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Strong Convergence of the Modified Projection and Contraction Methods for Split Feasibility Problem

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Abstract : In this work, we propose the modified projection and contraction methods for split feasibility problem. We prove the strong convergence theorems under mild conditions. Finally, we provide numerical experiments to show the efficiency of our proposed algorithm.

Keywords : split feasibility problem; CQ algorithm; Hilbert space; projection
2000 Mathematics Subject Classification : 65K05; 90C30; 90C25; 65K10

1 Introduction

In recent years, the split feasibility problem (SFP) by Censor and Elfving [6] has been intensively investigated, since many real-world problems can be reformulated in signal processing, image reconstruction [19], intensity-modulated radiation therapy[5] and many other applied fields. The SFP can be modeled as the problem of finding a point x^* in $\mathbb{R}^N$ such that:

$$x^* \in C \text{ and } Ax^* \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively and A is an $M \times N$ matrix. Throughout this paper, we denote by S the solution set

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of (1.1). The SFP was first studied in Euclidean spaces. Later, Xu [22] considered the SFP in Hilbert spaces.

In [3, 4], Byrne presented a new method called the CQ algorithm for solving the SFP, which generates a sequence $\{x_n\}$ recursively as follows :

$$x_{n+1} = P_C(x_n - \tau_n A^T(I - P_Q)Ax_n) \tag{1.2}$$

where P_C and P_Q denote the metric projections onto C and Q, respectively and the stepsize τ_n is chosen in the interval $(0, 2/\|A\|^2)$, where $\|A\|^2$ is the spectral radius of the operator A^TA.

Recently, there have been many authors considered to establish some convergence theorems for the SFP (see also [15, 14, 12, 16, 17, 18, 7]). In 2004, Yang [23] introduced a relaxed CQ algorithm for solving the SFP, where two half spaces are used instead of C and Q, respectively.

We note that both the CQ algorithm and the relaxed CQ algorithm use a fixed stepsize depending on the largest eigenvalue of the matrix A^TA, which is in general not an easy work in practice. Hence, one way to avoid this estimation was proposed by Qu and Xiu [13] by adopting an Armijo-line search in Euclidean spaces. Subsequently, Gibali et al. [9] introduced relaxation CQ algorithm with the Armijo-linesearch in real Hilbert spaces.

Korpelevich [11] and Antipin [1] proposed the following extragradient method:

$$\begin{aligned} y_n &= P_C(x_n - \tau_n F(x_n)) \\ x_{n+1} &= P_C(x_n - \tau_n F(y_n)) \end{aligned} \tag{1.3}$$

where $F = A^T(I - P_Q)A$ and the fixed stepsize $\tau_n \in (0, \dfrac{1}{\|F\|})$, which is a classical two-step method. The second one is to select the self-adaptively the stepsize $\tau_n > 0$ such that

$$\tau_n\|F(x_n) - F(y_n)\| \leq \mu\|x_n - y_n\|, \ \forall\mu \in (0, 1). \tag{1.4}$$

In [21], Tseng proposed the following extragradient methods:

$$\begin{aligned} y_n &= P_C(x_n - \tau_n F(x_n)) \\ x_{n+1} &= y_n + \tau_n(F(x_n) - F(y_n)) \end{aligned} \tag{1.5}$$

where $\tau_n \in (0, \dfrac{1}{\|F\|})$ or $\{\tau_n\}$ is selected self-adaptively. Subsequently, Zhao et al. [25] used Tseng's method (1.5) to solve the SFP. Recently, Dong et al. [8] proposed the modified projection and contraction methods and their relaxation variants to solve the SFP as follows:

Algorithm 1.1. *For any $\sigma > 0, \rho \in (0, 1)$ and $\mu \in (0, 1)$, take arbitrarily $x_1 \in \mathbb{R}^N$ and let*

$$y_n = P_C(x_n - \tau_n F(x_n)) \tag{1.6}$$

where $\tau_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\tau_n\|F(x_n) - F(y_n)\| \leq \mu\|x_n - y_n\|. \tag{1.7}$$

Define

$$x_{n+1} = x_n - \gamma \phi_n d(x_n, y_n) \tag{1.8}$$

where $\gamma \in (0, 2)$

$$d(x_n, y_n) = (x_n - y_n) - \tau_n (F(x_n) - F(y_n)) \tag{1.9}$$

and

$$\phi_n = \frac{\langle x_n - y_n, d(x_n, y_n) \rangle + \tau_n \|(I - P_Q)A(y_n)\|^2}{\|d(x_n, y_n)\|^2}. \tag{1.10}$$

It was proved that the sequence generated by Algorithm 1.1 converges to a solution in SFP.

In this work, motivated by Dong et al. [8], we propose the modified projection and contraction methods and relaxation to solve the SFP in real Hilbert spaces and prove some strong convergence theorems of the proposed under mild assumptions. Finally, we provide numerical experiments to show the efficiency of our proposed algorithm.

2 Preliminaries

In this section, we give some definitions and lemmas, which are used in the main results. Throughout this paper, we recall the following definitions:

(1) A mapping $T : H_1 \to H_1$ is said to be *firmly nonexpansive* if, for all $x, y \in H_1$,

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2. \tag{2.1}$$

(2) A function $f : H_1 \to \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) \tag{2.2}$$

for all $\lambda \in (0, 1)$ and $x, y \in H_1$.

