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Sum of Ultra Maximal Monotone Operators and Operators of Type (D) in Grothendieck Spaces

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Abstract : Here, assuming Brézis, Crandall and Pazy constraint qualification conditions, we prove that the closure of the sum of an ultra maximal monotone operator and an operator of type (D) is a maximal monotone operator in Banach spaces which satisfy Grothendieck and weakly compactly generated properties.

Keywords : Maximal monotone operator; monotone operator of type (D); Ultramaximal monotone; Moreau-Yosida regularization
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1 Introduction

Monotone operators are an important class of operators used in the study of modern nonlinear analysis and various types of optimization problems. The theory of monotone operators (multifunctions) was first introduced by George Minty [20] and later it was used substantially in proving existence results in partial differential equations by Felix Browder and his school [9, 10, 11, 12, 15, 16, 1, 2, 31]. In particular, maximal monotone operators have found their plethora of applications in partial differential equations, optimization problems, variational inequalities,

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and mathematical economics.

Among all the problems related to the monotone operators and maximal monotone operators, most studied and celebrated problem concerns the maximality of the sum of two maximal monotone operators [3, 4, 6, 7, 8, 11, 22, 24, 25, 26, 27, 29, 30]. It is observed that certain qualification constraints are required to prove the maximality of sum of monotone operators.

In 1970, Rockafellar [22] provided a solution for the maximality of the sum of two maximal monotone operators under the constraint, i.e., one of the domain must intersect the interior of the another one (Rockafellar's constraint). Rockafellar proved this results in reflexive spaces. If the domain of the maximal monotone operators have empty interior, then the previous results cannot be applied. One may note that there are many maximal monotone operators having domains with empty interior [11].

Almost at the same time, Crandall and Pazy gave another qualification constraint for the maximality of the sum of two monotone operators. Interestingly, this qualification constraint is suitable "in a certain sense" for handling the maximal monotone operators having domains with empty interior in Hilbert spaces. Further, this result was extended to reflexive Banach space with some restricted conditions by Brezis, Crandall and Pazy [11, 25]. Now, we formally state the Crandall-Pazy result. Let X be a nonzero reflexive Banach spaces, $A : X \rightrightarrows X^*$ and $B : X \rightrightarrows X^*$ be maximal monotone, and satisfy following constraint qualification conditions (Br*ézes*-Crandall-Pazy constraint), i.e.,

(i) $\operatorname{dom} A \subset \operatorname{dom} B$,

(ii) $|Bx| \leq k(\|x\|)|Ax| + C(\|x\|)$, where $k : [0, \infty) \to [0, 1)$ and $C : [0, \infty) \to [0, \infty)$.

Then $A + B$ is a maximal monotone operator [13, Theorem 4.3].

Here, we will show that the sum of an ultra maximal monotone operator and a (D) type monotone operator with conditions (i) and (ii) is maximal in Banach spaces which satisfy Grothendieck property (Grothendieck space) and weakly compactly generated property.

The remainder of this paper is organized as follows. In Section 2, we provide some auxiliary results and notions which will be used in our main results. Section 3, contains our main results.

2 Preliminaries

A real Banach space X is said to be Grothendieck [21] if every weak star convergence sequence is weakly convergent in X^*. Every reflexive Banach space is a Grothendieck space. But the converse is not true, e.g., the space of bounded nets on some directed set Γ, $l_\infty(\Gamma)$ [21]. The dual of X is denoted as X^*; X and X^* are paired by $\langle x, x^* \rangle = x^*(x)$ for $x \in X$ and $x^* \in X^*$. If necessary, we identify $X \subset X^{**}$ with its image under the canonical embedding of X into X^{**}. Weak and weak star convergence are denoted by the notation $\overset{w}{\to}$ and $\overset{w^*}{\to}$ respectively.

For a given subset C of X, we denote interior of C as int$C$, closure of C as $\overline{C}$, boundary of C as bdry C, convex hull of C as conv$C$ and affine hull of C as aff$C$. Also, we denote the distance function by $\text{dist}(x,C) := \inf_{c\in C} \|x-c\|$ and $|C| = \inf_{c\in C} \|c\|$. For any $C, D \subseteq X$, $C - D := \{|x-y|\ x \in C,\ y \in D\}$. For any $\alpha > 0$, $\alpha C := \{\alpha x| \quad x \in C\}$. Let $A : X \rightrightarrows X^*$ be a set-valued operator (also known as multifunction or point-to-set mapping) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$.The domain of A is denoted as $\text{dom}A := \{x \in X|\ Ax \neq \phi\}$ and the range of A is $\text{ran}A := \{x^* \in Ax|\ x \in \text{dom}A\}$. The graph of A is denoted as $\text{gra}A = \{(x,x^*) \in X \times X^*|\ x^* \in Ax\}$ and we define the inverse of $A : X \rightrightarrows X^*$ as $A^{-1} : X^* \rightrightarrows X$ by $A^{-1}(x^*) := \{x \in X : x^* \in A(x)\}$. A is said to be linear relation if gra$A$ is a linear subspace. The set-valued mapping $A : X \rightrightarrows X^*$ is said to be monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x,x^*), (y,y^*) \in \text{gra}A.$$

