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Two Non-Uniform Bounds in the Poisson Approximation of Sums of Dependent Indicators

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Abstract : We use the Stein-Chen method to obtain two formulas of non-uniform bounds for the errors in Poisson approximation to the distribution of sums of dependent random indicators. We also give some examples to illustrate some applications of the formulas obtained.

Keywords : Non-uniform bounds, Poisson distribution, dependent indicators, Stein-Chen method.

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1 Introduction

In the past few years, mathematicians and statisticians have developed a powerful technique known as the Stein-Chen method for approximating the distribution of a sum of random indicators [1-2,6-7,13,15]. In contrast to many asymptotic methods, this approximation carries with it explicit error bounds. Let X_α be a random indicator with the probability $P(X_\alpha = 1) = 1 - P(X_\alpha = 0) = p_\alpha$, where α ranges over some finite index set Γ, and let $W = \sum_{\alpha \in \Gamma} X_\alpha$ and $\lambda = \sum_{\alpha\in\Gamma} p_\alpha$. If $\Gamma = \{1, ..., n\}$ and X_α's are independent, then W has the Poisson binomial distribution, and in case where p_α's are identical to p, W has the binomial distribution with parameter n and p. It is well known that the Poisson distribution is a good model for counting the number of occurrences of rare, or exceptional, events in an experiment with many trials. That is, if the probabilities p_α's are small, then the distribution of W is approximately Poisson with parameter $\lambda = EW = \sum_{\alpha\in\Gamma} p_\alpha$. Many authors used the Stein-Chen method to investigate bounds for approximating the distribution of W. For examples, in the case where $X_1, ..., X_n$ are independent and $\lambda = \sum_{\alpha=1}^{n} p_\alpha$, Stein [15] gave an explicit uniform error bound

$$\left|P(W\in A)-\sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right|\le \min\{1,\lambda^{-1}\}\sum_{\alpha=1}^{n}p_\alpha^2 \tag{1.1}$$

in the approximation of the distribution of W by the Poisson distribution, where $A\subseteq\mathbb{N}\cup\{0\}$. Neammanee [13] then gave a non-uniform error bound

$$\left|P(W=w_0)-\frac{\lambda^{w_0}e^{-\lambda}}{w_0!}\right|\le \min\left\{\frac{1}{w_0},\lambda^{-1}\right\}\sum_{\alpha=1}^{n}p_\alpha^2 \tag{1.2}$$

in approximating the point probability of W by the Poisson probability, where $w_0\in\{1,...,n-1\}$. Teerapabolarn and Neammanee [19] gave a non-uniform error bound

$$\left|P(W\le w_0)-\sum_{k=0}^{w_0}\frac{\lambda^k e^{-\lambda}}{k!}\right|\le \lambda^{-1}(1-e^{-\lambda})\min\left\{1,\frac{e^{\lambda}}{w_0+1}\right\}\sum_{\alpha=1}^{n}p_\alpha^2 \tag{1.3}$$

in the approximation of the distribution function of W by the Poisson distribution function, where $w_0\in\{0,1,...,n\}$.

In the case of dependent indicator summands, we first suppose that, for each $\alpha\in\Gamma$, a neighborhood $B_\alpha\subsetneq\Gamma$ of α can be chosen so that X_α is independent of those X_β with $\beta\notin B_\alpha$. Let

$$b_1=\sum_{\alpha\in\Gamma}\sum_{\beta\in B_\alpha}p_\alpha p_\beta \tag{1.4}$$

and

$$b_2=\sum_{\alpha\in\Gamma}\sum_{\beta\in B_\alpha\setminus\{\alpha\}}E[X_\alpha X_\beta]. \tag{1.5}$$

Barbour, Holst and Janson [6] gave a uniform bound in the form of

$$\left|P(W\in A)-\sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right|\le \lambda^{-1}(1-e^{-\lambda})(b_1+b_2) \tag{1.6}$$

and Janson [9] used the coupling method to determine a uniform bound in the form of

$$\left|P(W\in A)-\sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right|\le \lambda^{-1}(1-e^{-\lambda})\sum_{\alpha\in\Gamma}p_\alpha E|W-W_\alpha^*|, \tag{1.7}$$

where W_α^* is a random variable that has the same distribution as $W-X_\alpha$ conditional on $X_\alpha=1$.

Non-uniform counterparts of the uniform bounds in (1.6) and (1.7) were obtained by Teerapabolarn and Neammanee. In [17], they gave two pointwise bounds, i.e.

$$\left| P(W = w_0) - \frac{\lambda^{w_0} e^{-\lambda}}{w_0!} \right| \le \min \left\{ \frac{1}{w_0}, \lambda^{-1} \right\} (b_1 + b_2) \tag{1.8}$$

and

$$\left| P(W = w_0) - \frac{\lambda^{w_0} e^{-\lambda}}{w_0!} \right| \le \min\{ \frac{1}{w_0}, \lambda^{-1}\} \sum_{\alpha \in \Gamma} p_\alpha E|W - W_\alpha^*|, \tag{1.9}$$

where $w_0 \in \{1, 2, ..., |\Gamma|\}$ and $|\Gamma|$ is the number of elements of Γ. They later discovered two non-uniform bounds for $A = \{0, 1, ..., w_0\}$ in [19], which say that

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \lambda^{-1}(1 - e^{-\lambda}) \min \left\{ 1, \frac{e^\lambda}{w_0 + 1} \right\} (b_1 + b_2) \tag{1.10}$$

and

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \lambda^{-1}(1 - e^{-\lambda}) \min \left\{ 1, \frac{e^\lambda}{w_0 + 1} \right\} \sum_{\alpha \in \Gamma} p_\alpha E|W - W_\alpha^*|. \tag{1.11}$$

In this paper, our goal is to find non-uniform bounds that are counterparts of (1.6) and (1.7) where A is any subset of $\{0, 1, ..., |\Gamma|\}$, and to illustrate some applications of these formulas.

In section 2, we present formulas of non-uniform bounds on Poisson approximation theorems based on two approaches of the Stein-Chen method, the local and coupling approaches. These theorems are applied to a wide collection of examples that reduce to questions about sums of possibly dependent random indicators in section 3.