(3) F is said to be monotone on C if

$$\langle F(x) - F(y), x - y \rangle \geq 0, \ \forall x, y \in C \tag{2.3}$$

(4) F is said to be τ_n-inverse strongly monotone (shortly, τ_n-ism) with $\tau_n > 0$ if

$$\langle F(x) - F(y), x - y \rangle \geq \tau_n \|F(x) - F(y)\|^2, \ \forall x, y \in C; \tag{2.4}$$

(5) F is said to be Lipschitz continuous on C with constant $\lambda > 0$ if

$$\|F(x) - F(y)\| \leq \lambda \|x - y\|, \ \forall x, y \in C. \tag{2.5}$$

(6) A mapping $f : H_1 \to H_1$ is said to be a contraction on H_1 if there exists a constant $a \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq a \|x - y\|, \ \forall x, y \in H_1. \tag{2.6}$$

(7) A differentiable function f is convex if and only if there holds the inequality:

$$f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle \tag{2.7}$$

for all $z \in H_1$.

(8) An element $g \in H_1$ is called a *subgradient* of $f : H_1 \to \mathbb{R}$ at x if

$$f(z) \geq f(x) + \langle g, z - x \rangle \tag{2.8}$$

for all $z \in H_1$, which is called the *subdifferentiable inequality.*

(9) A function $f : H_1 \to \mathbb{R}$ is said to be *subdifferentiable* at x if it has at least one subgradient at x.

(10) The set of subgradients of f at the point x is called the *subdifferentiable* of f at x, which is denoted by $\partial f(x)$.

(11) A function f is said to be *subdifferentiable* if it is subdifferentiable at all $x \in H_1$. If a function f is differentiable and convex, then its gradient and subgradient coincide.

(12) A function $f : H_1 \to \mathbb{R}$ is said to be *weakly lower semi-continuous* (shortly, w-lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n\to\infty} f(x_n). \tag{2.9}$$

We know that the orthogonal projection P_C from H_1 onto a nonempty closed convex subset $C \subset H_1$ is a typical example of a firmly nonexpansive mapping, which is defined by

$$P_C x := \arg\min_{y\in C} \|x - y\|^2 \tag{2.10}$$

for all $x \in H_1$.

Lemma 2.1. [2] *For any $x \in H$ and $z \in C$, then $z = P_C x$ if and only if*

$$\langle x - z, y - z \rangle \leq 0, \forall y \in C. \tag{2.11}$$

Lemma 2.2. [2] *Let C be a nonempty closed convex subset of a real Hilbert space H_1. Then, for any $x \in H_1$, the following assertions hold:*

(1) $\langle x - P_C x, z - P_C x \rangle \leq 0$ *for all $z \in C$;*

(2) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$ *for all $x, y \in H_1$;*

(3) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$ *for all $z \in C$.*

From Lemma 2.2, the operator $I - P_C$ is also firmly nonexpansive, where I denotes the identity operator, i.e., for any $x, y \in H_1$,

$$\langle (I - P_C)x - (I - P_C)y, x - y \rangle \geq \|(I - P_C)x - (I - P_C)y\|^2. \tag{2.12}$$

Lemma 2.3. *[10] Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{aligned} s_{n+1} &\leq (1-c_n)s_n + c_n\delta_n, \quad (2.13)\\ s_{n+1} &\leq s_n - \lambda_n + \varphi_n, \quad (2.14)\end{aligned}$$

for all $n \geq 1$ where $\{c_n\}$ is a sequence in $(0,1)$, $\{\lambda_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\varphi_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} c_n = \infty$;

(ii) $\lim_{n\to\infty} \varphi_n = 0$;

(iii) $\lim_{k\to\infty} \lambda_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *[24] The line rule (3.4) is well defined. Besides, $\tau' \leq \tau_n \leq \sigma$, where $\tau' = \min\{\sigma, \frac{\mu\rho}{L}\}$.*

Lemma 2.5. *[8] Let $\{x_n\}$ and $\{y_n\}$ be the iterations generated by Algorithm 1.1. Then we have*

$$\langle x_n - z, d(x_n, y_n)\rangle \geq \phi_n \|d(x_n, y_n)\|^2, \ \forall z \in S. \quad (2.15)$$

Lemma 2.6. *[8] Let $\{x_n\}$ and $\{y_n\}$ be the iterations generated by Algorithm 1.1. Then we have*

$$\langle x_n - y_n, d(x_n, y_n)\rangle \geq (1-\mu)\|x_n - y_n\|^2 \quad (2.16)$$

and

$$\phi_n \geq \frac{1-\mu}{1+\mu^2}. \quad (2.17)$$

3 Main Results

3.1 The modified projection and contraction methods

In this section, we introduce a projection algorithm using linesearch for the strong convergence theorem. We define the functions

$$F(x) = A^T(I - P_Q)A(x). \quad (3.1)$$

Algorithm 3.1. *Let $f : H \to H$ be a contraction. For any $\sigma > 0, \rho \in (0,1)$ and $\mu \in (0,1)$, choose an arbitrary initial guess $x_1 \in H$. Assume x_n and y_n have been constructed. Compute the sequence x_{n+1} via the formula*

$$\begin{aligned} y_n &= P_C(x_n - \tau_n F(x_n)) \quad (3.2)\\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) \quad (3.3)\end{aligned}$$

where $\gamma \in (0,2)$ *and* $\tau_n = \sigma\rho^{m_n}$ *and* m_n *is the smallest nonnegative integer such that*

$$\tau_n\|F(x_n) - F(y_n)\| \le \mu\|x_n - y_n\|, \tag{3.4}$$

$$d(x_n, y_n) = (x_n - y_n) - \tau_n(F(x_n) - F(y_n)) \tag{3.5}$$

and

$$\phi_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle + \tau_n\|(I - P_Q)Ay_n\|^2}{\|d(x_n, y_n)\|^2}. \tag{3.6}$$

Theorem 3.2. *Assume that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If* $S \ne \emptyset$, *then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to* $z = P_S f(z)$ *in* S.

