\}\\
&= \max\Big\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\Big\}\\
&\le \max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1})\Big\}.
\end{aligned} \tag{2.14}$$

Since (A,B) satisfies the P-property, we obtain

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n)\\
&\le H(Tx_{n-1},Tx_n)\\
&\le \alpha(x_{n-1},x_n)+H(Tx_{n-1},Tx_n)\\
&\le \psi\Big(\mathcal{M}(x_{n-1},x_n)\Big) \text{ for all } n\in\mathbb{N}.
\end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we have

$$\begin{aligned}
d(x_n,x_{n+1}) &\le \psi\Big(\mathcal{M}(x_{n-1},x_n)\Big)\\
&\le \psi\Big(\max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1})\Big\}\Big) \text{ for all } n\in\mathbb{N}.
\end{aligned} \tag{2.16}$$

If $x_{n_0}=x_{n_0+1}$, for some $n_0\in\mathbb{N}$, from (2.10), we obtain

$$D(x_{n_0},Tx_{n_0}) = D(x_{n_0+1},Tx_{n_0}) = d(A,B),$$

This means x_{n_0} is a best proximity point of T. Therefore, we suppose that

$$d(x_{n+1},x_n) > 0 \text{ for all } n\in\mathbb{N}\cup\{0\}. \tag{2.17}$$

If $\max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1})\Big\} = d(x_n,x_{n+1})$, then (2.16) implies

$$d(x_n,x_{n+1}) \le \psi\Big(d(x_n,x_{n+1})\Big) < d(x_n,x_{n+1}), \tag{2.18}$$

which is a contradiction. Therefore,

$$d(x_n,x_{n+1}) \le \psi\Big(\mathcal{M}(x_{n-1},x_n)\Big) \le \psi\Big(d(x_{n-1},x_n)\Big) \text{ for all } n \in \mathbb{N}. \tag{2.19}$$

By the monotonicity of ψ and by induction, it follows from (2.19) that

$$d(x_n,x_{n+1}) \le \psi^n\Big(d(x_0,x_1)\Big) \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \tag{2.20}$$

Suppose ϵ is any positive real number. There exists $N \in \mathbb{N}$ such that

$$\sum_{n\ge N} \psi^n\big(d(x_0,x_1)\big) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

If $m,n \in \mathbb{N}$ with $m > n \ge N$. By the triangle inequality, we have

$$\begin{aligned} d(x_n,x_m) &\le \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \\ &\le \sum_{n\ge N} \psi^n\big(d(x_0,x_1)\big) < \epsilon. \end{aligned}$$

Consequently $\lim_{m,n\to\infty} d(x_n,x_m) = 0$, which implies $\{x_n\}$ is a Cauchy sequence. Since X is complete, $x_n \to x^* \in X$. If (iv) holds, Then $Tx_n \to Tx^*$ as $n \to \infty$ and

$$d(A,B) = \lim_{n\to\infty} D(x_{n+1},Tx_n) = D(x^*,Tx^*).$$

Hence x^* is the best proximity point of T.

Next, assume that (v) holds. Then $\alpha(x_n,x^*) \ge 0$. If the following inequalities hold:

$$\frac{1}{2}D^*(x_n,Tx_n) > d(x_n,x^*) \ \text{ and } \ \frac{1}{2}D^*(x_{n+1},Tx_{n+1}) > d(x_{n+1},x^*),$$

for some $n \in \mathbb{N}$, then by the triangle inequality, (2.10) and definition of D^*, we obtain

$$\begin{aligned} d(x_n,x_{n+1}) &\le d(x_n,x^*) + d(x^*,x_{n+1}) \\ &< \frac{1}{2}\left[D^*(x_n,Tx_n) + D^*(x_{n+1},Tx_{n+1})\right] \\ &= \frac{1}{2}\left[D(x_n,Tx_n) + D(x_{n+1},Tx_{n+1}) - 2d(A,B)\right] \\ &\le \frac{1}{2}\left[d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})\right] \\ &\le d(x_n,x_{n+1}), \end{aligned}$$

which is a contradiction. Consequently, for any $n \in \mathbb{N}$, either

$$\frac{1}{2}D^*(x_n, Tx_n) \le d(x_n, x^*) \quad \text{or} \quad \frac{1}{2}D^*(x_{n+1}, Tx_{n+1}) \le d(x_{n+1}, x^*)$$

holds.