Let $A : X \rightrightarrows X^*$ be monotone and $(x,x^*) \in X \times X^*$. We say that (x,x^*) is monotonically related to gra$A$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y,y^*) \in \text{gra}A.$$

And a set-valued mapping A is said to be maximal monotone if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). In the other word A is maximal monotone if any $(x,x^*) \in X \times X^*$ is monotonically related to gra$A$ belongs to gra$A$. A monotone operator $A : X \rightrightarrows X^*$ is said to be ultramaximal monotone [5, 28] if A is maximally monotone with respect to $X^{**} \times X^*$.

Let $f : X \to]-\infty, +\infty]$ be a function, its domain is defined as $\text{dom}f := f^{-1}(\mathbb{R})$. f is said to be proper if $\text{dom}f \neq \phi$. Let f be any proper convex function. Then the subdifferential operator of f is defined as $\partial f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^*|\ \langle y - x, x^* \rangle + f(x) \leq f(y), \forall y \in X\}$. Similarly, for $\epsilon \geq 0$, the $\epsilon-$subdifferetial of f is defined by

$$\partial_\epsilon f = \{(x,x^*) : f(y) \geq f(x) + \langle y - x, x^* \rangle - \epsilon, \forall y \in X\}.$$

The duality map $J : X \to X^*$ is defined as $J = \partial(\frac{1}{2}\|.\|^2)$. Similarly, the $\epsilon-$duality mapping is defined as $J_\epsilon := \partial_\epsilon(\frac{1}{2}\|.\|^2)$. Using $f(x) = \frac{1}{2}\|x\|^2$ in the above definitions, we get

$$x^* \in J(x) \Leftrightarrow \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = \langle x, x^* \rangle$$

or equivalently,

$$J(x) = \{x^* \in X^* | \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Similarly,

$$J_\epsilon(x) = \{x^* \in X^*|\ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \leq \langle x, x^* \rangle + \epsilon\}.$$

The Gossez's monotone closure of J is defined by

$$\tilde{J} = J^{-1}_{X^*},$$

where J_{X^*} denotes as the duality map on X^*. For any two monotone operators A and B, the sum operator is defined as $A + B : X \rightrightarrows X^* : x \mapsto Ax + Bx = \{a^* + b^* | \ a^* \in Ax \ \text{ and } \ b^* \in Bx\}$.

For our convenience, we recall some fundamental properties of maximal monotone operators. Let $A : X \rightrightarrows X^*$ be maximally monotone. We say A is of dense type or type (D) [14] if for every $(x^{**}, x^*) \in X^{**} \times X^*$ with

$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle a - x^{**}, a^* - x^* \rangle \geq 0,$$

there exists a bounded net $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ in $\operatorname{gra} A$ such that $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ converges to (x^{**}, x^*) with respect to (weak$^*\times$ strong) norm and A is said to be of type negative infimum (NI) [23] if

$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle a - x^{**}, a^* - x^* \rangle \leq 0, \quad \forall \ \ (x^{**}, x^*) \in X^{**} \times X^*.$$

By Simons [24, Theorem 36.3(a)] and Marques Alves and Svaiter [19, Theorem 4.4] we see that these two operators coincide. By definition of ultramaximal monotone it is clear that every ultramaximal monotone operator is of type (D). But the converse is not true for more details refer [28]. For a maximal monotone operator $A : X \rightrightarrows X^*$ we will define $\tilde{A} : X^{**} \rightrightarrows X^*$ as

$$\tilde{A} = \{(x^{**}, x^*) \in X^{**} \times X^* : (x^{**}, x^*) \text{ is monotonically related to} \operatorname{gra} A\}.$$

When A is of type (D), $\tilde{A}$ is the unique maximal monotone extension on $X^{**} \times X^*$. Let us collect some well-known results for the forthcoming sections.

Fact 2.1. *[19, Lemma 3.2] Let $A : X \rightrightarrows X^*$ be monotone and $\mu > 0$. Then the following conditions are equivalent:*

1. $\overline{\operatorname{ran}(A(.+z_0) + \mu J_\epsilon)} = X^*$, *for any* $\epsilon > 0$, *and* $z_0 \in X$.
2. $\operatorname{ran}(A(.+z_0) + \mu J_\epsilon) = X^*$, *for any* $\epsilon > 0$, *and* $z_0 \in X$.

Fact 2.2. *[19, Lemma 3.3] Let $A : X \rightrightarrows X^*$ be monotone and $\mu > 0$. If*

$$\overline{\operatorname{ran} A(.+z_0) + \mu J_\epsilon} = X^*, \quad \forall \ \ \epsilon > 0, z_0 \in X$$

then $\overline{A}$, the closure of A in the norm-topology of $X \times X^*$, is maximal monotone and of type NI.