Proof. We set $z = P_S f(z)$. Then, by Lemma 2.5, we have

$$\begin{aligned}
\|x_n - \gamma\phi_n d(x_n, y_n) - z\|^2 &= \|x_n - z\|^2 - 2\gamma\phi_n\langle x_n - z, d(x_n, y_n)\rangle \\
&\quad +\gamma^2\phi_n^2\|d(x_n, y_n)\|^2 \\
&\le \|x_n - z\|^2 - 2\gamma\phi_n^2\|d(x_n, y_n)\|^2 \\
&\quad +\gamma^2\phi_n^2\|d(x_n, y_n)\|^2 \\
&= \|x_n - z\|^2 - \gamma(2-\gamma)\phi_n^2\|d(x_n, y_n)\|^2.
\end{aligned} \tag{3.7}$$

So, we obtain

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
= \; &\|\alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) - z\|^2 \\
= \; &\langle \alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) - z, x_{n+1} - z\rangle \\
= \; &\alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&+(1-\alpha_n)\langle x_n - \gamma\phi_n d(x_n, y_n) - z, x_{n+1} - z\rangle \\
\le \; &\alpha_n\|f(x_n) - f(z)\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&+(1-\alpha_n)\|x_n - \gamma\phi_n d(x_n, y_n) - z\|\|x_{n+1} - z\| \\
\le \; &\frac{1}{2}\alpha_n(\|f(x_n) - f(z)\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&+\frac{1}{2}(1-\alpha_n)(\|x_n - \gamma\phi_n d(x_n, y_n) - z\|^2 + \|x_{n+1} - z\|^2) \\
\le \; &\frac{1}{2}\alpha_n a\|x_n - z\|^2 + \frac{1}{2}\alpha_n\|x_{n+1} - z\|^2 + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&+\frac{1}{2}(1-\alpha_n)(\|x_n - z\|^2 - \gamma(2-\gamma)\phi_n^2\|d(x_n, y_n)\|^2 + \|x_{n+1} - z\|^2).
\end{aligned} \tag{3.8}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - z\|^2 \le \; &(1 - \alpha_n(1-a))\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&-(1-\alpha_n)\gamma(2-\gamma)\phi_n^2\|d(x_n, y_n)\|^2.
\end{aligned} \tag{3.9}$$

Next, we will show that $\{x_n\}$ is bounded. We see that

$$\begin{aligned}\|x_{n+1}-z\| &= \|\alpha_n f(x_n)+(1-\alpha_n)(x_n-\gamma\phi_n d(x_n,y_n))-z\| \\ &\leq \alpha_n\|f(x_n)-z\|+(1-\alpha_n)\|x_n-\gamma\phi_n d(x_n,y_n)-z\| \\ &\leq \alpha_n\|f(x_n)-f(z)\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|x_n-z\| \\ &\leq \alpha_n a\|x_n-z\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|x_n-z\| \\ &= (1-\alpha_n(1-a))\|x_n-z\|+\alpha_n\|f(z)-z\|. \end{aligned} \tag{3.10}$$

By induction, we can show that $\{x_n\}$ is bounded. Employing Lemma 2.3 and (3.9), we set

$$\begin{aligned} s_n &= \|x_n-z\|^2 \\ \varphi_n &= 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\ \delta_n &= \frac{2}{1-a}\langle f(z)-z, x_{n+1}-z\rangle \\ \lambda_n &= (1-\alpha_n)\gamma(2-\gamma)\phi_n^2\|d(x_n,y_n)\|^2 \\ c_n &= (1-a)\alpha_n. \end{aligned} \tag{3.11}$$

So, (3.9) reduces to the inequalities

$$s_{n+1} \leq (1-c_n)s_n+c_n\delta_n \tag{3.12}$$

$$s_{n+1} \leq s_n-\lambda_n+\varphi_n. \tag{3.13}$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\limsup_{k\to\infty}\lambda_{n_k}\leq 0. \tag{3.14}$$

It follows that

$$\limsup_{k\to\infty}(1-\alpha_{n_k})\gamma(2-\gamma)\phi_{n_k}^2\|d(x_{n_k},y_{n_k})\|^2\leq 0. \tag{3.15}$$

Using Lemma 2.6, we obtain

$$\lim_{k\to\infty}\|d(x_{n_k},y_{n_k})\|=0. \tag{3.16}$$

By (3.5) we see that

$$\begin{aligned}\|x_{n_k}-y_{n_k}\| &\leq \|d(x_{n_k},y_{n_k})\|+\tau_{n_k}\|F(x_{n_k})-F(y_{n_k})\| \\ &\leq \|d(x_{n_k},y_{n_k})\|+\mu\|x_{n_k}-y_{n_k}\|. \end{aligned} \tag{3.17}$$