2 A non-uniform bound on Poisson approximation

In 1972, Stein [14] introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation for dependent random variables. This method was adapted and applied to the Poisson approximation by Chen [7] in 1975. He used the Stein's method to find upper bounds for the error in approximating the distribution of a sum of dependent random indicators by the Poisson distribution. This method is usually referred to as the Stein-Chen method (or the Chen-Stein method). The idea of this method is based on the Stein's equation for Poisson distribution with parameter λ which says

$$\lambda f(w+1) + wf(w) = h(w) - \mathcal{P}_\lambda(h), \tag{2.1}$$

where $\mathcal{P}_\lambda(h) = e^{-\lambda}\sum_{l=0}^{\infty} h(l)\frac{\lambda^l}{l!}$ and f and h are bounded real-valued functions on $\mathbb{N}\cup\{0\}$. For $A \subseteq \{0,1,...,|\Gamma|\}$, let $h_A : \mathbb{N}\cup\{0\} \to \mathbb{R}$ be defined as

$$h_A(w) = \begin{cases} 1 & \text{if } \ w \in A, \\ 0 & \text{if } \ w \notin A. \end{cases} \tag{2.2}$$

It follows from Barbour, Holst and Janson [6, p.7] that the solution $U_\lambda h_A$ of (2.1) is of the form

$$U_\lambda h_A(w) = \begin{cases} (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(h_{A\cap C_{w-1}}) - \mathcal{P}_\lambda(h_A)\mathcal{P}_\lambda(h_{C_{w-1}})] & \text{if } \ w \geq 1, \\ 0 & \text{if } \ w = 0, \end{cases} \tag{2.3}$$

where $C_{w-1} = \{0,...,w-1\}$.

In this section, we use the Stein-Chen method to obtain two non-uniform error bounds in the Poisson approximation of the distribution of W which follows by the local and coupling approaches in subsections 2.1 and 2.2 respectively.

2.1 The local approach

The method of this approach exploits certain neighborhoods of dependency B_α associated with each $\alpha \in \Gamma$. That is, for each $\alpha \in \Gamma$, we have chosen $B_\alpha \subsetneq \Gamma$ as a neighborhood of α such that X_α and X_β are independent for all $\beta \notin B_\alpha$. We first state our main result obtained by this approach in Theorem 2.1 along with two lemmas necessary in proving the theorem. Its proof is then duely followed.

Theorem 2.1 Let $A \subseteq \{0,1,...,|\Gamma|\}$. Then

$$\left| P(W\in A) - \sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right| \leq \lambda^{-1}\min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\}(b_1+b_2), \tag{2.4}$$

and

$$\left| P(W=0) - e^{-\lambda}\right| \leq \lambda^{-2}(\lambda + e^{-\lambda} - 1)\max\{b_1, b_2\}, \tag{2.5}$$

where

$$\Delta(\lambda) = \begin{cases} e^{\lambda} + \lambda - 1 & \text{if } \lambda^{-1}(e^{\lambda} - 1) \leq M_A, \\ 2(e^{\lambda} - 1) & \text{if } \lambda^{-1}(e^{\lambda} - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w | C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w | w \in A\} & \text{if } 0 \notin A. \end{cases}$$

Lemma 2.1 Let $A \subseteq \{0, 1, ..., |\Gamma|\}$. Then the followings hold.
1. For any $s, t \in \mathbb{N}$,

$$|V_{\lambda} h_A(t, s)| \leq \sup_{w \geq 1} |V_{\lambda} h_A(w+1, w)||t - s|,$$

where $V_{\lambda} h_A(t, s) = U_{\lambda} h_A(t) - U_{\lambda} h_A(s)$.
2. For $w \geq 1$,

$$|V_{\lambda} h_A(w)| \leq \lambda^{-1} \min \left\{ 1, \lambda, \frac{\Delta(\lambda)}{M_A + 1} \right\}, \tag{2.6}$$

where $V_{\lambda} h_A(w) = V_{\lambda} h_A(w+1, w)$.

Proof.
1. Assume that $t > s$. Then

$$\begin{aligned} |V_{\lambda} h_A(t, s)| &= \left| \sum_{w=s}^{t-1} V_{\lambda} h_A(w+1, w) \right| \\ &\leq \sum_{w=s}^{t-1} |V_{\lambda} h_A(w+1, w)| \\ &\leq \sup_{w \geq 1} |V_{\lambda} h_A(w+1, w)||t - s|. \end{aligned}$$

2. From Stein [15, p.88], we have $|V_{\lambda} h_A(w)| \leq \lambda^{-1} \min\{1, \lambda\}$. So, it suffices to show that

$$|V_{\lambda} h_A(w)| \leq \frac{\lambda^{-1} \Delta(\lambda)}{M_A + 1}.$$

Let's consider three cases.

Case 1. $w \geq M_A + 1$.
Since $V_\lambda h_{\{t\}}(w) < 0$ for all $t \neq w$, we have

$$V_\lambda h_A(w) \leq \sum_{t \in A} V_\lambda h_{\{t\}}(w) \leq V_\lambda h_{\{w\}}(w)$$

and

$$\begin{aligned} V_\lambda h_A(w) &\geq V_\lambda h_{A \setminus \{w\}}(w) \\ &\geq V_\lambda h_{\{w\}^c}(w) \\ &= V_\lambda 1(w) - V_\lambda h_{\{w\}}(w) \\ &= -V_\lambda h_{\{w\}}(w). \end{aligned}$$

Hence

$$|V_\lambda h_A(w)| \leq V_\lambda h_{\{w\}}(w) \leq \frac{1}{w} \leq \frac{1}{M_A + 1} \leq \frac{\lambda^{-1}(e^\lambda - 1)}{M_A + 1} < \frac{\lambda^{-1}\Delta(\lambda)}{M_A + 1}.$$

Case 2. $w \leq M_A$ and $0 \notin A$.
Note first that if $w \in A$, then

$$V_\lambda h_A(w) \leq V_\lambda h_{\{w\}}(w)$$

and

$$V_\lambda h_A(w) \geq V_\lambda h_{\{w\}^c}(w) \geq -V_\lambda h_{\{w\}}(w).$$

Thus

$$|V_\lambda h_A(w)| \leq V_\lambda h_{\{w\}}(w) \leq \frac{1}{w} = \frac{1}{M_A} \leq \frac{\lambda^{-1}(e^\lambda - 1)}{M_A} \leq \frac{\lambda^{-1}\Delta(\lambda)}{M_A + 1}.$$