Remark 2.3. *For any $z \in X$, $A(.+z)$ is the translation of the operator A and the translation is given by $A(.+z) := A - (z, 0)$.*

Fact 2.4. *[28, Theorem 3.3] Let $A, B : X \rightrightarrows X^*$ be maximal monotone operators. Assume that $\bigcup_{\lambda>0} \lambda[\operatorname{dom} A - \operatorname{dom} B]$ is a closed subspace. Suppose that A is ultramaximally monotone, and that B is of type (NI). Then $A+B$ is ultramaximally monotone.*

Fact 2.5. *[28, Corollary 3.6] Let $A : X \rightrightarrows X^*$ be ultramaximally monotone. Then $A + J$ is ultramaximally monotone and* $\operatorname{ran}(A + J) = X^*$.

For a maximal monotone operator $B : X \rightrightarrows X^*$ of type (D), the Moreau-Yosida regularization and the resolvent of B, (see [18]) with regularization parameter $\lambda > 0$ are given by $B_\lambda : X \rightrightarrows X^*$ and $R_\lambda : X \rightrightarrows X^{**}$

$$B_\lambda = \big\{(x, x^*) \in X \times X^* : \exists z^{**} \in X^* s.t. \quad 0 \in \lambda x^* + \tilde{J}(z^{**} - x), x^* \in \tilde{B}(z^{**})\big\} \tag{2.1}$$

$$R_\lambda = \big\{(x, z^{**}) \in X \times X^* : \exists x^* \in X^* s.t. \quad 0 \in \lambda x^* + \tilde{J}(z^{**} - x), x^* \in \tilde{B}(z^{**})\big\}. \tag{2.2}$$

We recall some properties of B_λ.

Fact 2.6. *([19, Theorem 3.6] and [17, Theorem 4.4]) Let $B : X \rightrightarrows X^*$ be a maximal monotone of type (D), $\lambda > 0$. Then*

1. *B_λ is a maximal monotone of type (D).*
2. $\operatorname{dom}(B_\lambda) = X$.
3. *B_λ maps bounded sets into bounded sets.*

3 Main Results

We remind that we assume X as a real Grothendieck space. Also, we assume it satisfies weakly compactly generated property. A Banach space X is called as weakly compactly generated if there is a weakly compact set K in X such that $X = \overline{\operatorname{span}}(K)$. Let $B : X \rightrightarrows X^*$ be a maximal monotone operator of type (D), then by Fact 2.6, B_λ is a maximal monotone operator of type (D) and $\operatorname{dom} B_\lambda = X$ for $\lambda > 0$. For any ultramaximal monotone operator A, we have $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} B_\lambda) \neq \phi$. Therefore, $\bigcup_{t>0} t[\operatorname{dom} A - \operatorname{dom} B_\lambda]$ is a closed subspace of X, in fact, whole space X. Using Fact 2.4, $A + B_\lambda$ is a ultramaximal monotone operator and Fact 2.5

$$\operatorname{ran}(A + B_\lambda + J) = X^*. \tag{3.1}$$

The proof of the following lemma is in the same line as the proof of [11, Lemma 1.2].

Lemma 3.1. *Let $A : X \rightrightarrows X^*$ be a monotone operator. If $(x_n, x_n^*) \in \operatorname{gra} A$, $x_n \overset{w^*}{\to} x^{**}$ in X^{**}, $x_n^* \overset{w^*}{\to} x^*$ in X^* and*

$$\limsup_{m,n\to\infty} \langle x_n - x_m, x_n^* - x_m^* \rangle \leq 0. \tag{3.2}$$

Then $\langle x_n, x_n^* \rangle \longrightarrow \langle x^{**}, x^* \rangle$.

Proof. Using $(x_n, x_n^*), (x_m, x_m^*) \in \operatorname{gra} A$, monotonicity of A and (3.2), we get

$$\lim_{m,n\to\infty} \langle x_n - x_m, x_n^* - x_m^* \rangle = 0. \tag{3.3}$$

Let $\{n_i\}$ be a subsequence of $\{n\}$ such that $\langle x_{n_i}, {x_{n_i}}^* \rangle \to L$ (say). Thus, from (3.3), we have

$$\begin{aligned} 0 &= \lim_{n_i\to\infty} \left[\lim_{n_k\to\infty} \langle x_{n_i} - x_{n_k}, {x_{n_i}}^* - {x_{n_k}}^* \rangle \right] \\ &= \lim_{n_i\to\infty} \left[\langle x_{n_i}, {x_{n_i}}^* \rangle - \langle x_{n_i}, x^* \rangle - \langle x^{**}, {x_{n_i}}^* \rangle + L \right]. \end{aligned}$$

Since X is a Gronthendieck space, now we treat $x_{n_i}^* \overset{w^*}{\to} x^*$ as $x_{n_i}^* \overset{w}{\to} x^*$ in X^*. Thus,

$$\begin{aligned} 0 &= L - \langle x^{**}, x^* \rangle - \langle x^{**}, x^* \rangle + L \\ &= 2L - 2\langle x^{**}, x^* \rangle. \end{aligned}$$