It follows that

$$(1-\mu)\|x_{n_k}-y_{n_k}\|\leq\|d(x_{n_k},y_{n_k})\|. \tag{3.18}$$

From (3.16), we obtain

$$\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0. \tag{3.19}$$

Consider

$$\begin{aligned}
& \|x_{n_k+1} - x_{n_k}\| \\
= \;& \|\alpha_{n_k} f(x_{n_k}) + (1-\alpha_{n_k})(x_{n_k} - \gamma\phi_{n_k} d(x_{n_k}, y_{n_k})) - x_{n_k}\| \\
\leq \;& \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\alpha_{n_k})\|x_{n_k} - \gamma\phi_{n_k} d(x_{n_k}, y_{n_k}) - x_{n_k}\| \\
= \;& \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\alpha_{n_k})\gamma\phi_{n_k}\|d(x_{n_k}, y_{n_k})\| \\
\to \;& 0, \text{ as } k \to \infty.
\end{aligned} \tag{3.20}$$

By the definitions of $\{y_{n_k}\}$ and $d(x_{n_k}, y_{n_k})$, we get

$$y_{n_k} = P_C(y_{n_k} - (\tau_{n_k} F(y_{n_k}) - d(x_{n_k}, y_{n_k}))). \tag{3.21}$$

From Lemma 2.1, it follows that

$$\langle x - y_{n_k}, \tau_{n_k} F(y_{n_k}) - d(x_{n_k}, y_{n_k})\rangle \geq 0, \ \forall x \in C. \tag{3.22}$$

Take arbitrarily $z \in S \subset C$. By setting $x = z$ in (3.22), we have

$$\langle y_{n_k} - z, d(x_{n_k}, y_{n_k}) - \tau_{n_k} F(y_{n_k})\rangle \geq 0, \tag{3.23}$$

which implies that

$$\langle y_{n_k} - z, d(x_{n_k}, y_{n_k})\rangle \geq \tau_{n_k}\langle y_{n_k} - z, F(y_{n_k})\rangle \tag{3.24}$$

Since $\{x_{n_k}\}$ is bounded, the set $\omega_w(x_{n_k})$ is nonempty. Let $x^* \in \omega_w(x_{n_k})$ then there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightharpoonup x^*$.

Next, we show that x^* is a solution of the SFP. From (3.16) and the boundedness of $\{y_{n_k}\}$ implies

$$\begin{aligned}
\tau_{n_k}\|Ay_{n_k} - P_Q Ay_{n_k}\|^2 &\leq \tau_{n_k}\langle Ay_{n_k} - Az, (I-P_Q)Ay_{n_k} - (I-P_Q)Az\rangle \\
&= \tau_{n_k}\langle Ay_{n_k} - Az, (I-P_Q)Ay_{n_k}\rangle \\
&= \tau_{n_k}\langle y_{n_k} - z, A^T(I-P_Q)Ay_{n_k}\rangle \\
&= \tau_{n_k}\langle y_{n_k} - z, F(y_{n_k})\rangle.
\end{aligned} \tag{3.25}$$

By (3.24), (3.16) and Lemma 2.2, we have

$$\begin{aligned}
\|Ay_{n_k} - P_Q Ay_{n_k}\|^2 &\leq \frac{1}{\tau'}\langle y_{n_k} - z, d(x_{n_k}, y_{n_k})\rangle \\
&\leq \frac{1}{\tau'}\|y_{n_k} - z\|\|d(x_{n_k}, y_{n_k})\| \\
&\to 0.
\end{aligned} \tag{3.26}$$

Hence,

$$\lim_{k\to\infty}\|Ay_{n_k} - P_Q Ay_{n_k}\| = 0. \tag{3.27}$$

Thus $Ax^* \in Q$. From (3.1) and (3.27), it follows that $\lim_{k\to\infty} \|F(y_{n_k})\| = 0$. By (3.2) and Lemma 2.2 (3), we have

$$\begin{aligned} \|y_{n_k} - P_C(y_{n_k})\| &\leq \|x_{n_k} - y_{n_k} - \tau_{n_k} F(x_{n_k})\| \\ &\leq \|x_{n_k} - y_{n_k}\| + \tau_{n_k}\|F(x_{n_k})\| \\ &\leq \|x_{n_k} - y_{n_k}\| + \tau_{n_k}\|F(x_{n_k}) - F(y_{n_k})\| + \tau_{n_k}\|F(y_{n_k})\| \\ &\leq (1+\mu)\|x_{n_k} - y_{n_k}\| + \tau_{n_k}\|F(y_{n_k})\| \\ &\to 0 \text{ as } k \to \infty, \end{aligned} \tag{3.28}$$

which implies $x^* \in C$. From Lemma 2.2 (1), we obtain

$$\begin{aligned} \limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle &= \lim_{i\to\infty}\langle f(z) - z, x_{n_{k_i}} - z\rangle \\ &= \langle f(z) - z, x^* - z\rangle \\ &\leq 0. \end{aligned} \tag{3.29}$$

From (3.20) and (3.29), we obtain

$$\limsup_{k\to\infty}\langle f(z) - z, x_{n_k+1} - z\rangle \leq 0. \tag{3.30}$$

Hence, we get

$$\limsup_{k\to\infty} \delta_{n_k} \leq 0. \tag{3.31}$$

Using Lemma 2.3, we conclude that the sequence $\{x_n\}$ converges strongly to $z = P_S f(z)$. □

3.2 The modified relaxation projection and contraction methods

In this section, we introduce the modified relaxation projection and contraction methods, in which the closed convex subsets C and Q have particular structure.