On the other hand, if $w \notin A$, then $V_\lambda h_A(w) < 0$ and

$$\begin{aligned} 0 < -V_\lambda h_A(w) &= U_\lambda h_A(w) - U_\lambda h_A(w+1) \\ &= w! \lambda^{-(w+1)} e^\lambda \mathcal{P}_\lambda(h_A) \mathcal{P}_\lambda(h_{C_w}) - (w-1)! \lambda^{-w} e^\lambda \mathcal{P}_\lambda(h_A) \mathcal{P}_\lambda(h_{C_{w-1}}) \\ &= (w-1)! \lambda^{-(w+1)} e^\lambda \mathcal{P}_\lambda(h_A) [w \mathcal{P}_\lambda(h_{C_w}) - \lambda \mathcal{P}_\lambda(h_{C_{w-1}})] \\ &\leq e^\lambda w! \lambda^{-(w+1)} \mathcal{P}_\lambda(h_A) \\ &\leq e^\lambda w! \lambda^{-(w+1)} \mathcal{P}_\lambda(1 - h_{C_{M_A - 1}}) \end{aligned}$$

$$
\begin{aligned}
&= w! \sum_{k=M_A}^{\infty} \frac{\lambda^{k-(w+1)}}{k!} \\
&= \sum_{k=M_A}^{\infty} \frac{w!\lambda^{k-(w+1)}}{k(k-1)\cdots(k-w)[k-(w+1)]!} \\
&\le \frac{1}{M_A} \sum_{k=M_A}^{\infty} \frac{\lambda^{k-(w+1)}}{\binom{k-1}{w}[k-(w+1)]!} \\
&\le \frac{1}{M_A} \left\{ 1 + \frac{\lambda}{2!} + \frac{\lambda^2}{3!} + \cdots \right\} \\
&= \frac{\lambda^{-1}(e^{\lambda}-1)}{M_A} \\
&\le \frac{\lambda^{-1}\Delta(\lambda)}{M_A+1}.
\end{aligned}
$$

Case 3. $w \le M_A$ and $0 \in A$.
Since $0 < w\mathcal{P}_\lambda(h_{C_w}) - \lambda\mathcal{P}_\lambda(h_{C_{w-1}})$ and

$$
\begin{aligned}
V_\lambda h_A(w) &= w!\lambda^{-(w+1)}e^{\lambda}[\mathcal{P}_\lambda(h_{A\cap C_w}) - \mathcal{P}_\lambda(h_A)\mathcal{P}_\lambda(h_{C_w})] \\
&\quad - (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(h_{A\cap C_{w-1}}) - \mathcal{P}_\lambda(h_A)\mathcal{P}_\lambda(h_{C_{w-1}})] \\
&= (w-1)!\lambda^{-(w+1)}e^{\lambda}(1-\mathcal{P}_\lambda(h_A))[w\mathcal{P}_\lambda(h_{C_w}) - \lambda\mathcal{P}_\lambda(h_{C_{w-1}})] \\
&= (w-1)!\lambda^{-(w+1)}e^{\lambda}\mathcal{P}_\lambda(1-h_A)[w\mathcal{P}_\lambda(h_{C_w}) - \lambda\mathcal{P}_\lambda(h_{C_{w-1}})],
\end{aligned}
$$

we obtain, using the same argument as in the last inequality of Case 2,

$$
\begin{aligned}
0 < V_\lambda h_A(w) &= (w-1)!\lambda^{-(w+1)}e^{\lambda}\mathcal{P}_\lambda(1-h_A)[w\mathcal{P}_\lambda(h_{C_w}) - \lambda\mathcal{P}_\lambda(h_{C_{w-1}})] \\
&\le e^{\lambda}w!\lambda^{-(w+1)}\mathcal{P}_\lambda(1-h_A) \\
&\le e^{\lambda}w!\lambda^{-(w+1)}\mathcal{P}_\lambda(1-h_{C_{M_A}}) \\
&= \frac{\lambda^{-1}(e^{\lambda}-1)}{M_A+1} \\
&< \frac{\lambda^{-1}\Delta(\lambda)}{M_A+1}.
\end{aligned}
$$

Hence, from the three cases, we have proved (2.6). □

Lemma 2.2 Let $Z_\alpha = \sum_{\beta \in B_\alpha \setminus \{\alpha\}} X_\beta, Y_\alpha = W - X_\alpha - Z_\alpha = \sum_{\beta \notin B_\alpha} X_\beta$, and $f = U_\lambda h_A$. Then

1. $|E[p_\alpha(f(W+1) - f(Y_\alpha+1))]|$
$$\leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\} (p_\alpha^2 + p_\alpha E[Z_\alpha]) \text{ and}$$
2. $|E[X_\alpha(f(Y_\alpha + Z_\alpha + 1) - f(Y_\alpha+1))]|$
$$\leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\} E[X_\alpha Z_\alpha].$$

Proof.
1. By Lemma 2.1, we have

$$\begin{aligned}
&|E[p_\alpha(f(W+1) - f(Y_\alpha+1))]| \\
&\qquad \leq E|p_\alpha|f(Y_\alpha + Z_\alpha + X_\alpha + 1) - f(Y_\alpha+1))|| \\
&\qquad \leq \sup_{w \geq 1} |V_\lambda h_A(w)| p_\alpha E[X_\alpha + Z_\alpha] \\
&\qquad \leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\} (p_\alpha^2 + p_\alpha E[Z_\alpha]).
\end{aligned}$$

2. Proof is similar to that of 1. $\square$

Proof of Theorem 2.1 Let $Z_\alpha = \sum_{\beta \in B_\alpha \setminus \{\alpha\}} X_\beta, Y_\alpha = W - X_\alpha - Z_\alpha = \sum_{\beta \notin B_\alpha} X_\beta$ and $W_\alpha = W - X_\alpha$.

The inequality (2.5) was derived in [18]. It is now left to verify (2.4). Substituting $h = h_A$ in (2.1) yields, in expected values,

$$P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} = E[\lambda f(W+1) - W f(W)], \tag{2.7}$$

where $f = U_\lambda h_A$ is defined in (2.3).

By the fact that each X_α takes values 0 or 1, we can see that

$$\begin{aligned}
E[Wf(W)] &= \sum_{\alpha \in \Gamma} E[X_\alpha f(W_\alpha + 1)] \\
&= \sum_{\alpha \in \Gamma} E[X_\alpha f(Y_\alpha+1)] + \sum_{\alpha \in \Gamma} E[X_\alpha(f(Y_\alpha + Z_\alpha + 1) - f(Y_\alpha+1))].
\end{aligned}$$

Hence, by the independence of X_α and Y_α,

$$\begin{aligned}
&E[\lambda f(W+1) - Wf(W)] \\
&\quad = \sum_{\alpha\in\Gamma}\{E[p_\alpha(f(W+1) - f(Y_\alpha+1))] - E[X_\alpha(f(Y_\alpha+Z_\alpha+1) - f(Y_\alpha+1))] \\
&\quad\quad + E[p_\alpha f(Y_\alpha+1) - X_\alpha f(Y_\alpha+1)]\} \\
&\quad = \sum_{\alpha\in\Gamma}\{E[p_\alpha(f(W+1) - f(Y_\alpha+1))] - E[X_\alpha(f(Y_\alpha+Z_\alpha+1) - f(Y_\alpha+1))]\}
\end{aligned}$$

and, by Lemma 2.2 and equation (2.7), we have

$$\begin{aligned}
\left|P(W\in A) - \sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right| &= |E[\lambda f(W+1) - Wf(W)]| \\
&\le \lambda^{-1}\min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\}(b_1+b_2). \qquad \square
\end{aligned}$$

2.2 The coupling approach

When the dependence between the X_α's are global, we have an alternative approach for approximating the distribution of W. This approach is particularly useful when it is possible to construct, for each α, a random variable W_α^* on a common probability space with W such that W_α^* has the same distribution as $W - X_\alpha$ conditional on $X_\alpha = 1$. The main result via this approach is the following.