Then, we may choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{2}D^*(x_{n_k}, Tx_{n_k}) \le d(x_{n_k}, x^*) \quad \text{and} \quad \alpha(x_{n_k}, x_{n_k+1}) \ge 0,$$

for all $k \in \mathbb{N}$. By (2.2), we have

$$H(Tx_{n_k}, Tx^*) \le \alpha(x_{n_k}, x^*) + H(Tx_{n_k}, Tx^*) \le \psi\big(\mathcal{M}(x_{n_k}, x^*)\big). \tag{2.21}$$

Observe that

$$\begin{aligned}
\mathcal{M}(x_{n_k}, x^*) &= \max\Big\{d(x_{n_k}, x^*), \frac{D(x_{n_k}, Tx_{n_k}) + D(x^*, Tx^*)}{2} - d(A,B), \\
&\qquad \frac{D(x_{n_k}, Tx^*) + D(x^*, Tx_{n_k})}{2} - d(A,B)\Big\} \\
&\le \max\Big\{d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + D(x_{n_k+1}, Tx_{n_k}) + D(x^*, Tx^*)}{2} - d(A,B), \\
&\qquad \frac{d(x_{n_k}, x^*) + D(x^*, Tx^*) + d(x^*, x_{n_k+1}) + D(x_{n_k+1}, Tx_{n_k})}{2} - d(A,B)\Big\} \\
&= \max\Big\{d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + d(A,B) + D(x^*, Tx^*)}{2} - d(A,B), \\
&\qquad \frac{d(x_{n_k}, x^*) + D(x^*, Tx^*) + d(x^*, x_{n_k+1}) + d(A,B)}{2} - d(A,B)\Big\}.
\end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x^*) \ \le \ \frac{D(x^*, Tx^*) - d(A,B)}{2}. \tag{2.22}$$

Further,

$$\begin{aligned}
D(x^*, Tx^*) \ &\le \ d(x^*, x_{n_{k+1}}) + D(x_{n_k+1}, Tx_{n_k}) + d(Tx_{n_k}, Tx^*) \\
&= \ d(x^*, x_{n_k+1}) + d(A,B) + d(Tx_{n_k}, Tx^*),
\end{aligned}$$

which gives

$$D(x^*, Tx^*) - d(x^*, x_{n_k+1}) - d(A,B) \ \le \ d(Tx_{n_k}, Tx^*). \tag{2.23}$$

Taking the limit as $k \to \infty$ in (2.23), we obtain

$$D(x^*, Tx^*) - d(A,B) \ \le \ \lim_{k\to\infty} d(Tx_{n_k}, Tx^*). \tag{2.24}$$

Therefore, from (2.21), (2.22) and (2.24), we have

$$\begin{aligned} D(x^*,Tx^*)-d(A,B) &\leq \lim_{k\to\infty} d(Tx_{n_k},Tx^*) \\ &\leq \lim_{k\to\infty} H(T_{n_k},Tx^*) \\ &\leq \psi\big(\lim_{k\to\infty} \mathcal{M}(x_{n_k},x^*)\big) \\ &\leq \psi\left(\frac{D(x^*,Tx^*)-d(A,B)}{2}\right). \end{aligned}$$

Now, if $D(x^*,Tx^*)-d(A,B)>0$, we have that

$$D(x^*,Tx^*)-d(A,B) < \frac{D(x^*,Tx^*)-d(A,B)}{2}.$$

This is a contradiction. Hence, $D(x^*,Tx^*)=d(A,B)$. Therefore x^* is the best proximity point of T. □

Theorem 2.5. *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha: A\times A\to[-\infty,\infty)$ and let $T:A\to$ $\mathrm{K}(B)$ be a mapping satisfying (2.2) and the following conditions:*

(i) *$T(A_0)\subseteq B_0$ for each $x\in A_0$ and (A,B) satisfies the P-property;*

(ii) *T is an α^+-proximal admissible multimap;*

(iii) *there exist elements x_0,x_1 in A_0 such that*

$$D(x_1,Tx_0)=d(A,B), \quad \alpha(x_0,x_1)\geq 0; \tag{2.25}$$

(iv) *T is continuous, or*

(v) *A is α-regular, that is, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\geq 0$ and $x_n\to x\in A$ as $n\to\infty$, then $\alpha(x_n,x)\geq 0$ for all n.*

Then there exists an element $x^*\in A_0$ such that

$$D\big(x^*,Tx^*\big)=d(A,B).$$

The following result can be deduced easily from Theorem 2.4.