Hence, $L = \langle x^{**}, x^* \rangle$. Therefore, $\langle x_n, x_n^* \rangle \longrightarrow \langle x^{**}, x^* \rangle$. □

Lemma 3.2. *Let $B : X \rightrightarrows X^*$ be a maximal monotone operator of type (D). Let $x \in \operatorname{dom} B$ and $(x, b^*) \in \operatorname{gra} B_\lambda$. Then $\|b^*\| \le |B(x)|$.*

Proof. Since $(x, b^*) \in \operatorname{gra} B_\lambda$, then by the definition of B_λ, there exists $z^{**} \in X^{**}$ such that $(z^{**}, b^*) \in \operatorname{gra}(\tilde{B})$ and $0 \in \lambda b^* + \tilde{J}(z^{**} - x)$. By hypothesis, we have $x \in \operatorname{dom} B$. Let $a^* \in B(x)$. Then by definition of $\tilde{B}$,

$$\langle x - z^{**}, a^* - b^* \rangle \ge 0. \tag{3.4}$$

Since $0 \in \lambda b^* + \tilde{J}(z^{**} - x)$, then there exists $u^* \in \tilde{J}(z^{**} - x)$ such that $0 = \lambda b^* + u^*$ which implies that

$$b^* = -\frac{u^*}{\lambda}. \tag{3.5}$$

Using (3.5) in (3.4), we obtain

$$\begin{aligned} 0 &\le \langle x - z^{**}, a^* + \frac{u^*}{\lambda} \rangle \\ &= \langle x - z^{**}, a^* \rangle + \langle x - z^{**}, \frac{u^*}{\lambda} \rangle \\ &= -\frac{1}{\lambda} \langle z^{**} - x, u^* \rangle + \langle x - z^{**}, a^* \rangle. \end{aligned}$$

Since $u^* \in \tilde{J}(z^{**} - x)$, we have

$$0 \le -\frac{1}{\lambda} \|z^{**} - x\|^2 + \|z^{**} - x^*\| \|a^*\|. \tag{3.6}$$

By (3.5),

$$\|b^*\| = \frac{\|u^*\|}{\lambda} = \frac{\|z^{**} - x\|}{\lambda}.$$

Thus, by (3.6) $\|b^*\| \le \|a^*\|$. Since $a^* \in B(x)$ is arbitrary, $\|b^*\| \le |B(x)|$. □

We require the following proposition to prove the main result.

Proposition 3.3. *Let $A : X \rightrightarrows X^*$ be an ultramaximal monotone operator, $B : X \rightrightarrows X^*$ be a maximal monotone operator of type (D) with* $\operatorname{dom} A \subseteq \operatorname{dom} B$ *and for any $\lambda > 0$, the net (x_λ) satisfy, $x^* \in (A + B_\lambda + J)(x_\lambda)$, where $x^* \in X^*$. If $|B(x_\lambda)|$ is bounded as $\lambda \to 0$, then*

$$x^* \in \operatorname{ran}(A + B + J).$$

Proof. By assumption, (x_λ) satisfy $x^* \in (A + B_\lambda + J)(x_\lambda)$, where $x^* \in X^*$. There exists $(x_\lambda, x_\lambda^*) \in \operatorname{gra} A$, $(x_\lambda, b_\lambda^*) \in \operatorname{gra} B_\lambda$ and $(x_\lambda, w_\lambda^*) \in \operatorname{gra} J$ such that

$$x^* = x_\lambda^* + b_\lambda^* + w_\lambda^*. \tag{3.7}$$

Let $(y_0, y_0^*) \in \operatorname{gra} A$. By monotonicity of A, we have

$$\langle y_0 - x_\lambda, y_0^* - x_\lambda^* \rangle \geq 0.$$

By (3.7),

$$\begin{aligned}
&\langle y_0 - x_\lambda, y_0^* - x^* + b_\lambda^* + w_\lambda^* \rangle \geq 0 \\
\Rightarrow &\langle y_0 - x_\lambda, y_0^* - x^* \rangle \geq \langle x_\lambda - y_0, b_\lambda^* + w_\lambda^* \rangle \\
&\qquad\qquad\qquad\qquad \geq \langle x_\lambda - y_0, b_\lambda^* \rangle + \langle x_\lambda . w_\lambda^* \rangle - \langle y_0, w_\lambda^* \rangle \\
&\qquad\qquad\qquad\qquad = \langle x_\lambda - y_0, b_\lambda^* \rangle + \frac{1}{2}\|x_\lambda\|^2 + \frac{1}{2}\|w_\lambda^*\|^2 - \langle y_0, w_\lambda^* \rangle.
\end{aligned} \tag{3.8}$$