Theorem 2.2 Let $A \subseteq \{0, 1, ..., |\Gamma|\}$. Then

$$\left|P(W\in A) - \sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right| \le \lambda^{-1}\min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\}\sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|, \tag{2.8}$$

and

$$\left|P(W=0) - e^{-\lambda}\right| \le \lambda^{-2}(\lambda + e^{-\lambda} - 1)\sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|. \tag{2.9}$$

From Theorem 2.2, if $W \ge W_\alpha^*$ or $W - X_\alpha \le W_\alpha^*$ for every $\alpha \in \Gamma$, then we have more convenient forms in the following corollary.

Corollary 2.1 Let $A \subseteq \{0, 1, ..., |\Gamma|\}$. Then
1.

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \{\lambda - Var[W]\}, \tag{2.10}$$

and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq \lambda^{-2}(\lambda + e^{-\lambda} - 1)\{\lambda - Var[W]\}, \tag{2.11}$$

where $W \geq W_\alpha^*$ a.s. for every $\alpha \in \Gamma$ and
2.

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right|$$

$$\leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \left\{Var[W] - \lambda + 2 \sum_{\alpha \in \Gamma} p_\alpha^2 \right\}, \tag{2.12}$$

and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq \lambda^{-2}(\lambda + e^{-\lambda} - 1) \left\{Var[W] - \lambda + 2 \sum_{\alpha \in \Gamma} p_\alpha^2 \right\}, \tag{2.13}$$

where $W - X_\alpha \leq W_\alpha^*$ a.s. for every $\alpha \in \Gamma$.

Proof of Theorem 2.2 Let $f = U_\lambda h_A$. From Teerapabolarn, Neammanee and Chongcharoen [16], we have (2.9). Next we will show that (2.8) is also valid. Note that $E[Wf(W)] = \sum_{\alpha \in \Gamma} E[X_\alpha f(W)]$ and, for each α,

$$\begin{aligned}
E[X_\alpha f(W)] &= E[E[X_\alpha f(W)|X_\alpha]] \\
&= E[X_\alpha f(W)|X_\alpha = 0]P(X_\alpha = 0) + E[X_\alpha f(W)|X_\alpha = 1]P(X_\alpha = 1) \\
&= E[f(W)|X_\alpha = 1]P(X_\alpha = 1) \\
&= p_\alpha E[f(W_\alpha^* + 1)].
\end{aligned}$$

Thus

$$\begin{aligned}
E[\lambda f(W + 1) - Wf(W)] &= \sum_{\alpha \in \Gamma} p_\alpha E[f(W + 1)] - \sum_{\alpha \in \Gamma} p_\alpha E[f(W_\alpha^* + 1)] \\
&= \sum_{\alpha \in \Gamma} p_\alpha E[f(W + 1) - f(W_\alpha^* + 1)].
\end{aligned}$$

By Lemma 2.1, we have

$$
\begin{aligned}
|E[\lambda f(W+1) - Wf(W)]| &\leq \sum_{\alpha\in\Gamma} p_\alpha E|f(W+1) - f(W_\alpha^*+1)| \\
&\leq \lambda^{-1}\min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\} \sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|.
\end{aligned}
$$

Hence, by (2.7), the estimate (2.8) immediately follows. □

Remarks.

1. We note that, for $\Gamma = \{1, 2, ..., n\}$, if $X_1, X_2, ..., X_n$ are independent then $b_1 = \sum_{\alpha=1}^{n} p_\alpha^2$ and b_2 vanishes, since $Z_\alpha = 0$ for all α. So the Poisson local estimates in Theorem 2.1 reduce to

$$
\left|P(W\in A) - \sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right| \leq \lambda^{-1}\min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\}\sum_{\alpha=1}^{n} p_\alpha^2, \tag{2.14}
$$

where $A \subseteq \{0, 1, ..., n\}$.

2. If the coupling (W, W_α^*) in Theorem 2.2 can be constructed in such a way that $E|W - W_\alpha^*|$ is small, then the result via the coupling approach yields good Poisson approximation. For instance, with independent X_i's, one can take the original space such that $W_\alpha^* = W - X_\alpha$. So $E|W - W_\alpha^*| = E[X_\alpha] = p_\alpha$ and the Poisson coupling estimate reduce to the same result as (2.14).

3. It is well-known that the Poisson estimate works best in many applications when λ is small. It is then worth noting that if $\lambda \leq 1$ then

$$
\min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\} = \begin{cases} \lambda & \text{if } M_A \leq 1, \\ \dfrac{e^\lambda + \lambda - 1}{M_A+1} & \text{if } M_A \geq 2. \end{cases} \tag{2.15}
$$

3 Applications of the local approach

There are many applications of the Poisson local estimate which were proposed by many authors during the past few years, e.g., the birthday problem and the longest head run in Arratia, Goldstein and Gordon [1-2], applications in the theory of random graphs in Barbour, Holst and Janson[6], the problem of estimating statistical significance in sequence comparison in Goldstein and Waterman [8], sequence comparison significance in Waterman and Vingron [20], applications to time series analysis in Kim [10] and the somatic cell hybrid model in Lange [11]. In this section, we present some examples of this approach that are useful in applications of Theorems 2.1.

