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Fixed Point Theorems for Generalized Weakly Contractive Mappings in S-metric Spaces

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Abstract : In this paper, we introduce the notion of generalized weakly contractive mappings in S-metric spaces and prove the existence and uniqueness of fixed point for such mappings in complete S metric spaces. These theorems generalize many previously obtained fixed point results. An example is given to illustrate the main result.

Keywords : Generalized weakly contractive mappings; S-Metric Space
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1 Introduction and preliminaries

Metric spaces are very important in various areas of mathematics such as analysis, topology, applied mathematics etc. So various generalizations of metric spaces have been studied and several fixed point results. Recently, Sedghi, Shobe and Aliouche have defined the concept of an S-metric space [1]. This notion is a generalization of a G-metric space [2] and a D^*-metric space [3]. Some papers dealing with fixed point theorems for mappings satisfying certain contractive conditions on S-metric spaces can be referred in [4–8].

Alber and Guerre-Delabriere [9] introduced weakly contractive maps in Hilbert spaces as a generalization of contraction maps and established a fixed point theorem in the Hilbert spaces. Rhoades [10] extended this idea to Banach spaces and proved existence of fixed points of weakly contractive self maps in Banach space setting. Dutta and Choudhury [11] proved the existence and uniqueness of fixed points and generalized the results of Alber and Guerre-Delabriere and Khan Swaleh and Sessa [12]. Since then different types of weakly contractive maps have been considered in several works to establish the existence of fixed points. For more works on weakly contractive maps we refer [13–15].

Motivated by the above studies, we extend the notion of generalized weakly contractive mappings to S-metric spaces and define a new type of contractive mappings.

Now we provide some preliminaries and basic definitions which we use throughout this paper.

Definition 1.1 ([12]). *An* **altering distance function** *is a function* $\psi : [0, \infty) \to [0, \infty)$ *which satisfies*
(i) ψ *is continuous (ii)* ψ *is non-decreasing and (iii)* $\psi(t) = 0$ *if and only if* $t = 0$.

We denote the class of all altering distance functions by Ψ.

We denote $\Phi = \{\phi : [0, \infty) \to [0, \infty)$ *: (i)* ϕ *is continuous and (ii)* $\phi(t) = 0$ *if and only if* $t = 0\}$.

In the following, Dutta and Choudhury [11] established the existence of fixed points of (ψ, ϕ)-weakly contractive maps involving two altering distance functions ψ and ϕ in complete metric spaces.

Theorem 1.2 ([11]). *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a self-map of* X. *If there exist* $\psi, \phi \in \Psi$ *such that*

$$\psi\big(d(Tx, Ty)\big) \ \leq \ \psi\big(d(x, y)\big) - \phi\big(d(x, y)\big)$$

for all $x, y \in X$. *Then* T *has a unique fixed point.*

In 2012, Sedghi, Shobe and Aliouche [1] introduced S-metric spaces as follows:

Definition 1.3 ([1]). *Let* X *be a nonempty set. An* S*-metric on* X *is a function* $S : X^3 \to [0, \infty)$ *that satisfies the following conditions, for each* $x, y, z, a \in X$.

(S1) $S(x, y, z) \geq 0$;

(S2) $S(x, y, z) = 0$ *if and only if* $x = y = z$;

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then the pair (X, S) *is called an* S*-metric space.*

The following is an intuitive geometric example for S-metric spaces.

Example 1.4 ([1]). *Let* $\mathbb{R}$ *be the real line. Then*

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in \mathbb{R}$ *is an* S*-metric on* R. *This* S*-metric on* $\mathbb{R}$ *is called the* **usual** **S-metric** *on* $\mathbb{R}$.

Lemma 1.5 ([1]). *Let* (X, S) *be an* S*-metric space. Then* $S(x, x, y) = S(y, y, x)$ *for all* $x, y \in X$.

Remark 1.6. *Let (X, S) be an S-metric space. From Definition 1.3 we have,*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$

then

$$\frac{1}{3}S(x, x, z) \leq \max\left\{S(x, x, y), S(y, y, z)\right\}$$

for all $x, y, z \in X$.