Note that,

$$\begin{aligned}
&(\|y_0\| - \|w_\lambda^*\|)^2 \geq 0 \\
\Rightarrow &\|y_0\|^2 + \|w_\lambda^*\|^2 - 2\|y_0\|\|w_\lambda^*\| \geq 0 \\
\Rightarrow &\frac{1}{2}\|y_0\|^2 + \frac{1}{2}\|w_\lambda^*\|^2 \geq \|y_0\|\|w_\lambda^*\| \geq \langle y_0, w_\lambda^* \rangle.
\end{aligned}$$

Equation (3.8) implies that

$$\langle y_0 - x_\lambda, y_0^* - x^* \rangle \geq \langle x_\lambda - y_0, b_\lambda^* \rangle + \frac{1}{2}\|x_\lambda\|^2 - \frac{1}{2}\|y_0\|^2.$$

Thus,

$$\|x_\lambda\|^2 \leq 2\langle y_0 - x_\lambda, y_0^* - x^* \rangle + 2\langle y_0 - x_\lambda, b_\lambda^* \rangle + \|y_0\|^2. \tag{3.9}$$

Now, we show that $\|b_\lambda^*\|$ is bounded as $\lambda \to 0$. Using $(x_\lambda, b_\lambda^*) \in \operatorname{gra} B_\lambda$ and $\operatorname{dom} A \subseteq \operatorname{dom} B$ we have $x_\lambda \in \operatorname{dom} B$. Then by Lemma 3.2, $\|b_\lambda^*\| \leq |B(x_\lambda)|$. By hypothesis, $\|b_\lambda^*\|$ is bounded and thus, Equation (3.9) shows that $\|x_\lambda\|$ is bounded as $\lambda \to 0$. Also, $\|w_\lambda^*\|$ is bounded. Therefore, $\|x_\lambda^*\| = \|x^* - b_\lambda^* - w_\lambda^*\|$ is bounded. Since X satisfies both Grothendieck and weakly compactly generated property. Thence, by [21, Theorem 4.9 (iii)] and Amir-Lindenstrauss Theorem [21, Theorem 4.8], let (x_{λ_n}), $(x_{\lambda_n}^*)$, $(b_{\lambda_n}^*)$ and $(w_{\lambda_n}^*)$ subsequences of (x_λ), (x_λ^*), (b_λ^*) and (w_λ^*)

such that $x_{\lambda_n} \to x_0$ in the weak topology of X and $x^*_{\lambda_n} \to x^*_0$, $b^*_{\lambda_n} \to b^*_0$, $w^*_{\lambda_n} \to w^*_0$ in the weak* topology of X^* as $\lambda_n \to 0$. By (3.7), for $\lambda = \lambda_n, \lambda_m$,

$$\langle x_{\lambda_m} - x_{\lambda_n}, x^*_{\lambda_m} + b^*_{\lambda_m} + w^*_{\lambda_m} - (x^*_{\lambda_n} + b^*_{\lambda_n} + w^*_{\lambda_n})\rangle = 0. \tag{3.10}$$

Thus,

$$\begin{aligned}\langle x_{\lambda_m} - x_{\lambda_n}, &x^*_{\lambda_m} - x^*_{\lambda_n}\rangle \\ &= -\langle x_{\lambda_m} - x_{\lambda_n}, b^*_{\lambda_m} - b^*_{\lambda_n}\rangle - \langle x_{\lambda_m} - x_{\lambda_n}, w^*_{\lambda_m} - w^*_{\lambda_n}\rangle. \end{aligned}\tag{3.11}$$

Since $b^*_{\lambda_m} \in B_{\lambda_m}(x_{\lambda_m})$ and $b^*_{\lambda_n} \in B_{\lambda_n}(x_{\lambda_n})$, then there exists $z^{**}_{\lambda_m}, z^{**}_{\lambda_n} \in X^{**}$ such that $(z^{**}_{\lambda_m}, b^*_{\lambda_m}) \in \operatorname{gra} \tilde{B}$, $(z^{**}_{\lambda_n}, b^*_{\lambda_n}) \in \operatorname{gra} \tilde{B}$ and $0 \in \lambda_m b^*_{\lambda_m} + \tilde{J}(z^{**}_{\lambda_m} - x_{\lambda_m})$ and $0 \in \lambda_n b^*_{\lambda_n} + \tilde{J}(z^{**}_{\lambda_n} - x_{\lambda_n})$. By monotonicity of $\tilde{B}$,

$$\langle z^{**}_{\lambda_m} - z^{**}_{\lambda_n}, b^*_{\lambda_m} - b^*_{\lambda_n}\rangle \geq 0 \tag{3.12}$$

and by duality mapping of $\tilde{J}$, $\|z^{**}_{\lambda_m} - x_{\lambda_m}\| = \lambda_m \|b^*_{\lambda_m}\|$. Thus, $\lim_{m\to\infty} \|z^{**}_{\lambda_m} - x_{\lambda_m}\| = 0$ ($\|b_{\lambda_m}\|$ is bounded as $\lambda_m \to 0$). Similarly, $\lim_{n\to\infty} \|z^{**}_{\lambda_n} - x_{\lambda_n}\| = 0$. Therefore,