Example 3.1 (The birthday problem)

Suppose n balls (people) are uniformly and independently distributed into d boxes (days of the year). The birthday problem involves finding an approximate distribution of the number of boxes that receive k or more balls for some fixed positive integer k. To get started, let the index set Γ be the collection of all sets of trials $\alpha \subset \{1, 2, ..., n\}$ having $|\alpha| = k$ elements, where $\{1, 2, ..., n\}$ is a set of n balls. Let X_α be the indicator of the event that the balls indexed by α all fall into the same box probability $p_\alpha = P(X_\alpha = 1) = d^{1-k}$. The number of sets of k balls that fall into the same box is given by $W = \sum\limits_{\alpha\in\Gamma} X_\alpha$. It seems reasonable to approximate W as a Poisson random variable with mean $\lambda = E[W]$ if p_α's are small. Since all p_α are identical, we have

$$\lambda = |\Gamma| p_\alpha = \binom{n}{k} d^{1-k}.$$

We now define the neighborhood B_α of α so that X_α is independent of those X_β with β not in B_α by taking $B_\alpha = \{\beta \in \Gamma : \alpha\cap\beta \neq \varnothing\}$. We observe that X_α and X_β are independent if $\alpha \cap \beta = \varnothing$. Since $|B_\alpha| = \binom{n}{k} - \binom{n-k}{k}$, we have

$$\begin{aligned} b_1 &= |\Gamma||B_\alpha| p_\alpha^2 \\ &= \lambda |B_\alpha| d^{1-k}. \end{aligned}$$

For a given α, we have $1 \le |\alpha \cap \beta| \le k - 1$ for $\beta \in B_\alpha \setminus \{\alpha\}$ and

$$\begin{aligned} b_2 &= \binom{n}{k} \sum_{j=1}^{k-1} \binom{k}{j} \binom{n-k}{k-j} d^{1+j-2k} \\ &= \lambda b, \end{aligned}$$

where $b = \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} d^{j-k}$. By (2.4) and (2.5), we have

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \left(|B_\alpha| d^{1-k} + b\right),$$

where $A \subseteq \left\{0, 1, ..., \binom{n}{k}\right\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq \lambda^{-1}(\lambda + e^{-\lambda} - 1) \max\left\{|B_\alpha| d^{1-k}, b\right\}.$$

If $k = 3$, $n = 50$ and $d = 365$, we have $\lambda = \binom{50}{3}(365)^{-2} = 0.14711953$, $|B_\alpha| = \binom{50}{3} - \binom{47}{3} = 3385$ and $b = 3\binom{47}{2}(365)^{-2} + 3(47)(365)^{-1} = 0.41064365$. So, by (2.15), a non-uniform bound for approximating the distribution of the number of sets of three balls that fall into the same box is

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \begin{cases} 0.064152 & \text{if } M_A \leq 1, \\ \dfrac{0.133263}{M_A + 1} & \text{if } M_A \geq 2, \end{cases}$$

where $A \subseteq \{0, 1, ..., 19600\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq 0.0287784.$$

That is, the probability that no box gets three or more balls is

$$0.8344124 \leq P(W = 0) \leq 0.8919692.$$

Example 3.2 (A random graph problem)

Consider the n-dimensional unit cube $[0, 1]^n$. Suppose that each of the $n2^{n-1}$ edges is independently assigned one of two equally likely orientations. Let Γ be the set of all 2^n vertices, and for each $\alpha \in \Gamma$, let X_α be the indicator that vertex α has all of its edges directed inward with the probability $p_\alpha = P(X_\alpha = 1) = 2^{-n}$. Let $W = \sum_{\alpha \in \Gamma} X_\alpha$ be the number of vertices at which all n edges point inward. Its distribution seems reasonable to be approximated by Poisson distribution with mean $\lambda = E[W] = 1$ when n is large.

Following Arratia, Goldstein and Gordon [2], we define the neighborhood of α so that X_α and X_β are independent for all $\beta \notin B_\alpha$ by setting the set $B_\alpha = \{\beta \in \Gamma : |\alpha - \beta| = 1\}$. We note that X_α is independent of those

X_β with $|\alpha-\beta|>1$, and also $E[X_\alpha X_\beta]=0$ for $|\alpha-\beta|=1$, hence $b_2=0$. Since $|B_\alpha|=n$, we have

$$\begin{aligned} b_1 &= |\Gamma||B_\alpha|p_\alpha^2 \\ &= n2^{-n}. \end{aligned}$$

It follows from Theorem 2.1 and (2.15) that

$$\left|P(W\in A)-\sum_{k\in A}\frac{e^{-1}}{k!}\right| \le \begin{cases} n2^{-n} & \text{if } M_A\le 1, \\ \dfrac{n2^{-n}e}{M_A+1} & \text{if } M_A\ge 2, \end{cases}$$

where $A\subseteq\{0,1,...,2^{n-1}\}$ and

$$\left|P(W=0)-e^{-1}\right| \le ne^{-1}2^{-n}.$$

For $n=10$, a non-uniform bound for approximating the distribution of the number of vertices at which all n edges point inward is of the form

$$\left|P(W\in A)-\sum_{k\in A}\frac{e^{-1}}{k!}\right| \le \begin{cases} 0.0097656 & \text{if } M_A\le 1, \\ \dfrac{0.0265457}{M_A+1} & \text{if } M_A\ge 2, \end{cases}$$

where $A\subseteq\{0,1,...,512\}$ and

$$\left|P(W=0)-e^{-1}\right| \le 0.0035926.$$

Example 3.3 (Sequence alignments problem)

Let $\mathcal{C}_1,...,\mathcal{C}_{n+m-1}$ be independent random variables with common distribution μ over a finite alphabet $\mathcal{A}$. Let $\omega=a_1a_2...a_m$ be a specific sequence of m letters of $\mathcal{A}$ without self overlaps: that is, for all $k=1,2,...,m-1$, $a_1a_2...a_k\neq a_{m-k+1}a_{m-k+2}...a_m$.

Now we define X_α to be a random indicator $I(\mathcal{C}_\alpha=a_1,...,\mathcal{C}_{\alpha+m-1}=a_m)$, for $\alpha\in\Gamma=\{1,...,n\}$, with the probability $p=E[X_\alpha]=\prod_{l=1}^{m}P(\mathcal{C}_{\alpha+l-1}=a_l)$. Let $W=\sum_{\alpha=1}^{n}X_\alpha$ be the number of times that the sequence ω appears in the long random string $\mathcal{C}_1,...,\mathcal{C}_{n+m-1}$, and its distribution can be approximated by Poisson distribution with parameter $\lambda=np$ when p is small. Following Barbour [4], the X_α's are locally dependent in the sense that X_α

is independent of those X_β with $\beta \notin B_\alpha$, where $B_\alpha = \{\beta \in \Gamma : |\alpha - \beta| < m\}$. Since the self overlaps do not occur in ω, $b_2 = 0$, and

$$\begin{aligned} b_1 &= |\Gamma||B_\alpha|p^2 \\ &\le \lambda(2m-1)p. \end{aligned}$$

Substituting these values into Theorem 2.1, a non-uniform error bound in approximating the distribution of the number of times that the sequence ω appears in the long random string $\mathcal{C}_1, ..., \mathcal{C}_{n+m-1}$ is of the form

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le (2m-1)p \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\},$$

where $A \subseteq \{0, 1, ..., n\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \le \lambda^{-1}(\lambda + e^{-\lambda} - 1)(2m-1)p.$$

The bounds are small if p is small. If $\lambda \le 1$, then by (2.15) we have

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \begin{cases} \lambda(2m-1)p & \text{if } M_A \le 1, \\ \dfrac{(e^\lambda + \lambda - 1)(2m-1)p}{M_A + 1} & \text{if } M_A \ge 2, \end{cases}$$

where $A \subseteq \{0, 1, ..., n\}$.