Theorem 2.6. *Let X,A,A_0 and B be the same as in Theorem 2.4. Assume that $T:A\to CL(B)$ satisfies the conditions (i)-(v) in Theorem 2.4 and*

$$\alpha(p,q)+H(Tp,Tq)\leq\psi\big(\mathcal{M}(p,q)\big)$$

holds for all $p,q\in A$. Then there exists an element $x^*\in A_0$ such that

$$D\big(x^*,Tx^*\big)=d(A,B).$$

Therefore we also obtain the following theorem.

Theorem 2.7. *Let X, A, A_0 and B be the same as in Theorem 2.4. Assume that $T : A \to K(B)$ satisfies the conditions (i)-(v) in Theorem 2.4 and*

$$\alpha(p,q) + H(Tp,Tq) \le \psi\big(\mathcal{M}(p,q)\big)$$

holds for all $p,q \in A$. Then there exists an element $x^* \in A_0$ such that

$$D\big(x^*, Tx^*\big) = \mathrm{dist}(A,B).$$

Theorem 2.8. *Let X, A, A_0, and B be as in Theorem 2.4. Assume that $T : A \to CL(B)$ satisfies (2.3) and the following assertions:*

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;*

(ii) *T is an α^+-proximal admissible multimap;*

(iii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$D(x_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0,x_1) \ge 0; \tag{2.26}$$

(iv) *T is continuous, or*

(v) *A is α-regular, that is, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \ge 0$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \ge 0$ for all n.*

Then there exists an element $x^* \in A_0$ such that

$$D(x^*, Tx^*) = d(A,B).$$

Proof. As in the proof of Theorem 2.4, we can construct a sequence $\{x_n\}$ satifying

$$D(x_{n+1}, Tx_n) = d(A,B), \tag{2.27}$$

and

$$\frac{1}{2}D^*\big(x_{n-1}, Tx_{n-1}\big) \le d(x_{n-1}, x_n) \text{ and } \alpha(x_{n-1}, x_n) \ge 0 \quad \text{for all } n \in \mathbb{N}.$$

Now (2.3) implies

$$\theta\big(H(Tx_{n-1}, Tx_n)\big) \le \alpha(x_{n-1}, x_n) + \theta\big(H(Tx_{n-1}, Tx_n)\big) \le \big[\theta\big(\mathcal{M}(x_{n-1}, x_n)\big)\big]^{\beta}. \tag{2.28}$$

From Theorem 2.4, we obtain

$$\mathcal{M}(x_{n-1}, x_n) \le \max\big\{d(x_{x-1}, x_n), d(x_n, x_{n+1})\big\} \tag{2.29}$$

and

$$d(x_n, x_{n+1}) \le H(Tx_{n-1}, Tx_n) \quad \text{for all } n \in \mathbb{N}.$$

Therefore from (2.28) and (2.29), we get

$$\begin{aligned}\theta\big(d(x_n,x_{n+1})\big) &\le \theta\big(H(Tx_{n-1},Tx_n)\big)\\ &\le \big[\theta\big(\mathcal{M}(x_{n-1},x_n)\big)\big]^{\beta}\\ &\le \big[\theta\big(\max\big\{d(x_{x-1},x_n),d(x_n,x_{n+1})\big\}\big)\big]^{\beta} \text{ for all } n\in\mathbb{N}.\end{aligned} \tag{2.30}$$

Now if $\max\big\{d(x_{n-1},x_n),d(x_n,x_{n+1})\big\}=d(x_n,x_{n+1})$, then from (2.30) we get

$$\theta\big(d(x_x,x_{n+1})\big)\le\big[\theta\big(d(x_n,x_{n+1})\big)\big]^{\beta}<\theta\big(d(x_n,x_{n+1})\big),$$

which is a contradiction. Therefore, we have

$$\theta\big(d(x_x,x_{n+1})\big)\le\big[\theta\big(d(x_{n-1},x_n)\big)\big]^{\beta}\quad\text{for all } n\in\mathbb{N}. \tag{2.31}$$