For the SFP, we assume that the convex sets C and Q satisfy the following conditions:

(A1) The set C is given by

$$C = \{x \in H_1 : c(x) \leq 0\}, \tag{3.32}$$

where $c : H_1 \to \mathbb{R}$ is a convex function and C is a nonempty set. The set Q is given by

$$Q = \{y \in H_2 : q(y) \leq 0\}, \tag{3.33}$$

where $q : H_2 \to \mathbb{R}$ is a convex function and Q is a nonempty set. Assume that c and q are subdifferentiable on C and Q, respectively, and c and q are bounded on bounded sets. Note that this condition is automatically satisfied in finite dimensional spaces.

For any $x \in H_1$, at least one subgradient $\xi \in \partial c(x)$ can be calculated, where $\partial c(x)$ is defined as follows:

$$\partial c(x) = \{z \in H_1 : c(u) \geq c(x) + \langle u - x, z\rangle, \forall u \in H_1\}. \tag{3.34}$$

For any $y \in \mathbb{R}^M$, at least one subgradient $\eta \in \partial q(y)$ can be calculated, where

$$\partial q(x) = \{w \in H_2 : q(u) \geq q(y) + \langle v - y, w\rangle, \forall v \in H_2\}. \tag{3.35}$$

Define the sets C_n and Q_n by the following half-spaces:

$$C_n = \{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n\rangle \leq 0\}, \tag{3.36}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in H_2 : q(Ax_n) + \langle \eta_n, y - Ax_n\rangle \leq 0\}, \tag{3.37}$$

where $\eta_n \in \partial q(Ax_n)$.

By the definition of the subgradient, it is clear that $C \subseteq C_n$ and $Q \subseteq Q_n$. The projections onto C_n and Q_n are easy to compute since C_n and Q_n are two half-spaces.

Define $f_n(x) = \frac{1}{2}\|(I - P_{Q_n})A(x)\|^2$ and

$$F_n(x) = A^T(I - P_{Q_n})A(x). \tag{3.38}$$

Algorithm 3.3. *Let $f : H \to H$ be a contraction. For any $\sigma > 0, \rho \in (0,1)$ and $\mu \in (0,1)$, choose an arbitrary initial guess x_1. Assume x_n and y_n have been constructed. Compute the sequence x_{n+1} via the formula*

$$\begin{aligned} y_n &= P_{C_n}(x_n - \tau_n F_n(x_n)) \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) \end{aligned} \tag{3.39}$$

where $\gamma \in (0,2)$ and $\tau_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\tau_n\|F_n(x_n) - F_n(y_n)\| \leq \mu\|x_n - y_n\|, \tag{3.40}$$

$$d(x_n, y_n) = (x_n - y_n) - \tau_n(F_n(x_n) - F_n(y_n)) \tag{3.41}$$

and

$$\phi_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle + \tau_n\|(I - P_{Q_n})Ay_n\|^2}{\|d(x_n, y_n)\|^2}. \tag{3.42}$$

Theorem 3.4. *Assume that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $S \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 3.3 converges strongly to $z = P_S f(z)$ in S.*

Proof. As in the proof of Theorem 3.2, we see that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1-\alpha_n(1-a))\|x_n-z\|^2+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle \\ &\quad -(1-\alpha_n)\gamma(2-\gamma)\phi_n^2\|d(x_n,y_n)\|^2. \end{aligned} \tag{3.43}$$

Moreover, the sequence $\{x_n\}$ is bounded and $\|x_n-y_n\|\to 0$. Let x^* be a cluster point of $\{x_n\}$ with $\{x_{n_k}\}$ converging to x^*. From (3.19), it follows that $\{y_{n_k}\}$ also converges to x^*.

Now, we show that x^* is a solution of the SFP. In fact, since $y_{n_k}\in C_{n_k}$, by the definition of $\{C_{n_k}\}$, we have

$$c(x_{n_k})+\langle \xi_{n_k},y_{n_k}-x_{n_k}\rangle\leq 0, \tag{3.44}$$

where $\xi_{n_k}\in\partial c(x_{n_k})$. Since ∂c is bounded and (3.19), we have

$$\begin{aligned} c(x_{n_k}) &\leq \langle\xi_{n_k},x_{n_k}-y_{n_k}\rangle \\ &\leq \|\xi_{n_k}\|\|y_{n_k}-x_{n_k}\| \\ &\to 0 \text{ as } k\to\infty. \end{aligned} \tag{3.45}$$

which implies $c(x^*)\leq 0$, i.e., $x^*\in C$. As in Theorem 3.2, we can show that $\|A_{y_{n_k}}-P_{Q_{n_k}}(A_{y_{n_k}})\|\to 0$ as $k\to\infty$. Since $P_{Q_{n_k}}(Ay_{n_k})\in Q_{n_k}$, we have

$$q(Ay_{n_k})+\langle\eta_{n_k},P_{Q_{n_k}}(Ay_{n_k})-Ay_{n_k}\rangle\leq 0 \tag{3.46}$$

where $\eta_{n_k}\in\partial q(Ay_{n_k})$. From (3.27), we obtain

$$\begin{aligned} q(Ay_{n_k}) &\leq \|\eta_{n_k}\|\|P_{Q_{n_k}}(Ay_{n_k})-Ay_{n_k}\| \\ &\to 0 \text{ as } k\to\infty. \end{aligned} \tag{3.47}$$

Similarly, we have $q(Ax^*)\leq 0$, i.e., $Ax^*\in Q$. Thus x^* is a solution of the SFP.