$$\begin{aligned}\langle x_{\lambda_m} - x_{\lambda_n}, &b^*_{\lambda_m} - b^*_{\lambda_n}\rangle \\ &= \langle x_{\lambda_m} - z^{**}_{\lambda_m} + z^{**}_{\lambda_n} - x_{\lambda_n}, b^*_{\lambda_m} - b^*_{\lambda_n}\rangle + \langle z^{**}_{\lambda_m} - z^{**}_{\lambda_n}, b^*_{\lambda_m} - b^*_{\lambda_n}\rangle \\ &\geq \langle x_{\lambda_m} - z^{**}_{\lambda_m}, b^*_{\lambda_m} - b^*_{\lambda_n}\rangle + \langle z^{**}_{\lambda_n} - x_{\lambda_n}, b^*_{\lambda_m} - b^*_{\lambda_n}\rangle (\text{by } (3.12)) \\ &\geq -\|x_{\lambda_m} - z^{**}_{\lambda_m}\|(\|b^*_{\lambda_m} + \|b^*_{\lambda_n}\|) - \|z^{**}_{\lambda_n} - x_{\lambda_n}\|(\|b^*_{\lambda_m}\| + \|b^*_{\lambda_n}\|). \end{aligned}\tag{3.13}$$

Since $\langle x_{\lambda_m} - x_{\lambda_n}, w^*_{\lambda_m} - w^*_{\lambda_n}\rangle \geq 0$. Then by appealing (3.13) in (3.11), we get

$$\langle x_{\lambda_m} - x_{\lambda_n}, x^*_{\lambda_m} - x^*_{\lambda_n}\rangle \leq \|x_{\lambda_m} - z^{**}_{\lambda_m}\|(\|b^*_{\lambda_m} + \|b^*_{\lambda_n}\|) + \|z^{**}_{\lambda_n} - x_{\lambda_n}\|(\|b^*_{\lambda_m}\| + \|b^*_{\lambda_n}\|).$$

Therefore,

$$\lim_{m,n\to\infty} \langle x_{\lambda_m} - x_{\lambda_n}, x^*_{\lambda_m} - x^*_{\lambda_n}\rangle \leq 0. \tag{3.14}$$

If we replace $x_{\lambda_n} \to x_0$ in the weak topology of X instead of the weak star topology, then the Lemma 3.1 holds. Thus, by Lemma 3.1, $\langle x_{\lambda_n}, x^*_{\lambda_n}\rangle \to \langle x_0, x^*_0\rangle$. By monotonicity of A, we get

$$\langle x_\lambda - y_0, x^*_\lambda - y^*_0\rangle \geq 0, \quad \forall (y_0, y^*_0) \in \operatorname{gra} A.$$

By passing limit along the subsequence,

$$\langle x_0 - y_0, x^*_0 - y^*_0\rangle \geq 0, \quad \forall (y_0, y^*_0) \in \operatorname{gra} A.$$

By maximal monotonicity of A, we have $x^*_0 \in A(x_0)$. Again by the same argument and from the monotonicity of A, we obtain

$$\langle x_{\lambda_m} - x_{\lambda_n}, w^*_{\lambda_m} - w^*_{\lambda_n}\rangle \leq -\langle x_{\lambda_m} - x_{\lambda_n}, b^*_{\lambda_m} - b^*_{\lambda_n}\rangle.$$

Thus,

$$\lim_{m,n\to\infty}\langle x_{\lambda_m}-x_{\lambda_n}, w^*_{\lambda_m}-w^*_{\lambda_n}\rangle \le 0.$$

Hence, by Lemma 3.1, $\langle x_{\lambda_n}, w_{\lambda_n}\rangle \to \langle x_0, w^*_0\rangle$. Since $\|.\|$ is lower semi-continuous in weak and weak* topology, $\liminf \|x_{\lambda_n}\| \ge \|x_0\|$ and $\liminf \|w^*_{\lambda_n}\| \ge \|w^*_0\|$.

$$\begin{aligned}\frac{1}{2}\|x_0\|^2+\frac{1}{2}\|w^*_0\|^2 &\le \frac{1}{2}\liminf \|w^*_{\lambda_n}\|^2+\frac{1}{2}\liminf\|x_{\lambda_n}\|^2\\ &\le \liminf\langle x_{\lambda_n}, w^*_{\lambda_n}\rangle = \langle x_0, w^*_0\rangle.\end{aligned}$$