Example 3.4 (A random graph problem)

Consider a graph with n nodes created by randomly connecting some pairs of nodes by edges. If the connection probability per pair is p, then all pairs in a triple of nodes are connected with probability p^3. Let Γ be the set of all triples of nodes in the random graph. Define $X_\alpha = 1$ if the triple of nodes α is connected to form a triangle and $X_\alpha = 0$ otherwise. Then the probability $p_\alpha = P(X_\alpha = 1) = p^3$. Let $W = \sum_{\alpha \in \Gamma} X_\alpha$, then W is the number of such triangles in the random graph. If p is small, W is approximately Poisson with mean $\lambda = |\Gamma|p^3 = \binom{n}{3}p^3$.

We now choose the neighborhood B_α of α such that X_α and X_β are independent for $\beta \notin B_\alpha$ by taking $B_\alpha = \{\beta : |\alpha \cap \beta| \ge 2\}$. Note that, for $\alpha \ne \beta$, $E[X_\alpha X_\beta] = P(X_\alpha = 1, X_\beta = 1) = p^5$ and $|B_\alpha| = 3(n-3) + 1$. Hence

$$\begin{aligned} b_1 &= |\Gamma||B_\alpha|p_\alpha^2 \\ &= \lambda|B_\alpha|p^3 \\ &= (3(n-3)+1)\lambda p^3 \end{aligned}$$

and

$$\begin{aligned} b_2 &= |\Gamma||B_\alpha - 1|p^5 \\ &= \lambda|B_\alpha - 1|p^2 \\ &= 3(n-3)\lambda p^2. \end{aligned}$$

By applying Theorem 2.1, a non-uniform bound for the error in Poisson approximation to the distribution of the number of triangles in the random graph is in the term of

$$\left|P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!}\right| \leq \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} 3(n-3)p^2(1+p) + p^3,$$

where $A \subseteq \left\{0, 1, ..., \binom{n}{3}\right\}$ and

$$\left|P(W = 0) - e^{-\lambda}\right| \leq \lambda^{-1}(\lambda + e^{-\lambda} - 1)\max\left\{(3(n-3)+1)p^3, 3(n-3)p^2\right\}.$$

For $n = 30$ and $p = 0.05$, we have $\lambda = \binom{30}{3}(0.05)^3 = 0.5075$ and, by (2.15),

$$\left|P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!}\right| \leq \begin{cases} 0.10797 & \text{if } M_A \leq 1, \\ \dfrac{0.24863}{M_A + 1} & \text{if } M_A \geq 2, \end{cases}$$

where $A \subseteq \{0, 1, ..., 4060\}$ and

$$\left|P(W = 0) - e^{-\lambda}\right| \leq 0.0436916.$$

Example 3.5 (The longest perfect head run)

Consider an infinite sequence $Y_1, Y_2, ...$ of independent random indicators with common success probability p. Let $\Gamma = \{1, ..., n\}$ and fix a positive integer value t. Let X_α be the indicator of the event that a success run of length t or longer begins at position α. Note that $X_1 = \prod_{k=1}^{t} Y_k$ and for $\alpha = 2$ to n,

$$X_j = (1 - Y_{j-1}) \prod_{k=j}^{j+t-1} Y_k.$$

Let $W = \sum_{\alpha \in \Gamma} X_\alpha$ be the number of such success runs starting in the first n positions. The Poisson heuristic suggests that W is approximately Poisson with mean $\lambda = E[W] = p^t[(n-1)(1-p) + 1]$.

Following Arratia, Goldstein and Gordon [2], we define a neighborhood B_α of α by setting $B_\alpha = \{\beta \in \Gamma : |\beta - \alpha| \le t\}$. We observe that X_α is independent of those X_β for $\beta \notin B_\alpha$ and $E[X_\alpha X_\beta] = 0$, hence $b_2 = 0$ and

$$\begin{aligned} b_1 &= \sum_{\alpha \in \Gamma} \sum_{\beta \in B_\alpha} p_\alpha p_\beta \\ &= p^{2t} + 2tp^{2t}(1-p) + [2nt - t^2 + n - 3t - 1]p^{2t}(1-p)^2 \\ &\le \frac{\lambda^2(2t+1)}{n} + 2\lambda p^t. \end{aligned}$$

A non-uniform bound, from Theorem 2.1, in approximating the distribution of the number of success runs starting in the first n positions by Poisson distribution is of the form

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \left[\frac{\lambda(2t+1)}{n} + 2p^t\right],$$

where $A \subseteq \{0, 1, ..., n\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \le \lambda^{-1}(\lambda + e^{-\lambda} - 1) \left[\frac{\lambda(2t+1)}{n} + 2p^t\right].$$

Assume that $n = 100$, $p = 0.5$ and $t = 10$, we have $\lambda = 0.0493164$ and, by (2.15), a non-uniform bound is

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \begin{cases} 0.0006071 & \text{if } M_A \le 1, \\ \dfrac{0.0012294}{M_A + 1} & \text{if } M_A \ge 2, \end{cases}$$

where $A \subseteq \{0, 1, ..., 100\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \le 0.0002986.$$

4 Applications of the coupling approach

As a local approach, many applications for the coupling approach in Poisson approximation of a sum of random indicators were proposed by several authors. Some examples are random graphs problems in Barbour [3-4], Stein [15], Barbour, Holst and Janson [6] and Janson [9], the random allocation problem in Mikhailov [12], occupancy and urn models in

Barbour, Holst and Janson [6] and ménage problem, birthday problem and biggest random gap problem in Lange [11]. In this section, we present some examples that are useful in applications of this approach.