Following the line of the proof of Theorem 3.2 we get that $\{x_n\}$ converges strongly to $z=P_Sf(z)$. □

4 Numerical Experiments

In this section, we present some numerical examples and illustrate its performance by using Algorithm 3.1 in Theorem 3.2 and Algorithm 3.3 in Theorem 3.4.

Example 4.1. *Let*

$$\begin{aligned} C &= \{(x_1,x_2,x_3)\in\mathbb{R}^3:\|(x_1,x_2,x_3)-(0.5,0,0)\|_2\leq 10\}, \\ Q &= \{(y_1,y_2,y_3)\in\mathbb{R}^3:(15,0,0)\leq(y_1,y_2,y_3)\leq(25,0,0)\}, \end{aligned}$$

and $A = \begin{pmatrix} -1 & 0 & -9 \\ 5 & 9 & 1 \\ -1 & 0 & 1 \end{pmatrix}$. *Choose* $\alpha_n = \dfrac{1}{100n}$, *for all* $n \in \mathbb{N}$ *and* $f(x) = \dfrac{1}{2}x$ *where* $x \in \mathbb{R}^3$. *The stopping criterion is defined by* $E_n = \|x_{n+1} - x_n\|_2 < 10^{-4}$.

We consider four cases as follows:

Case 1: $x_1 = (-2, 1, 0)$, $\sigma = 1$, $\rho = 0.5$, $\mu = 0.6$ *and* $\gamma = 1.5$.
Case 2: $x_1 = (-1, 0, 3)$, $\sigma = 2$, $\rho = 0.6$, $\mu = 0.7$ *and* $\gamma = 0.5$.
Case 3: $x_1 = (-4, 0, 2)$, $\sigma = 3$, $\rho = 0.2$, $\mu = 0.3$ *and* $\gamma = 1.9$.
Case 4: $x_1 = (0, -2, 1)$, $\sigma = 4$, $\rho = 0.9$, $\mu = 0.5$ *and* $\gamma = 0.3$.

Using Algorithm 3.1 in Theorem 3.2, we obtain the following results:

Table 1: Algorithm 3.1 with different cases.

	Case 1	*Case 2*	*Case 3*	*Case 4*
No. of Iter.	*159*	*101*	*266*	*119*
cpu (Time)	*0.0682*	*0.0593*	*0.0814*	*0.2459*

The convergence behavior of the error E_n *for each Cases is shown in Figure 1-4, respectively.*