Note that,

$$\frac{1}{2}\|x_0\|^2+\frac{1}{2}\|w^*_0\|^2 \ge \langle x_0, w^*_0\rangle.$$

Hence, $w^*_0 \in J(x_0)$. Now it remains to show that $b^*_0 \in B(x_0)$. Since $x_{\lambda_n} \to x_0$ in weak topology of X and $\lim \|z^{**}_{\lambda_n} - x_{\lambda_n}\| = 0$, then $z^{**}_{\lambda_n} \to x^{**}_0$ in the weak star topology of X^*. Let $(y, y^*) \in \operatorname{gra} B$. By definition of $\tilde{B}$,

$$\langle z^{**}_{\lambda_n} - y, b^*_{\lambda_n} - y^*\rangle \ge 0. \tag{3.15}$$

It is known that $\langle z^{**}_{\lambda_n}, y^*\rangle \to \langle x_0, y^*\rangle$, $\langle y, b^*_{\lambda_n}\rangle \to \langle y, b^*_0\rangle$. Therefore, it is enough to show that

$$\langle z^{**}_{\lambda_n}, b^*_{\lambda_n}\rangle \to \langle x_0, b^*_0\rangle.$$

Note that

$$\begin{aligned}&\|\langle z^{**}_{\lambda_n}, b^*_{\lambda_n}\rangle - \langle x^{**}_0, b^*_0\rangle\|\\ &\quad= \|\langle z^{**}_{\lambda_n}, b^*_{\lambda_n}\rangle - \langle x_{\lambda_n}, b^*_{\lambda_n}\rangle + \langle x_{\lambda_n}, b^*_{\lambda_n}\rangle - \langle x^{**}_0, b^*_0\rangle\|\\ &\quad\le \|\langle z^{**}_{\lambda_n} - x_{\lambda_n}, b^*_{\lambda_n}\rangle\| + \|\langle x_{\lambda_n}, b^*_{\lambda_n}\rangle - \langle x^{**}_0, b^*_0\rangle\|\\ &\quad\le \|z^{**}_{\lambda_n} - x_{\lambda_n}\|\|b^*_{\lambda_n}\| + \|\langle x_{\lambda_n}, b^*_{\lambda_n}\rangle - \langle x^{**}_0, b^*_0\rangle\|.\end{aligned} \tag{3.16}$$

To complete the proof, we must show that $\|\langle x_{\lambda_n}, b^*_{\lambda_n}\rangle - \langle x_0, b^*_0\rangle\| \to 0$. By (3.7), we get that

$$\begin{aligned}\langle x_{\lambda_n}, b^*_{\lambda_n}\rangle &= \langle x_{\lambda_n}, x^* - x^*_{\lambda_n} - w^*_{\lambda_n}\rangle\\ &= \langle x_{\lambda_n}, x^*\rangle - \langle x_{\lambda_n}, x^*_{\lambda_n}\rangle - \langle x_{\lambda_n}, w^*_{\lambda_n}\rangle.\end{aligned}$$

As $(x_{\lambda_n}) \to x_0$ with respect to weak topology of X and $\langle x_{\lambda_n}, x^*_{\lambda_n}\rangle \to \langle x_0, x^*_0\rangle$ and $\langle x_{\lambda_n}, w^*_{\lambda_n}\rangle \to \langle x_0, w^*_0\rangle$, we have

$$\langle x_{\lambda_n}, b^*_{\lambda_n}\rangle \to \langle x_0, x^* - x^*_0 - w^*_0\rangle.$$

By Passing of the limit along the subsequence of (3.7), we get $x^* = x^*_0 + b^*_0 + w^*_0$. Thus, $\langle x_{\lambda_n}, b^*_{\lambda_n}\rangle \to \langle x_0, b^*_0\rangle$ and hence, by passing the limit in (3.16), we obtain

$$\langle z^{**}_{\lambda_n}, b^*_{\lambda_n}\rangle \to \langle x_0, b^*_0\rangle.$$

Therefore, for $\lambda_n \to 0$, (3.15) implies $\langle x_0 - y, b^*_0 - y^*\rangle \ge 0$. Since $(y, y^*) \in \operatorname{gra} B$ is arbitrary, then by maximal monotonicity of B, we have $b^*_0 \in B(x_0)$. Hence, $x^* \in (A + B + J)(x_0)$. □

Now we will prove the main result.

Theorem 3.4. *Let $A : X \rightrightarrows X^*$ be an ultramaximal monotone operator, $B : X \rightrightarrows X^*$ be a maximal monotone operator of type (D) and*

1. $\operatorname{dom} A \subset \operatorname{dom} B$,
2. $|B(x)| \le K(\|x\|)|A(x)| + C(\|x\|)$

where $K(r)$ and $C(r)$ are non-decreasing functions of r. Assume $K(r) < 1$ for every r. Then $\overline{A+B}$ is a maximal monotone operator of type (D). Moreover, if $A+B$ is a closed monotone operator, then $A+B$ is a maximal monotone operator of type (D).