Example 4.1 (The hypergeometric distribution)
Suppose a random sample of size n is chosen without replacement from a finite population containing N elements of two types ($n < N$), of which m are of type A and $N-m$ ($\neq 0$) are of type B. Let the random variable W be the number of type A elements in the sample. It is well-known that W has the hypergeometric distribution $\mathcal{H}(N,n,m)$ where, for $\max\{0, n+m-N\} \leq k \leq \min\{n,m\}$,

$$P(W=k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}.$$

For each $\alpha \in \{1,...,n\}$, let $X_\alpha = 1$ if the α'th element in the sample is of type A and $X_\alpha = 0$ otherwise, and let $W = \sum_{i=1}^{n} X_i$. We then have the probability $P(X_i = 1) = \frac{m}{N}$ and $\lambda = E[W] = \frac{nm}{N}$. If $\frac{m}{N}$ and $\frac{n}{N}$ are small then it seems reasonable to approximate the distribution of W by Poisson distribution with mean λ. For the hypergeometric distribution, the variance of W is

$$Var[W] = \frac{N-n}{N-1} \cdot \frac{nm}{N}\left(1 - \frac{m}{N}\right).$$

We shall then construct the coupled random variable W_α^* which has the same distribution as $W - X_\alpha$ conditional on $X_\alpha = 1$. Consider the number of type A elements in the sample other than the α'th element conditional on $X_\alpha = 1$, which is obtained by swapping the α'th element chosen, if it is of type B, for a randomly chosen an element of type A. Following Barbour [4], let

$$W_\alpha^* = W - X_\alpha - \sum_{\beta=1,\beta\neq\alpha}^{n} X_\beta I_\beta,$$

where I_β is the indicator of the event that the β'th element in the sample is chosen to be swapped with the α'th. Then W_α^* has the same distribution as $W - X_\alpha$ conditional on $X_\alpha = 1$, and we observe that $W \geq W_\alpha^*$ for every $\alpha \in \{1, ..., n\}$. Thus, by (2.10) and (2.11),

$$\begin{aligned}
\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| &\leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \{\lambda - Var[W]\} \\
&= \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \left[\frac{n+m-1}{N-1} - \frac{nm}{N(N-1)}\right] \\
&\leq \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \left(\frac{n+m-1}{N-1}\right),
\end{aligned}$$

where $A \subseteq \{0, ..., \min\{m, n\}\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq \lambda^{-1}(\lambda + e^{-\lambda} - 1) \left(\frac{n+m-1}{N-1}\right).$$

The bounds are both small provided that $\frac{m}{N}$ and $\frac{n}{N}$ are small. Suppose $N = 1000$, $m = 20$ and $n = 10$. We have $\lambda = 0.20$ and, by (2.15), non-uniform error bounds in Poisson approximation to the hypergeometric distribution is of the form

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \begin{cases} 0.0058 & \text{if } M_A \leq 1, \\ \dfrac{0.0122}{M_A + 1} & \text{if } M_A \geq 2, \end{cases}$$

where $A \subseteq \{0, ..., 10\}\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq 0.00272.$$

Example 4.2 (The classical occupancy problem)

Let m balls be thrown independently of each other into n boxes where each ball has probability $1/n$ of falling into the α'th box. Let $X_\alpha = 1$ if the α'th box is empty and $X_\alpha = 0$ otherwise, then $W = \sum_{\alpha=1}^{n} X_\alpha$ is the number of empty boxes. The probability $P(X_i = 1)$ is $(1 - 1/n)^m$ and $\lambda = E[W] = n(1 - 1/n)^m$. Since $E[X_\alpha X_\beta] = (1 - 2/n)^m \neq (1 - 1/n)^{2m} = E[X_\alpha]E[X_\beta]$ for $\alpha \neq \beta$, X_α's are not independent. The distribution of W can be approximated by a Poisson distribution with parameter λ if $(1 - 1/n)^m$ is small, or m/n is large.

We now construct W_α^* such that W_α^* is distributed as $W - X_\alpha$ conditional on $X_\alpha = 1$. Consider the number of empty boxes other than the α'th box conditional on $X_\alpha = 1$, which is obtained by throwing each ball in the α'th box, if it is not empty, into one of the other boxes in such a way that the probability of a ball falling into box β, $\beta \neq \alpha$, is $1/(n-1)$. Thus, we can take

$$W_\alpha^* = W - X_\alpha - \sum_{\beta=1,\beta\neq\alpha}^{n} X_\beta I_\beta,$$

where I_β is the indicator of the event that there is at least one ball from the α'th box falling into the empty β'th box. Then W_α^* has the same distribution as $W - X_\alpha$ conditional on $X_\alpha = 1$, and for every $\alpha \in \{1, ..., n\}$, $W \geq W_\alpha^*$. So, by (2.10) and (2.11), a non-uniform bound for approximating the distribution of the number of empty boxes by Poisson distribution is of the form

$$\left| P(W \in A) - \sum_{k\in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \left\{\lambda - (n-1)\left(\frac{n-2}{n-1}\right)^m\right\},$$

where $A \subseteq \{0, ..., n\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq \lambda^{-1}(\lambda + e^{-\lambda} - 1)\left\{\lambda - (n-1)\left(\frac{n-2}{n-1}\right)^m\right\}.$$

For $m = 50$ and $n = 10$, we get $\lambda = 0.051538$ and, by (2.15), a non-uniform bound for this approximation is

$$\left| P(W \in A) - \sum_{k\in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \begin{cases} 0.00137 & \text{if } M_A \leq 1, \\ \dfrac{0.00278}{M_A + 1} & \text{if } M_A \geq 2, \end{cases}$$

where $A \subseteq \{0, ..., 10\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq 0.0006742.$$

Example 4.3 (A random graph problem)

A random graph $G(n, p)$ is a graph on n labeled vertices $\{1, 2, \ldots, n\}$ where each possible edge $\{\alpha, \beta\}$ is present randomly and independently with probability p, $0 < p < 1$. Let $E_{\alpha\beta}$ be the independent edge indicator of the event that edge $\{\alpha, \beta\} \in G(n, p)$, then $P(E_{\alpha\beta} = 1) = p$. Let $X_\alpha = 1$

if vertex α is an isolated vertex in $G(n,p)$ and $X_\alpha = 0$ otherwise. Then $W = \sum_{\alpha=1}^{n} X_\alpha$ is the number of isolated vertices in $G(n,p)$. We now have the probability $p_\alpha = P(X_\alpha = 1) = (1-p)^{n-1}$, $\lambda = E[W] = n(1-p)^{n-1}$ and $Var[W] = \lambda + n(n-1)(1-p)^{2n-3} - \lambda^2$. Since $E[X_\alpha X_\beta] \neq E[X_\alpha]E[X_\beta]$ for $\alpha \neq \beta$, it indicates that X_α's are not independent.