Therefore from (2.31), we get

$$\begin{aligned}1\le\theta\big(d(x_x,x_{n+1})\big) &\le \big(\theta\big(d(x_{n-1},x_n)\big)\big)^{\beta}\\ &\le \big(\big(\theta\big(d(x_{n-2},x_{n-1})\big)^{\beta}\big)^{\beta}\\ &\cdots\\ &\le \big(\theta\big(d(x_0,x_1)\big)\big)^{\beta^n}.\end{aligned} \tag{2.32}$$

Letting $n\to\infty$ in (2.32), we obtain

$$\lim_{n\to\infty}\theta\big(d(x_n,x_{n+1})\big)=1, \tag{2.33}$$

and since $\theta\in\Theta$, we have

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0. \tag{2.34}$$

Again since $\theta\in\Theta$, there exists $0<r<1$ and $0<l\le\infty$ with

$$\lim_{n\to\infty}\frac{\theta\big(d(x_n,x_{n+1})\big)-1}{\big[d(x_n,x_{n+1})\big]^r}=l. \tag{2.35}$$

Assume that $l<\infty$. Let $C=\frac{l}{2}$. Thus there exists $n_0\in\mathbb{N}$ such that

$$\Big|\frac{\theta\big(d(x_n,x_{n+1})\big)-1}{\big[d(x_n,x_{n+1})\big]^r}-l\Big|\le C\quad\text{for all } n\ge n_0.$$

Hence

$$\frac{\theta\big(d(x_n,x_{n+1})\big)-1}{\big[d(x_n,x_{n+1})\big]^r}\ge l-C=C\quad\text{for all } n\ge n_0,$$

and also

$$n\big[d(x_n,x_{n+1})\big]^r\le nK\big[\theta\big(d(x_n,x_{n+1})\big)-1\big]\quad\text{for all } n\ge n_0,$$

where $K = \dfrac{1}{C}$. If $l = \infty$, then there exists $n_0 \in \mathbb{N}$,

$$\frac{\theta\big(d(x_n, x_{n+1})\big) - 1}{\big[d(x_n, x_{n+1})\big]^r} \geq C \quad \text{for all } n \geq n_0,$$

which implies

$$n\big[d(x_n, x_{n+1})\big]^r \leq nK\big[\theta\big(d(x_n, x_{n+1})\big) - 1\big] \quad \text{for all } n \geq n_0,$$

where $K = \dfrac{1}{C}$. Hence, in all cases there exist $K > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\big[d(x_n, x_{n+1})\big]^r \leq nK\big[\theta\big(d(x_n, x_{n+1})\big) - 1\big] \quad \text{for all } n \geq n_0. \tag{2.36}$$

From (2.33) and (2.36), letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} n\big[d(x_n, x_{n+1})\big]^r = 0. \tag{2.37}$$

It follows from (2.37) that there is $n_1 \in \mathbb{N}$ with

$$n\big[d(x_n, x_{n+1})\big]^r \leq 1 \quad \text{for all } n > n_1.$$

This implies

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}} \quad \text{for all } n > n_1.$$

If $m > n > n_1$, then by the triangle inequality, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1})\big) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.$$

Since $0 < r < 1, \sum_{i=n}^{\infty} \dfrac{1}{i^{1/r}} < \infty$. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$, which means that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Suppose that (iv) holds. Thus $Tx_n \to Tx^*$ as $n \to \infty$, which implies

$$d(A, B) = \lim_{n\to\infty} \big(D(x_{n+1}, Tx_n)\big) = D(x^*, Tx^*),$$

as required. Next, assume that (v) holds. As in the proof of Theorem 2.4, we can deduce that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\frac{1}{2}D^*(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, x^*) \text{ and } \alpha(x_{n_k}, x_{n_k+1}) \geq 0,$$

for all $k \in \mathbb{N}$. By (2.3) we have

$$\theta\big(H(Tx_{n_k}, Tx^*)\big) \leq \big[\theta\big(\mathcal{M}(x_{n_k}, x^*)\big)\big]^\beta < \theta\big(\mathcal{M}(x_{n_k}, x^*)\big),$$

where $0 \leq \beta < 1, \theta \in \Theta$. This implies

$$H(Tx_{n_k}, Tx^*) \leq \mathcal{M}(x_{n_k}, x^*). \tag{2.38}$$

As in Theorem 2.4, we obtain

$$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x^*) \ \leq \ \frac{D(x^*, Tx^*) - d(A,B)}{2} \tag{2.39}$$

and

$$D(x^*, Tx^*) - d(A,B) \ \leq \ \lim_{k\to\infty} d(Tx_{n_k}, Tx^*). \tag{2.40}$$