Definition 1.7 ([1]). *Let (X, S) be an S-metric space.*

(i) *A sequence $\{x_n\} \subset X$ is said to* converge to $x \in X$ *if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.*

(ii) *A sequence $\{x_n\} \subset X$ is called a* Cauchy sequence *if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.*

(iii) *The S-metric space (X, S) is said to be* complete *if every Cauchy sequence is a convergent sequence*

Lemma 1.8 ([1]). *Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$, then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

Lemma 1.9. *Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{1.1}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon \quad \textit{and} \quad S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \varepsilon \tag{1.2}$$

and

(i) $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon$

(ii) $\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) = \varepsilon$

(iii) $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \varepsilon$

(iv) $\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \varepsilon$

Proof. From (S3) in Definition of S-metric we have

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \tag{1.3}$$

By (1.2) we have

$$\varepsilon \leq S(x_{m_k}, x_{m_k}, x_{n_k}) \leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + \varepsilon.$$

Let $k \to \infty$ and using (1.1)

$$\varepsilon \ \leq \ \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) \ \leq \ \varepsilon.$$

Then $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon$.

Next, we proof that $\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) = \varepsilon$. From (1.3) and using (1.2) we have

$$\varepsilon \leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + \varepsilon.$$

Let $k \to \infty$, using (1.1)

$$\varepsilon \leq \lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \leq \varepsilon.$$

Therefore, (ii) is true.

Next, from (S3) and Lemma 1.5 we have

$$\begin{aligned} S(x_{m_k}, x_{m_k}, x_{n_k}) - 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) &\leq S(x_{m_k}, x_{m_k}, x_{n_k-1}) \\ &\leq 2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + S(x_{m_k}, x_{m_k}, x_{n_k}). \end{aligned}$$

Let $k \to \infty$, using (1.1)

$$\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \varepsilon.$$

Therefore, (iii) is true.

For the last item, from (S3) and Lemma 1.5 we have

$$\begin{aligned} S(x_{m_k}, x_{m_k}, x_{n_k}) &\leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{n_k}, x_{n_k}, x_{m_k-1}) \\ &\leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) \\ &\quad + S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}). \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) &\leq 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) \\ &\leq 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + 2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) \\ &\quad + S(x_{m_k}, x_{m_k}, x_{n_k}). \end{aligned} \tag{1.5}$$

Let $k \to \infty$ in (1.4),(1.5), using (i) and (1.1) we have

$$\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \varepsilon.$$

□

2 Main Results

Definition 2.1. *Suppose that a mapping $T : X \to X$, where X be an S-metric space. If there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi\big(S(Tx, Ty, Tz)\big) \ \leq \ \psi\big(m(x, y, z)\big) - \phi\big(m(x, y, z)\big), \tag{2.1}$$

where

$$\begin{aligned} m(x, y, z) = \max\Big\{ & S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz), \\ & \tfrac{1}{3}S(Tx, Tx, y), \tfrac{1}{3}S(Ty, Ty, z), \tfrac{1}{3}S(Tz, Tz, x), \\ & \tfrac{1}{6}\big(S(Tx, Tx, y) + S(Ty, Ty, z) + S(Tz, Tz, x)\big)\Big\} \end{aligned} \tag{2.2}$$

for all $x, y, z \in X$, then T is called an generalized weakly contractive map on X.

Theorem 2.2. *Let (X, S) be a complete S-metric space and let T be an generalized weakly contractive map. Then T has a unique fixed point x^* and T is continuous at x^*.*

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T and we are through. Now we assume that $x_n \neq x_{n+1}$ for all n. By substituting $x = y = x_{n-1}, z = x_n$ in (2.1), we have

