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B-Well-Posedness for Set Optimization Problems Involving Set Order Relations

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Abstract: In this paper, both pointwise and global B-well-posedness for set optimization problems involving three kinds of set order relations are investigated. We give characterizations and sufficient and/or necessary conditions of these types of well-posedness. Moreover, pointwise L-well-posedness and relationships between these kinds of pointwise well-posedness are studied.

Keywords: B-well-posedness; generalized B-well-posedness; pointwise B-well-posedness; pointwise L-well-posedness; set optimization problem
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1 Introduction

Set order relations were firstly introduced by Kuroiwa et al. in [1] and then they were generalized in [2]. These concepts gave a new way, so-called set approach, to formulate the optimal of set-valued optimization problems [3]. In this approach, all images of the set-valued objective mapping were compared by set order relations [4,5], and hence it is a truely natural and practical approach. Therefore, this field

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has attracted a great deal of attention of researchers although it is a young direction in optimization. Many interesting and important results have been obtained in different topics in this area [6–11].

Well-posedness was originally proposed by Tikhonov in [12]. This concept requires two conditions, namely the uniqueness of solution and the convergence of each minimizing sequence to the unique solution. In other words, whenever we are able to compute approximately the optimal value then we automatically do approximate the optimal solution. So, well-posedness plays an important role in both theory results and numerical methods, and hence many mathematicians have paid much attention on this topic (see e.g., [13–15] and the reference therein). Later on, generalizations of Tikhonov well-posedness were introduced and studied widely. One of these extensions is the so-called B-well-posedness proposed by Bednarczuck for vector optimization problems in [16]. After that, this notion has been intensively considered for various problems related to optimization [17–21]

Studying on well-posedness for set optimization problems was initialed by Zhang et al. in [22]. The authors obtained sufficient, necessary conditions and characterizations for set optimization problems involving the lower set less relation to be well-posed by the scalarization method. After that, some different types of well-posedness for these problems introduced and investigated [23–26]. In 2013, as the first authors concerned B-well-posedness for set optimization problems, Long and Peng [24] introduced three types of B-well-posedness for set optimization problems involving upper set less relations $\leq^u$ and established some relations among these kinds of B-well-posedness. Moreover, the authors also provided necessary and sufficient conditions of these notions for set optimization problems. To extend the research in [24], Han and Huang [8] studied B-well-posedness for set optimization problems involving set order relations $\leq^l$ and $\leq^u$. They gave characterizations for the generalized l-B-well-posedness and the generalized u-B-well-posedness and provided the semicontinuity of solution mapping.

As mentioned in [2,27] that among three kinds of set order relations introduced in [1], the set less relation $\leq^s$ is generalized and more appropriate in practical problems than both the lower and upper set less relations; and it also occupies an important role in relationships with other new order relations for sets proposed in [2] which are more useful in real world. Moreover, to the best of our knowledge, there is no paper devoted to well-posedness for set optimization problems involving the set less relation, and hence well-posedness properties for such problems are deserved to study more. Consequently, we aim to investigate both pointwise and global B-well-posedness as well as pointwise L-well-posedness for set optimization problems involving three kinds of set order relations.

The outline of this paper is as follows. In Sect. 2, some concepts and results used in what follows are recalled. Sect. 3 studies global B-well-posedness for set optimization problems, including B-well-posedness and generalized B-well-posedness. Relationships between them are discussed. Moreover, sufficient conditions of B-well-posedness for such problems are provided. In Sect. 4, we focus to pointwise B-well-posedness. Characterizations as well as relationships between pointwise B-well-posedness and global B-well-posedness are studied. In the last section, Sect. 5, pointwise L-well-posedness is investigated. Then, relationships between it and pointwise B-well-posedness are researched.