If $D(x^*, Tx^*) - d(A,B) > 0$, therefore from (2.38), (2.39) and (2.40), we obtain

$$D(x^*, Tx^*) - d(A,B) \ \leq \ \frac{D(x^*, Tx^*) - d(A,B)}{2},$$

which is a contradiction. Therefore $D(x^*, Tx^*) = d(A,B)$, as required. □

Theorem 2.9. *Let X, A, A_0, and B be as Theorem 2.4. Assume that $T : A \to K(B)$ satisfies* (2.3) *and the assertions (i)-(v) in Theorem 2.8. Then there exists an element $x^* \in A_0$ such that*

$$D(x^*, Tx^*) = d(A,B).$$

Theorem 2.10. *Let X, A, A_0 and B be the same as in Theorem 2.4. Assume that $T : A \to CL(B)$ satisfies the conditions (i)-(v) in Theorem 2.8 and*

$$\alpha(p,q) + \theta\big(H(Tp,Tq)\big) \leq \big[\theta\big(\mathcal{M}(p,q)\big)\big]^{\beta}$$

holds for all $p, q \in A$, where $\alpha : A \times A \to [-\infty, \infty), 0 \leq \beta < 1, \theta \in \Theta$. Then there exists an element $x^* \in A_0$ such that

$$D\big(x^*, Tx^*\big) = d(A,B).$$

Theorem 2.11. *Let X, A, A_0 and B be the same as in Theorem 2.4. Assume that $T : A \to K(B)$ satisfies the conditions (i)-(v) in Theorem 2.8 and*

$$\alpha(p,q) + \theta\big(H(Tp,Tq)\big) \leq \big[\theta\big(\mathcal{M}(p,q)\big)\big]^{\beta}$$

holds for all $p, q \in A$, where $\alpha : A \times A \to [-\infty, \infty), 0 \leq \beta < 1, \theta \in \Theta$. Then there exists an element $x^* \in A_0$ such that

$$D\big(x^*, Tx^*\big) = d(A,B).$$

Example 2.12. *Let $X = [0,\infty) \times [0,\infty)$ be a product space endowed with the usual metric d. Suppose that $A = \big\{(\frac{1}{2}, x) : 0 \leq x < \infty\big\}$ and $B = \{(0,x) : 0 \leq x < \infty\}$.*

Define $T : A \to \mathrm{CL}(B)$ *by*

$$T\left(\tfrac{1}{2}, a\right) = \begin{cases} \left\{(0, \frac{x}{2}) : 0 \le x \le a\right\} & \text{if } a \le 1 \\ \left\{(0, x^2) : 0 \le x \le a^2\right\} & \text{if } a > 1, \end{cases} \tag{2.41}$$

and define $\alpha : A \times A \to [-\infty, \infty)$ *by*

$$\alpha(x, y) = \begin{cases} 0 & \text{if } x, y \in \left\{(\frac{1}{2}, a) : 0 \le a \le 1\right\} \\ -\infty & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \dfrac{t}{2}$ *for all* $t \ge 0$. *Note that* $A_0 = A, B_0 = B$, *and* $Tx \subseteq B_0$ *for each* $x \in A_0$. *Also, the pair* (A, B) *satisfies the* P*-property. For* $x_0, x_1 \in \left\{(\frac{1}{2}, x) : 0 \le x \le 1\right\}$; *then* $Tx_0, Tx_1 \subseteq \left\{(0, \frac{x}{2}) : 0 \le x \le 1\right\}$. *Consider* $u_1 \in Tx_0, u_2 \in Tx_1$ *and* $w_1, w_2 \in A$ *such that* $d(w_1, u_1) = d(A, B)$ *and* $d(w_2, u_2) = d(A, B)$. *Then we have* $w_1, w_2 \in \left\{(\frac{1}{2}, x) : 0 \le x \le \frac{1}{2}\right\}$, *so* $\alpha(w_1, w_2) = 0$. *Therefore,* T *is an* α^+*-proximal admissible map. For* $x_0 = (\frac{1}{2}, 1) \in A_0$ *and* $u_1 = (0, \frac{1}{2}) \in Tx_0 \in B_0$, *we have* $x_1 = (\frac{1}{2}, \frac{1}{2}) \in A_0$ *such that*

$$d(x_1, u_1) = \mathrm{dist}(A, B), \quad \alpha(x_0, x_1) = \alpha\left((\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2})\right) = 0.$$