$$\begin{aligned} \psi\big(S(x_n, x_n, x_{n+1})\big) &= \psi\big(S(Tx_{n-1}, Tx_{n-1}, Tx_n)\big) \\ &\leq \psi\big(m(x_{n-1}, x_{n-1}, x_n)\big) - \phi\big(m(x_{n-1}, x_{n-1}, x_n)\big), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} & m(x_{n-1}, x_{n-1}, x_n) \\ &= \max\Big\{ S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_n, x_n, Tx_n), \\ &\qquad \tfrac{1}{3}S(Tx_{n-1}, Tx_{n-1}, x_{n-1}), \tfrac{1}{3}S(Tx_{n-1}, Tx_{n-1}, x_n), \tfrac{1}{3}S(Tx_n, Tx_n, x_{n-1}), \\ &\qquad \tfrac{1}{6}\big(S(Tx_{n-1}, Tx_{n-1}, x_{n-1}) + S(Tx_{n-1}, Tx_{n-1}, x_n) + S(Tx_n, Tx_n, x_{n-1})\big)\Big\} \\ &= \max\Big\{ S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\qquad \tfrac{1}{3}S(x_n, x_n, x_{n-1}), \tfrac{1}{3}S(x_n, x_n, x_n), \tfrac{1}{3}S(x_{n+1}, x_{n+1}, x_{n-1}), \\ &\qquad \tfrac{1}{6}\big(S(x_n, x_n, x_{n-1}) + S(x_n, x_n, x_n) + S(x_{n+1}, x_{n+1}, x_{n-1})\big)\Big\} \\ &= \max\Big\{ S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \tfrac{1}{3}S(x_n, x_n, x_{n-1}), \\ &\qquad \tfrac{1}{3}S(x_{n+1}, x_{n+1}, x_{n-1}), \tfrac{1}{6}\big(S(x_n, x_n, x_{n-1}) + S(x_{n+1}, x_{n+1}, x_{n-1})\big)\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \max\Big\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1}), \tfrac{1}{3}S(x_{n+1},x_{n+1},x_{n-1}),\\
&\qquad \tfrac{1}{6}\big(S(x_n,x_n,x_{n-1}) + S(x_{n+1},x_{n+1},x_{n-1})\big)\Big\}\\
&\qquad \Big(\because \tfrac{1}{3}S(x_n,x_n,x_{n-1}) = \tfrac{1}{3}S(x_{n-1},x_{n-1},x_n) \le S(x_{n-1},x_{n-1},x_n)\Big)\\
&= \max\Big\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1}), \tfrac{1}{6}\big(S(x_n,x_n,x_{n-1}) + S(x_{n+1},x_{n+1},x_{n-1})\big)\Big\}\\
&\qquad \Big(\because \tfrac{1}{3}S(x_{n+1},x_{n+1},x_{n-1}) \le \max\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1})\}\Big)\\
&= \max\Big\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1}))\Big\} \qquad (2.4)\\
&\qquad \Big(\because \tfrac{1}{6}S(x_n,x_n,x_{n-1}) + \tfrac{1}{6}S(x_{n+1},x_{n+1},x_{n-1}) \le \tfrac{M_n}{2} + \tfrac{M_n}{2},\\
&\qquad\qquad \text{where } M_n = \max\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1})\}\Big).
\end{aligned}
$$

From (2.3) and (2.4) we have

$$\psi\big(S(x_n,x_n,x_{n+1})\big) \le \psi\big(M_n\big) - \phi\big(M_n\big), \tag{2.5}$$

where $M_n = \max\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1})\}$. Suppose that, for some n, $M_n = S(x_n,x_n,x_{n+1})$. Therefore from (2.5), it follows $\phi\big(S(x_n,x_n,x_{n+1})\big) = 0$. Hence $x_n = x_{n+1}$, a contradiction since x_n and x_{n+1} are distinct elements. Thus, $M_n = S(x_{n-1},x_{n-1},x_n)$. Hence, from (2.5), we have

$$
\begin{aligned}
\psi\big(S(x_n,x_n,x_{n+1})\big) &\le \psi\big(S(x_{n-1},x_{n-1},x_n)\big) - \phi\big(S(x_{n-1},x_{n-1},x_n)\big)\\
&< \psi\big(S(x_{n-1},x_{n-1},x_n)\big)
\end{aligned} \tag{2.6}
$$

Now, by the non-decreasing property of ψ , it follows that $S(x_n,x_n,x_{n+1}) \le S(x_{n-1},x_{n-1},x_n)$ for all $n \in \mathbb{N}$. Therefore $\{S(x_n,x_n,x_{n+1})\}$ is a decreasing sequence of positive real numbers. Hence there exists $r \ge 0$ such that

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = r. \tag{2.7}$$

On letting $n \to \infty$ in (2.6) and using (2.7), we obtain

$$\psi(r) \;\le\; \psi(r) - \phi(r), \tag{2.8}$$

so that $\phi(r) = 0$. Hence $r = 0$.