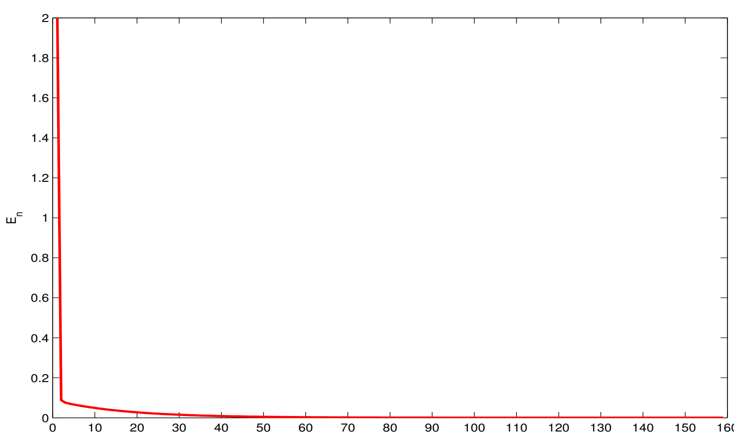

Figure 1: Error plotting E_n for Case 1 in Example 4.1

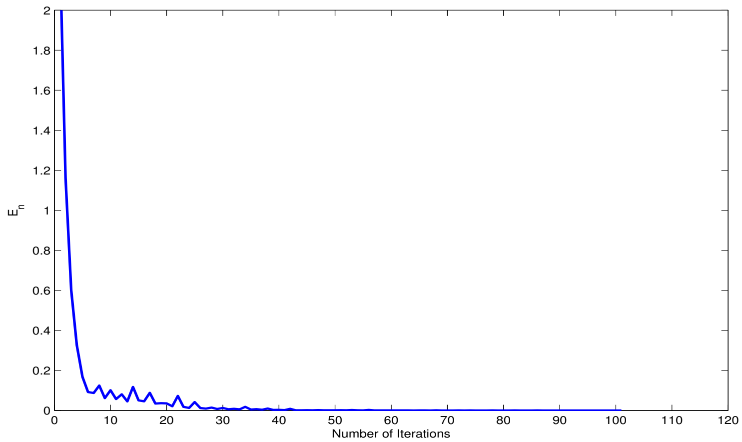

Figure 2: Error plotting E_n for Case 2 in Example 4.1

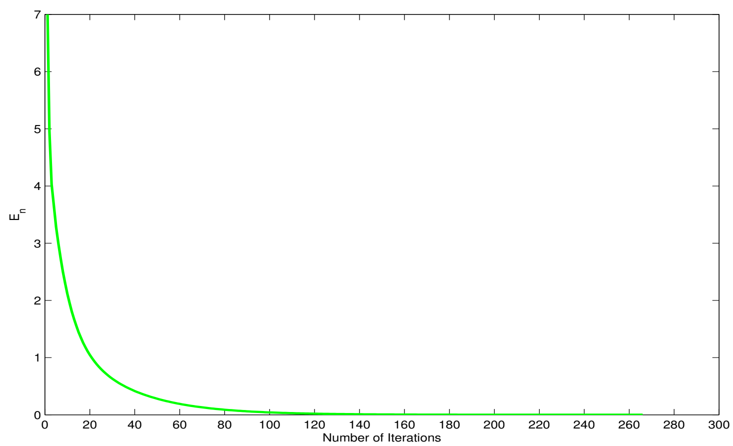

Figure 3: Error plotting E_n for Case 3 in Example 4.1

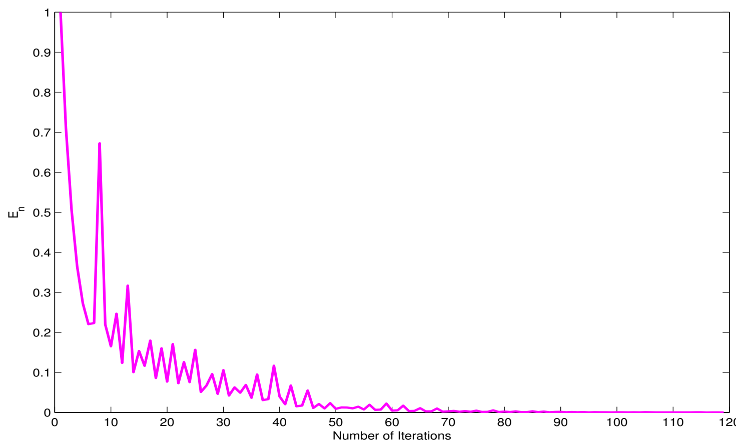

Figure 4: Error plotting E_n for Case 4 in Example 4.1

Example 4.2. *Consider the following LASSO problem [20]:*

$$\min\{\frac{1}{2}\|Ax-b\|^2 : x \in \mathbb{R}^5, \|x\|_1 \le \tau\}, \tag{4.1}$$

where $A = \begin{pmatrix} 1 & 3 & 2 & 1 & 0 \\ 5 & 6 & 1 & -1 & 1 \\ 4 & 2 & 3 & 0 & -2 \\ 0 & 2 & -2 & 1 & 9 \\ 0 & -1 & 3 & 0 & 1 \end{pmatrix}$, $b = (6, 12, 9, 0, 1)$. *We define* $C = \{x \in \mathbb{R}^5 : \|x\|_1 \le \tau\}$ *and* $Q = \{b\}$. *Since the projection onto the closed convex* C *does not have a closed form solution and so we make use of the subgradient projection. Define a convex function* $c(x) = \|x\|_1 - \tau$ *and denote the level set* C_n *by :*

$$C_n = \{x \in \mathbb{R}^5 : c(x_n) + \langle \xi_n, x - x_n \rangle \le 0\}, \tag{4.2}$$

where $\xi_n \in \partial c(x_n)$. *Then the orthogonal projection onto* C_n *can be calculated by the following:*

$$P_{C_n}(x) = \begin{cases} x, & \text{if } c(x_n) + \langle \xi_n, x - x_n \rangle \le 0, \\ x - \dfrac{c(x_n) + \langle \xi_n, x - x_n \rangle}{\|\xi_n\|^2}\xi_n, & \text{otherwise.} \end{cases} \tag{4.3}$$

It is worth noting that the subdifferential ∂c *at* x_n *is*

$$\partial c(x_n) = \begin{cases} 1, & \text{if } x_n > 0, \\ [-1, 1], & \text{if } x_n = 0, \\ -1, & \text{if } x_n < 0. \end{cases} \tag{4.4}$$

The iteration process is stopped when the following criteria satisfied $\|x_{n+1} - x_n\| < 10^{-4}$. *Choose* $\alpha_n = \dfrac{1}{100n}$, *for all* $n \in \mathbb{N}$ *and let* $f(x) = \dfrac{1}{2}x$.

We consider four cases as follows:

Case 1: $x_1 = (-1, 1, 1, 0, 1)$, $\sigma = 1$, $\rho = 0.5$, $\mu = 0.2$ *and* $\gamma = 1.5$.
Case 2: $x_1 = (0, -1, 3, 0, 5)$, $\sigma = 2$, $\rho = 0.4$, $\mu = 0.3$ *and* $\gamma = 0.9$.
Case 3: $x_1 = (1, 9, -2, 0, 5)$, $\sigma = 3$, $\rho = 0.7$, $\mu = 0.6$ *and* $\gamma = 1.9$.
Case 4: $x_1 = (-5, 0, 1, 3, 2)$, $\sigma = 4$, $\rho = 0.2$, $\mu = 0.9$ *and* $\gamma = 0.3$.