Proof. We can suppose that $(0,0) \in \operatorname{gra} A$ and $(0,0) \in \operatorname{gra} B$. Since B is a maximal monotone operator of type (D), for any $\lambda > 0$, Fact 2.6 implies that B_λ is a maximal monotone operator of type (D). Then by Fact 2.4 and Fact 2.6, $A + B_\lambda$ is an ultramaximal monotone. Thus, by Fact 2.5,

$$\operatorname{ran}(A + B_\lambda + J) = X^*. \tag{3.17}$$

To prove $\overline{A+B}$ is maximal monotone, by Fact 2.1 and 2.2, it is enough to prove that for any $\epsilon > 0$,

$$\operatorname{ran}(A + B + J_\epsilon) = X^*.$$

If Fact 2.2 is valid for $\mu = 1$, then it is valid for any $\mu > 0$ [19, Lemma 3.3]. For any $x^* \in X^*$, there exists $x_\lambda, x^*_\lambda, b^*_\lambda$ and w^*_λ such that

$$x^* = x^*_\lambda + b^*_\lambda + w^*_\lambda \tag{3.18}$$

where $(x_\lambda, x^*_\lambda) \in \operatorname{gra} A$, $(x_\lambda, b^*_\lambda) \in \operatorname{gra} B_\lambda$ and $(x_\lambda, w^*_\lambda) \in \operatorname{gra} J$. By Proposition 3.3, it remains to show that $|B(x_\lambda)|$ is bounded as $\lambda \to 0$. Since $(0,0) \in \operatorname{gra} \tilde{B}$ and $(0,0) \in \operatorname{gra} \tilde{J}$, then $(0,0) \in \operatorname{gra} B_\lambda$. By monotonicity of A, we get $\langle x_\lambda, x^*_\lambda\rangle \ge 0$. Using (3.18), we get $\langle x_\lambda, x^* - b^*_\lambda - w^*_\lambda\rangle \ge 0$. By monotonicity of B_λ, $\langle x_\lambda, b^*_\lambda\rangle \ge 0$ and definition J, we have $\langle x_\lambda, x^*\rangle \ge \|x_\lambda\|^2$ which shows that $\|x_\lambda\|$ is bounded. Thus,

$$\begin{aligned}
&\|x_\lambda\|^2 \le \|x_\lambda\|\|x^*\| \le 2\|x_\lambda\|\|x^*\| \\
\Rightarrow&\|x_\lambda\|^2 - \|x_\lambda\|\|x^*\| + \|x^*\|^2 \le \|x^*\| \\
\Rightarrow&(\|x_\lambda\| - \|x^*\|)^2 \le \|x^*\|^2 \\
\Rightarrow&\|x_\lambda\| \le 2\|x^*\|.
\end{aligned} \tag{3.19}$$

Therefore, $\|w^*_\lambda\| \le 2\|x^*\|$. Since $x_\lambda \in \operatorname{dom} A \subseteq \operatorname{dom} B$, then by (3.18) and Lemma 3.2, we get

$$\begin{aligned}
|A(x_\lambda)| &\le \|x^*_\lambda\| \\
&\le \|x^*\| + \|b^*_\lambda\| + \|w^*_\lambda\|. \\
&\le \|x^*\| + |B(x_\lambda)| + \|w^*_\lambda\| \\
&\le \|x^*\| + K(\|x_\lambda\|)|A(x_\lambda)| + C(\|x_\lambda\|) + \|w^*_\lambda\| \\
&\le \|x^*\| + K(2\|x^*\|)|A(x_\lambda)| + C(2\|x^*\|) + 2\|x^*\|. \text{ by (3.19).}
\end{aligned} \tag{3.20}$$

By assuming $r = 2\|x^*\|$ in (3.20), we get

$$\begin{aligned}
&|A(x_\lambda)| \leq 3\|x^*\| + K(r)|A(x_\lambda)| + C(r) \\
\Rightarrow &(1 - K(r))|A(a_\lambda)| \leq 3\|x^*\| + C(r) \\
\Rightarrow &|A(x_\lambda)| \leq \frac{3\|x^*\|}{1 - K(r)} + \frac{C(r)}{1 - K(r)} \quad (\because K(r) < 1)
\end{aligned} \tag{3.21}$$

Therefore, using hypothesis (2), (3.21) and (3.19), we get

$$\begin{aligned}
|B(x_\lambda)| &\leq K(\|x_\lambda\|)|A(x_\lambda)| + C(\|x_\lambda\|) \\
&\leq K(r)\frac{3\|x^*\|}{1 - K(r)} + \frac{C(r)}{1 - K(r)} + C(r).
\end{aligned}$$

Hence, $|B(x_\lambda)|$ is bounded. This complete the proof. □

4 Conclusion

We prove that the sum of an ultra maximal monotone operator and an operator of type (D) is a maximal monotone operator in Banach spaces which satisfy Grothendieck and weakly compactly generated properties. We extend the results to nonreflexive spaces assuming certain conditions those are automatically satisfied in reflexive spaces. This approach shows a new way of handling the nonreflexive scenario without assuming reflexivity directly. Therefore, for extending the results to general Banach spaces, we only require to relax above two conditions, i.e., Grothendieck and weakly compactly generated conditions.

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