We now show that $\{x_n\}$ is an Cauchy sequence. If possible, suppose that $\{x_n\}$ is not Cauchy. Therefore by Lemma 1.9, there exists an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that $S(x_{m_k},x_{m_k},x_{n_k}) \ge \varepsilon, S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \varepsilon$ satisfying the identities (i) to (iv) of Lemma 1.9. Taking $x = y = x_{m_k-1}, z = x_{n_k-1}$ and applying the inequality (2.1), we have

$$\psi\big(S(x_{m_k},x_{m_k},x_{n_k})\big) = \psi\big(S(Tx_{m_k-1},Tx_{m_k-1},Tx_{n_k-1})\big) \\ \leq \psi\big(m(x_{m_k-1},x_{m_k-1},x_{n_k-1})\big) - \phi\big(m(x_{m_k-1},x_{m_k-1},x_{n_k-1})\big), \tag{2.9}$$

where

$$m(x_{m_k-1},x_{m_k-1},x_{n_k-1}) \\ = \max\Big\{S(x_{m_k-1},x_{m_k-1},x_{n_k-1}), S(x_{m_k-1},x_{m_k-1},x_{m_k}), S(x_{n_k-1},x_{n_k-1},x_{n_k}), \\ \tfrac{1}{3}S(x_{m_k},x_{m_k},x_{m_k-1}), \tfrac{1}{3}S(x_{m_k},x_{m_k},x_{n_k-1}), \tfrac{1}{3}S(x_{n_k},x_{n_k},x_{m_k-1}), \\ \tfrac{1}{6}\big(S(x_{m_k},x_{m_k},x_{m_k-1}) + S(x_{m_k},x_{m_k},x_{n_k-1}) + S(x_{n_k},x_{n_k},x_{m_k-1})\big)\Big\}. \tag{2.10}$$

On letting $k \to \infty$ in (2.10) and using Lemma 1.26, we get

$$m(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \max\{\varepsilon, \tfrac{1}{3}\varepsilon\} = \varepsilon,$$

then

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

a contradiction. Hence $\{x_n\}$ is Cauchy. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

We now show that x^* is a fixed point of T. Here by Lemma 1.8 we note that $\lim_{n\to\infty} S(x_n,x_n,Tx^*) = S(x^*,x^*,Tx^*)$. We now consider

$$\psi\big(S(Tx^*,Tx^*,x_n)\big) = \psi\big(S(Tx^*,Tx^*,Tx_{n-1})\big) \\ \leq \psi\big(m(x^*,x^*,x_{n-1})\big) - \phi\big(m(x^*,x^*,x_{n-1})\big), \tag{2.11}$$

where

$$m(x^*,x^*,x_{n-1}) \\ = \max\Big\{S(x^*,x^*,x_{n-1}), S(x^*,x^*,Tx^*), S(x_{n-1},x_{n-1},x_n), \\ \tfrac{1}{3}S(Tx^*,Tx^*,x^*), \tfrac{1}{3}S(Tx^*,Tx^*,x_{n-1}), \tfrac{1}{3}S(x_n,x_n,x^*), \\ \tfrac{1}{6}\big(S(Tx^*,Tx^*,x^*) + S(Tx^*,Tx^*,x_{n-1}) + S(x_n,x_n,x^*)\big)\Big\}. \tag{2.12}$$

On letting $n \to \infty$ in (2.12), we get

$$m(x^*,x^*,x_{n-1}) = \max\{S(x^*,x^*,x^*), S(x^*,x^*,Tx^*)\} = S(x^*,x^*,Tx^*),$$

then

$$\psi\big(S(x^*,x^*,Tx^*)\big) \leq \psi\big(S(x^*,x^*,Tx^*)\big) - \phi\big(S(x^*,x^*,Tx^*)\big),$$

so that $\phi\big(S(x^*,x^*,Tx^*)\big) = 0$. Hence $x^* = Tx^*$.