Table 2: Algorithm 3.3 with different cases.

	Case 1	*Case 2*	*Case 3*	*Case 4*
No. of Iter.	*272*	*44*	*228*	*54*
cpu (Time)	*0.1444*	*0.0283*	*0.2045*	*0.0204*

The convergence behavior of the error E_n for each Cases is shown in Figure 5-8, respectively.

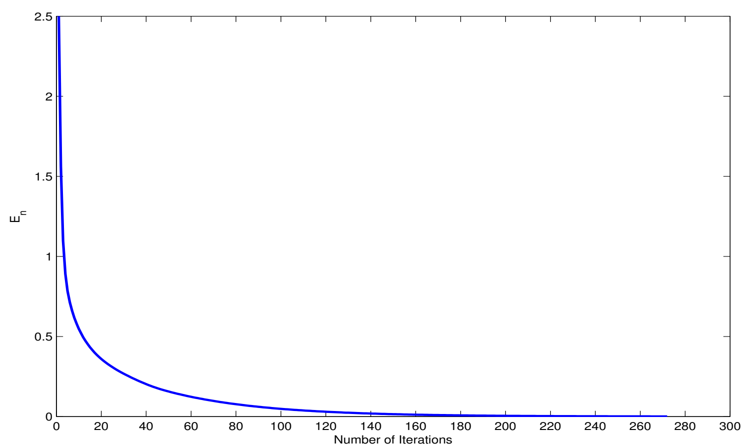

Figure 5: Error plotting E_n for Case 1 in Example 4.2

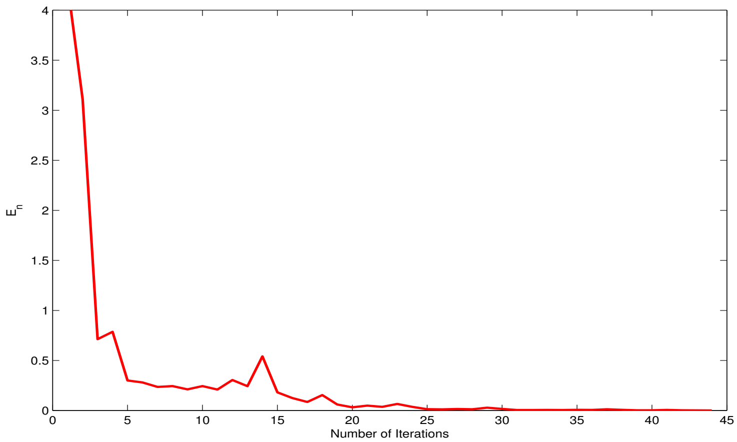

Figure 6: Error plotting E_n for Case 2 in Example 4.2

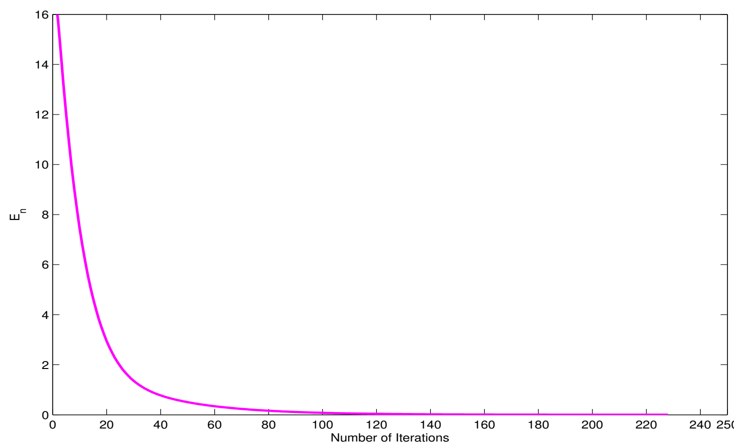

Figure 7: Error plotting E_n for Case 3 in Example 4.2

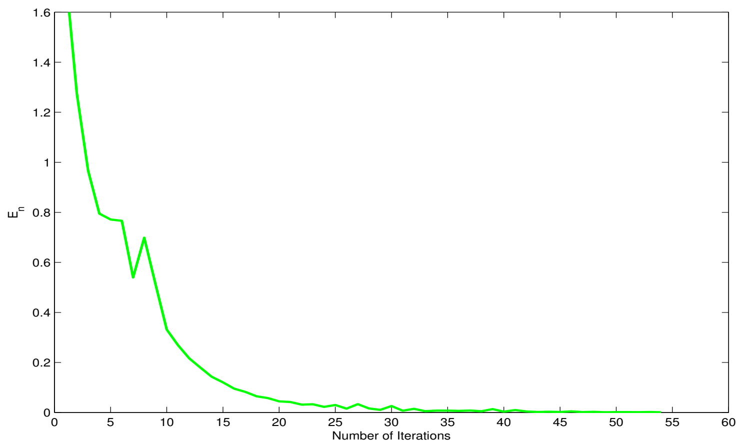

Figure 8: Error plotting E_n for Case 4 in Example 4.2

5 Conclusions

This paper discusses the strong convergence of the modified projection and contraction methods for split feasibility problem. Numerical experiments show the efficiency of our algorithm.

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