We then construct W_α^* by considering the number of isolated vertices in $G(n,p)$ other than the α'th vertex conditional on $X_\alpha = 1$, which is obtained by deleting all the edges $\{\alpha, \beta\}$ $(1 \leq \beta \leq n,\ \beta \neq \alpha)$ in $G(n,p)$. Following Barbour [4], we let

$$W_\alpha^* = W - X_\alpha + \sum_{\beta=1,\beta\neq\alpha}^{n} E_{\alpha\beta} \prod_{\gamma\neq\alpha,\beta} (1 - E_{\beta\gamma}),$$

where $\sum_{\beta=1,\beta\neq\alpha}^{n} E_{\alpha\beta} \prod_{\gamma\neq\alpha,\beta} (1 - E_{\beta\gamma})$ is the number of isolated vertices which are connected to the vertex α. Then W_α^* has the same distribution as $W - X_\alpha$ conditional on $X_\alpha = 1$, and we observe that $W_\alpha^* \geq W - X_\alpha$ for every $\alpha \in \{1, ..., n\}$. Thus, by (2.12) and (2.13), a non-uniform bound for approximating the distribution of the number of isolated vertices in $G(n,p)$ by Poisson distribution is of the form

$$\begin{aligned}
&\left| P(W \in A) - \sum_{k\in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \\
&\qquad \leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \left\{ Var[W] - \lambda + 2\sum_{\alpha=1}^{n} p_\alpha^2 \right\} \\
&\qquad \leq \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} [(n-2)p + 1]e^{-(n-2)p},
\end{aligned}$$

where $A \subseteq \{0, 1, ..., n\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq \lambda^{-1}(\lambda + e^{-\lambda} - 1)[(n-2)p + 1]e^{-(n-2)p}.$$

The bounds are small whenever np is large. If $n = 100$ and $p = 0.1$, then we have $\lambda = 0.00295$ and, by (2.15), a non-uniform bound of the error in the Poisson approximation of the distribution of W is of the form

$$\left| P(W \in A) - \sum_{k\in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \begin{cases} 0.000001770 & \text{if } M_A \leq 1, \\ \dfrac{0.000003546}{M_A + 1} & \text{if } M_A \geq 2, \end{cases}$$

where $A \subseteq \{0, 1, ..., 100\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq 0.000000883.$$

Example 4.4 (The matching problem)

Suppose that n cards (numbered $1, 2, ..., n$) are placed at random onto n numbered places, labelled $1, 2, \ldots, n$, on a table in such a way that each place receives one card. We say that a match occurs at the α'th place if the card numbered α is placed there. For each $\alpha \in \{1, ..., n\}$, let $X_\alpha = 1$ if the card numbered α is at the α'th place and 0 otherwise. Then the probability $P(X_\alpha = 1) = \dfrac{1}{n}$. Let $W = \sum_{\alpha=1}^{n} X_\alpha$ be the total number of matches. We observe that the distribution of W seems reasonable to be approximated by Poisson distribution with mean $\lambda = E[W] = 1$ when n is large.

We have to construct the coupled random variable W_α^* from the number of matches excluding the match at α'th place conditional on $X_\alpha = 1$, which is obtained by swapping the card at the α'th place, if match at α'th place is not occurs, with the card numbered α at the β'th place. Thus, we can set

$$W_\alpha^* = W - X_\alpha + \sum_{\beta=1, \beta \neq \alpha}^{n} I_\beta,$$

where I_β is the indicator of the event that the card at the β'th place is chosen to be swapped with the card at α'th place and the card at α'th place numbered β. Then W_α^* has the same distribution as $W - X_\alpha$ conditional on $X_\alpha = 1$, and for every $\alpha \in \{1, ..., n\}$, $W_\alpha^* \geq W - X_\alpha$. So, by (2.12), (2.13) and (2.15), a non-uniform bound in approximating the distribution of the number of matches is of the form

$$\left| P(W \in A) - \sum_{k \in A} \frac{e^{-1}}{k!} \right| \leq \begin{cases} \dfrac{2}{n} & \text{if } M_A \leq 1, \\ \dfrac{2e}{n(M_A + 1)} & \text{if } M_A \geq 2, \end{cases}$$

where $A \subseteq \{0, 1, ..., n\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq \frac{2e^{-1}}{n}.$$

The bounds are small whenever n is large. For $n = 100$, we obtain non-uniform bound for this approximation in the form of

$$\left|P(W \in A) - \sum_{k \in A} \frac{e^{-1}}{k!}\right| \leq \begin{cases} 0.02 & \text{if } M_A \leq 1, \\ \dfrac{0.0544}{M_A + 1} & \text{if } M_A \geq 2, \end{cases}$$

where $A \subseteq \{0, 1, ..., 100\}$ and

$$\left|P(W = 0) - e^{-\lambda}\right| \leq 0.00736.$$

Example 4.5 (The m*é*nage problem)

The classical m*é*nage problem asks for the number of seatings of n married couples at a round table, with men and women alternating, so that no one sits next to his or her partner. More generally, we may ask for the probability that a random seating gives exactly k couples sitting together. We number the seats around the table from 1 to $2n$, and let $X_\alpha = 1$ if a couple occupies seats α and $\alpha + 1$ and $X_\alpha = 0$ otherwise. Then W, the number of couples sitting next to each other, can be represented by $W = \sum_{\alpha=1}^{2n} X_\alpha$, where $X_{2n+1} = X_1$ and, by symmetry, $p_\alpha = P(X_\alpha = 1) = 1/n$, and $\lambda = E[W] = 2$.

To construct the coupled random variable W_α^*, we exchange the person in seat $\alpha + 1$ with the spouse of the person in seat α and then count the number of adjacent spouse pairs, excluding the pair now occupying seats α and $\alpha + 1$. From Lange [11, p.251], he bounded the term $E|W - W_\alpha^*|$ by $\frac{6(n-2)}{n(n-1)}$, i.e., $E|W - W_\alpha^*| \leq \frac{6(n-2)}{n(n-1)}$. By applying Theorem 2.2, a non-uniform bound in approximating the distribution of the number of couples sitting next to each other by Poisson distribution with mean 2 is of the form

$$\left|P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!}\right| \leq \begin{cases} \dfrac{6(n-2)}{n(n-1)} & \text{if } M_A \leq 8, \\ \dfrac{6(n-2)(e^2+1)}{n(n-1)(M_A+1)} & \text{if } M_A \geq 9, \end{cases}$$

where $A \subseteq \{0, 1, ..., 2n\}$ and

$$\left|P(W = 0) - e^{-\lambda}\right| \leq \frac{3(1 + e^{-2})(n-2)}{n(n-1)}.$$

A non-uniform bound for this approximation when $n = 100$ is

$$\left| P(W \in A) - \sum_{k \in A} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \begin{cases} 0.0594 & \text{if } M_A \leq 8, \\ \dfrac{0.4983}{M_A + 1} & \text{if } M_A \geq 9, \end{cases}$$

where $A \subseteq \{0, 1, ..., 200\}$ and

$$\left| P(W = 0) - e^{-\lambda} \right| \leq 0.03372.$$

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