We now prove uniqueness of fixed point. If u and v are two fixed points of T with $u \neq v$, then we consider

$$\psi\big(S(u,u,v)\big) = \psi\big(S(Tu,Tu,Tv)\big) \leq \psi\big(m(u,u,v)\big) - \phi\big(m(u,u,v)\big), \tag{2.13}$$

where

$$\begin{aligned} m(u,u,v) = \max\Big\{ & S(u,u,v), S(u,u,Tu), S(v,v,Tv), \tfrac{1}{3}S(Tu,Tu,u), \tfrac{1}{3}S(Tu,Tu,v), \\ & \tfrac{1}{3}S(Tv,Tv,u), \tfrac{1}{6}\big(S(Tu,Tu,u)+S(Tu,Tu,v)+S(Tv,Tv,u)\big)\Big\} \\ = {} & S(u,u,v). \end{aligned} \tag{2.14}$$

Then

$$\psi\big(S(u,u,v)\big) \leq \psi\big(S(u,u,v)\big) - \phi\big(S(u,u,v)\big) < \psi\big(S(u,u,v)\big),$$

a contradiction. Therefore $u = v$.

Finally we prove that T is continuous at x^*. Let $\{x_n\}$ be a sequence in X such that $x_n \to x^*$ as $n \to \infty$. We consider

$$\psi\big(S(x^*,x^*,Tx_n)\big) = \psi\big(S(Tx^*,Tx^*,Tx_n)\big) \leq \psi\big(m(x^*,x^*,x_n)\big) - \phi\big(m(x^*,x^*,x_n)\big), \tag{2.15}$$

where

$$\begin{aligned} & m(x^*,x^*,x_n) \\ & = \max\Big\{ S(x^*,x^*,x_n), S(x^*,x^*,Tx^*), S(x_n,x_n,x_{n+1}), \\ & \qquad \tfrac{1}{3}S(Tx^*,Tx^*,x^*), \tfrac{1}{3}S(Tx^*,Tx^*,x_n), \tfrac{1}{3}S(x_{n+1},x_{n+1},x^*), \\ & \qquad \tfrac{1}{6}\big(S(Tx^*,Tx^*,x^*) + S(Tx^*,Tx^*,x_n) + S(x_{n+1},x_{n+1},x^*)\big)\Big\} \\ & = \max\{S(x^*,x^*,x_n), S(x_n,x_n,x_{n+1})\}. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we obtain

$$\begin{aligned} \psi\big(S(x^*,x^*,Tx_n)\big) \leq {} & \psi\big(\max\{S(x^*,x^*,x_n), S(x_n,x_n,x_{n+1})\}\big) \\ & - \phi\big(\max\{S(x^*,x^*,x_n), S(x_n,x_n,x_{n+1})\}\big), \end{aligned} \tag{2.17}$$

Now by taking the limits on both sides of (2.17), we have

$$\lim_{n\to\infty} \psi\big(S(Tx^*,Tx^*,Tx_n)\big) = 0.$$

This implies that $\psi\big(\lim_{n\to\infty} S(Tx^*,Tx^*,Tx_n)\big) = 0$, since ψ is continuous. Now by property of ψ we have $\lim_{n\to\infty} S(Tx^*,Tx^*,Tx_n) = 0$. Hence $Tx_n \to Tx^*$ as $n \to \infty$. Therefore, T is continuous at x^*. This completes the proof of the theorem. □

Theorem 2.3. *Let (X, S) be a complete S-metric space and let $T : X \to X$ be a self-mapping. If there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that, for all $x, y, z \in X$*

$$\psi\big(S(Tx, Ty, Tz)\big) \leq \psi\big(m(x, y, z)\big) - \phi(m(x, y, z)),$$

where

$$m(x, y, z) = k \max\Big\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz), S(Tx, Tx, y), S(Ty, Ty, z), S(Tz, Tz, x)\Big\},$$

where $0 \leq k < 1/3$. Then T has a unique fixed point x^* and T is continuous at x^*.