Note that $d(A, B) = \dfrac{1}{2}$, $A_0 = \left\{(\frac{1}{2}, x) : 0 \le x < \infty\right\}$ *and* $B_0 = \{(0, x) : 0 \le x < \infty\}$. *Let* $d(x_1, y_1) = d(A, B) = \dfrac{1}{2}$ *and* $d(x_2, y_2) = d(A, B) = \dfrac{1}{2}$, *where* $x_1 = (\dfrac{1}{2}, u_1), x_2 = (\dfrac{1}{2}, u_2) \in A_0$ *and* $y_1 = (0, v_1), y_2 = (0, v_2) \in B_0$. *Then*

$$\frac{1}{2} + |u_1 - v_1| = \frac{1}{2}$$

and

$$\frac{1}{2} + |u_2 - v_2| = \frac{1}{2}$$

so $|u_1 - v_1| = 0$ *and* $|u_2 - v_2| = 0$. *So, we have* $v_1 = u_1$ *and* $v_2 = u_2$. *This shows that* $d(x_1, x_2) = d(y_1, y_2)$. *So* (A, B) *satifies the* P*-property.*

Notice that $T(A_0) \subset B_0$. *Assume* $\dfrac{1}{2}D^*(p, Tp) \le d(p, q)$ *and* $\alpha(p, q) \ge 0$, *for* $p, q \in A$. *Then*

$$\begin{cases} p = (\frac{1}{2}, 1), q = (\frac{1}{2}, \frac{1}{2}) & or \\ p = (\frac{1}{2}, \frac{1}{2}), q = (\frac{1}{2}, 1) & or \\ q = (\frac{1}{2}, 0), p = (\frac{1}{2}, \frac{1}{2}) & or \\ q = (\frac{1}{2}, 1), p = (\frac{1}{2}, 0). \end{cases}$$

Since $d(Tp,Tq) = d(Tq,Tp)$ *and* $\mathcal{M}(p,q) = \mathcal{M}(q,p)$ *for all* $p,q \in A$*, we can suppose that*

$$(p,q) = ((\frac{1}{2},1),(\frac{1}{2},\frac{1}{2})) \ \ or \ \ (p,q) = ((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0)).$$

Now we consider the following cases:

(i) *if* $(p,q) = ((\frac{1}{2},1),(\frac{1}{2},\frac{1}{2})) \in A_0$*, then*

$$H(T(\frac{1}{2},1),T(\frac{1}{2},\frac{1}{2})) \leq \frac{1}{4} = \psi(d((\frac{1}{2},1),(\frac{1}{2},\frac{1}{2})) \leq \psi(\mathcal{M}(p,q)).$$

(ii) *if* $(p,q) = ((\frac{1}{2},\frac{1}{2})),(\frac{1}{2},0)) \in A_0$*, then*

$$H(T(\frac{1}{2},\frac{1}{2}),T(\frac{1}{2},0)) \leq \frac{1}{4} = \psi(d((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0)) \leq \psi(\mathcal{M}(p,q)).$$

Consequently, we have

$$\frac{1}{2}D^*(p,Tp) \leq d(p,q) \Rightarrow \alpha(p,q) + H(Tp,Tq) \leq \psi(\mathcal{M}(p,q))$$

If $x,y \in \left\{(\frac{1}{2},a) : 0 \leq a \leq 1\right\}$*, then we have*

$$\alpha(x,y) + H\left(Tx,Ty\right) = 0 + \frac{|x-y|}{2} = \frac{1}{2}d(x,y) = \psi\left(d(x,y)\right) \leq \psi\big(\mathcal{M}(x,y)\big).$$

Therefore

$$\alpha(x,y) + H\left(Tx,Ty\right) \ \leq \ \psi\big(\mathcal{M}(x,y)\big).$$

Hence, T *is an* $\alpha^+\psi$*-proximal contractive multimap. Moreover, if* $\{x_n\}$ *is a sequence in* A *such that* $\alpha(x_n,x_{n+1}) \geq 0$ *for all* n *and* $x_n \to x \in A$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k},x) \geq 0$ *for all* k*. Therefore, all the conditions of Theorem 2.4 hold true and* T *has the best proximity point. Here* $p = (\frac{1}{2},0)$ *is the best proximity point of* T*.*

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