Proof. Let $\lambda = 3k$; then $0 \leq \lambda < 1$. Consider

$$\begin{aligned} &k \max\Big\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz), \\ &\quad S(Tx, Tx, y), S(Ty, Ty, z), S(Tz, Tz, x)\Big\} \\ &= \lambda \max\Big\{\frac{1}{3}S(x, y, z), \frac{1}{3}S(x, x, Tx), \frac{1}{3}S(y, y, Ty), \\ &\quad \frac{1}{3}S(z, z, Tz), \frac{1}{3}S(Tx, Tx, y), \frac{1}{3}S(Ty, Ty, z), \frac{1}{3}S(Tz, Tz, x)\Big\} \\ &\leq \lambda m(x, y, z). \end{aligned} \tag{2.18}$$

From Theorem 2.2, we can see that T has a unique fixed point x^* and T is continuous at x^*. □

Example 2.4. *Let $X = [0, \frac{6}{5}]$ with the usual S-metric given in Example 1.4. Let us define the function $T : X \to X$ as*

$$Tx = \begin{cases} \frac{x}{5} & \text{if } x \in [0, 1] \\ x - \frac{4}{5} & \text{if } x \in (1, \frac{6}{5}] \end{cases} \tag{2.19}$$

for all $x \in X$. We now define functions $\psi, \phi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \tfrac{3}{2}t \text{ for all } t \geq 0 \quad \text{and} \quad \phi(t) = \begin{cases} \frac{6t}{5} & \text{if } t \in [0, 1] \\ \frac{3t}{2(t+1)} & \text{if } t > 1. \end{cases}$$

We now show that T satisfies the inequality (2.1)*.*

Case I. $x, y, z \in [0, 1]$*. We assume, without loss of generality, that $x > y > z$*

$$S(Tx, Ty, Tz) = S(\tfrac{x}{5}, \tfrac{y}{5}, \tfrac{z}{5}) = \tfrac{1}{5}(|x-z|+|y-z|) \quad \text{and} \quad m(x, y, z) \geq S(x, y, z) = |x-z|+|y-z|$$

Subcase i) $|x-z|+|y-z| \in [0,1]$.

$$\begin{aligned}\psi\big(S(Tx,Ty,Tz)\big) &= \tfrac{3}{2}\big(\tfrac{1}{5}(|x-z|+|y-z|)\big) = \tfrac{3}{10}(|x-z|+|y-z|) = \tfrac{3}{10}S(x,y,z)\\ &\le \tfrac{3}{10}m(x,y,z) = \tfrac{3}{2}m(x,y,z) - \tfrac{6}{5}m(x,y,z)\\ &= \psi\big(m(x,y,z)\big) - \phi\big(m(x,y,z)\big).\end{aligned}$$

Subcase ii) $|x-z|+|y-z| > 1$.

$$\begin{aligned}\psi\big(S(Tx,Ty,Tz)\big) &= \tfrac{3}{2}\big(\tfrac{1}{5}(|x-z|+|y-z|)\big)\\ &\le \tfrac{3}{2}\Big(|x-z|+|y-z| - \frac{|x-z|+|y-z|}{1+|x-z|+|y-z|}\Big)\\ &= \tfrac{3}{2}\Big(S(x,y,z) - \frac{S(x,y,z)}{1+S(x,y,z)}\Big) = \tfrac{3}{2}\Big(\frac{S(x,y,z)^2}{1+S(x,y,z)}\Big)\\ &\le \tfrac{3}{2}\Big(\frac{m(x,y,z)^2}{1+m(x,y,z)}\Big) = \tfrac{3}{2}m(x,y,z) - \frac{3m(x,y,z)}{2(1+m(x,y,z))}\\ &= \psi\big(m(x,y,z)\big) - \phi\big(m(x,y,z)\big).\end{aligned}$$

Case II. $x,y,z \in (1,\frac{6}{5}]$. *We assume, without loss of generality, that* $x>y>z$

$$S(Tx,Ty,Tz) = S(x-\tfrac{4}{5}, y-\tfrac{4}{5}, z-\tfrac{4}{5}) = |x-z|+|y-z| \le \tfrac{2}{5}$$

and

$$m(x,y,z) \ge S(x,x,Tx) = 2|x-(x-\tfrac{4}{5})| = \tfrac{8}{5}$$

$$\begin{aligned}\psi\big(S(Tx,Ty,Tz)\big) &= \tfrac{3}{2}\big(|x-z|+|y-z|\big) \le \tfrac{3}{2}\times\tfrac{2}{5} = \tfrac{3}{5} \le \tfrac{96}{65} = \tfrac{3}{2}\Big(\tfrac{8}{5}-\tfrac{8}{13}\Big)\\ &= \tfrac{3}{2}\Big(S(x,x,Tx) - \frac{S(x,x,Tx)}{1+S(x,x,Tx)}\Big) = \tfrac{3}{2}\Big(\frac{S(x,x,Tx)^2}{1+S(x,x,Tx)}\Big)\\ &\le \tfrac{3}{2}\Big(\frac{m(x,y,z)^2}{1+m(x,y,z)}\Big) = \tfrac{3}{2}m(x,y,z) - \frac{3m(x,y,z)}{2(1+m(x,y,z))}\\ &= \psi\big(m(x,y,z)\big) - \phi\big(m(x,y,z)\big).\end{aligned}$$

Case III. $x \in (1,\frac{6}{5}]$ *and* $y,z \in [0,1]$. *We assume, without loss of generality, that* $y>z$

$$S(Tx,Ty,Tz) = S(x-\tfrac{4}{5}, \tfrac{y}{5}, \tfrac{z}{5}) = |x-\tfrac{4}{5}-\tfrac{z}{5}| + |\tfrac{y}{5}-\tfrac{z}{5}| = x+\tfrac{y}{5}-\tfrac{2z}{5}-\tfrac{4}{5} \le \tfrac{3}{5}$$

and

$$m(x,y,z) \ge S(x,x,Tx) = 2|x-(x-\tfrac{4}{5})| = \tfrac{8}{5}$$

$$\psi\big(S(Tx,Ty,Tz)\big) = \tfrac{3}{2}\big(x+\tfrac{y}{5}-\tfrac{2z}{5}-\tfrac{4}{5}\big) \le \tfrac{3}{2}\times\tfrac{3}{5} = \tfrac{9}{10} \le \tfrac{96}{65}.$$

Similar to Case II, we obtain $\psi\big(S(Tx,Ty,Tz)\big) \le \psi\big(m(x,y,z)\big) - \phi\big(m(x,y,z)\big)$.

Case IV. *$x, y \in (1, \frac{6}{5}]$ and $z \in [0,1]$. We assume, without loss of generality, that $x > y$*

$$S(Tx,Ty,Tz) = S(x-\tfrac{4}{5}, y-\tfrac{4}{5}, \tfrac{z}{5}) = |x-\tfrac{4}{5}-\tfrac{z}{5}|+|y-\tfrac{4}{5}-\tfrac{z}{5}| = x+y-\tfrac{2z}{5}-\tfrac{8}{5} \leq \tfrac{4}{5}$$

and

$$m(x,y,z) \geq S(x,x,Tx) = 2|x-(x-\tfrac{4}{5})| = \tfrac{8}{5}$$

$$\psi\big(S(Tx,Ty,Tz)\big) = \tfrac{3}{2}\big(x+y-\tfrac{2z}{5}-\tfrac{8}{5}\big) \leq \tfrac{3}{2}\times\tfrac{4}{5} = \tfrac{6}{5} \leq \tfrac{96}{65}.$$

Similar to Case II, we obtain $\psi\big(S(Tx,Ty,Tz)\big) \leq \psi\big(m(x,y,z)\big) - \phi\big(m(x,y,z)\big)$.
Case V. *$x, y \in [0,1]$ and $z \in (1, \frac{6}{5}]$. We assume, without loss of generality, that $x > y$*

$$\begin{aligned} S(Tx,Ty,Tz) = S(\tfrac{x}{5}, \tfrac{y}{5}, z-\tfrac{4}{5}) &= |\tfrac{x}{5}-(z-\tfrac{4}{5})|+|\tfrac{y}{5}-(z-\tfrac{4}{5})| \\ &= |\tfrac{4}{5}-(z-\tfrac{x}{5})|+|\tfrac{4}{5}-(z-\tfrac{y}{5})| \\ &= z-\tfrac{x}{5}-\tfrac{4}{5}+z-\tfrac{y}{5}-\tfrac{4}{5} \\ &= 2z-\tfrac{x+y}{5}-\tfrac{8}{5} \leq \tfrac{4}{5} \end{aligned}$$

and

$$m(x,y,z) \geq S(z,z,Tz) = 2|z-(z-\tfrac{4}{5})| = \tfrac{8}{5}$$

$$\psi\big(S(Tx,Ty,Tz)\big) = \tfrac{3}{2}\big(2z-\tfrac{x+y}{5}-\tfrac{8}{5}\big) \leq \tfrac{3}{2}\times\tfrac{4}{5} = \tfrac{6}{5} \leq \tfrac{96}{65}.$$

Similar to Case II, we obtain $\psi\big(S(Tx,Ty,Tz)\big) \leq \psi\big(m(x,y,z)\big) - \phi\big(m(x,y,z)\big)$.
Case VI. *$x \in [0,1]$ and $y, z \in (1, \frac{6}{5}]$.*
Subcase i) *$y > z$.*

$$\begin{aligned} S(Tx,Ty,Tz) = S(\tfrac{x}{5}, y-\tfrac{4}{5}, z-\tfrac{4}{5}) &= |\tfrac{x}{5}-(z-\tfrac{4}{5})|+|y-\tfrac{4}{5}-(z-\tfrac{4}{5})| \\ &= |\tfrac{4}{5}-(z-\tfrac{x}{5})|+|y-z| \\ &= z-\tfrac{x}{5}-\tfrac{4}{5}+y-z \\ &= y-\tfrac{x}{5}-\tfrac{4}{5} \leq \tfrac{2}{5} \end{aligned}$$

and

$$m(x,y,z) \geq S(y,y,Ty) = 2|y-(y-\tfrac{4}{5})| = \tfrac{8}{5}$$

$$\psi\big(S(Tx,Ty,Tz)\big) = \tfrac{3}{2}\big(y-\tfrac{x}{5}-\tfrac{4}{5}\big) \leq \tfrac{3}{2}\times\tfrac{2}{5} = \tfrac{3}{5} \leq \tfrac{96}{65}.$$

Similar to Case II, we obtain $\psi\big(S(Tx,Ty,Tz)\big) \leq \psi\big(m(x,y,z)\big) - \phi\big(m(x,y,z)\big)$.
Subcase ii) *$z > y$.*

$$\begin{aligned} S(Tx,Ty,Tz) = S(\tfrac{x}{5}, y-\tfrac{4}{5}, z-\tfrac{4}{5}) &= |\tfrac{x}{5}-(z-\tfrac{4}{5})|+|y-\tfrac{4}{5}-(z-\tfrac{4}{5})| \\ &= |\tfrac{4}{5}-(z-\tfrac{x}{5})|+|y-z| \\ &= z-\tfrac{x}{5}-\tfrac{4}{5}+z-y \\ &= 2z-y-\tfrac{x}{5}-\tfrac{4}{5} \leq \tfrac{3}{5} \end{aligned}$$

and

$$m(x,y,z) \geq S(z,z,Tz) = 2|z-(z-\tfrac{4}{5})| = \tfrac{8}{5}$$

$$\psi\big(S(Tx,Ty,Tz)\big) = \tfrac{3}{2}\big(y-\tfrac{x}{5}-\tfrac{4}{5}\big) \leq \tfrac{3}{2}\times\tfrac{3}{5} = \tfrac{9}{10} \leq \tfrac{96}{65}.$$

Similar to Case II, we obtain $\psi\big(S(Tx,Ty,Tz)\big) \leq \psi\big(m(x,y,z)\big) - \phi\big(m(x,y,z)\big)$. *From all the above cases, we conclude that* T *is an generalized weakly contractive map on* X. *Therefore,* T,ψ *and* ϕ *satisfy all the hypotheses of Theorem 2.2 and* T *has a unique fixed point* $u=0$.

Remark 2.5. *In Example 2.4, we observe that Theorem 2.3 cannot be applied since for all* $x,y,z \in [0,1]$, *we have* $m(x,y,z) = S(x,y,z)$

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