2 Preliminaries

Let X and Y be normed spaces. We denote the closed unit ball of Y by B_Y. Let K be a closed convex pointed cone in Y with $\text{int}K \neq \emptyset$, where $\text{int}K$ denotes the interior of K. Orderings induced by cone K in the space Y are defined as the following

$$x \leq_K y \Leftrightarrow y - x \in K,$$

$$x <_K y \Leftrightarrow y - x \in \text{int}K.$$

To compare two subsets of Y, we use set order relations introduced in [2,5,28]. We list here three kinds of set order relations used in this paper. Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y. For $A, B \in \mathcal{P}(Y)$, lower set less relation, upper set less relation and set less relation, respectively, are defined by

$$A \leq^l B \text{ if and only if } B \subset A + K,$$

$$A \leq^u B \text{ if and only if } A \subset B - K,$$

$$A \leq^s B \text{ if and only if } A \subset B - K \text{ and } B \subset A + K.$$

Definition 2.1. [2] We say that the binary relation $\leq$ is

(i) compatible with the addition if and only if $A \leq B$ and $D \leq E$ imply $A + D \leq B + E$ for all $A, B, D, E \in \mathcal{P}(Y)$.

(ii) compatible with the multiplication with a nonnegative real number if and only if $A \leq B$ implies $\lambda A \leq \lambda B$ for all scalars $\lambda \geq 0$ and all $A, B \in \mathcal{P}(Y)$.

(iii) compatible with the conlinear structure of $\mathcal{P}(Y)$ if and only if it is compatible with both the addition and the multiplication with a nonnegative real number.

Proposition 2.1. [2]

(i) The set order relations $\leq^l$, $\leq^u$ and $\leq^s$ are pre-order (i.e., these relations are reflexive and transitive).

(ii) The set order relations $\leq^l$, $\leq^u$ and $\leq^s$ are compatible with the conlinear structure of $\mathcal{P}(Y)$.

(iii) In general, the set order relations $\leq^l$, $\leq^u$ and $\leq^s$ are not antisymmetric; more precisely, for arbitrary sets $A, B \in \mathcal{P}(Y)$ we have

$$(A \leq^l B \text{ and } B \leq^l A) \Leftrightarrow (A + K = B + K),$$

$$(A \leq^u B \text{ and } B \leq^u A) \Leftrightarrow (A - K = B - K),$$

$$(A \leq^s B \text{ and } B \leq^s A) \Leftrightarrow (A + K = B + K \text{ and } A - K = B - K).$$

For $\alpha \in \{u, l, s\}$, we say that

$$A \sim^{\alpha} B \text{ if and only if } A \leq^{\alpha} B \text{ and } B \leq^{\alpha} A.$$

Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values on X, we denote $F(M) = \cup_{x \in M} F(x)$. For each $\alpha \in \{u, l, s\}$, we consider the following set optimization problem

$$(\mathrm{P}_\alpha) \qquad \alpha\text{-Min}\, F(x)$$
$$\text{subject to} \qquad x \in M,$$

where M is a nonempty subset of X. A point $\bar{x} \in M$ is said to be an α-minimal solution of (P_α) if for any $x \in M$ such that $F(x) \leq^{\alpha} F(\bar{x})$, then $F(\bar{x}) \leq^{\alpha} F(x)$. The set of all α-minimal solutions of (P_α) is called the solution set of (P_α) and denoted by $S_{\alpha\text{-Min}\, F}$.

Remark 2.2. It can be seen that if $\bar{x} \in S_{\alpha\text{-Min}\, F}$ and $F(\bar{x}) \sim^{\alpha} F(x)$ for some $x \in M$, then $x \in S_{\alpha\text{-Min}\, F}$.

Next, we recall definitions of semicontinuity for a set-valued mapping and their properties used in the sequel.

Definition 2.2. [29] A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(i) upper semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for any open subset V of Y with $F(x_0) \subset V$ there is a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$;

(ii) lower semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for any open subset V of Y with $F(x_0) \cap V \neq \emptyset$ there is a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;

(iii) lower (upper) semicontinuous on a subset D of X if it is lower (upper) semicontinuous at every $x \in D$;

where $\mathrm{Dom}F = \{x \in X \mid F(x) \neq \emptyset\}$.

Lemma 2.3. [30] Let $F : X \rightrightarrows Y$ be a set-valued mapping.

(i) F is lower semicontinuous at $x_0 \in \mathrm{Dom}F$ if for every $\{x_n\}$ converging to x_0 and for every $y \in F(x_0)$ there exists $\{y_n\}$ with $y_n \in F(x_n)$ such that $\{y_n\}$ converges to y.

(ii) If $F(x_0)$ is compact and F is upper semicontinuous at $x_0 \in \mathrm{Dom}F$, then for every $\{x_n\}$ converging to x_0 and $y_n \in F(x_n)$ there exist $y_0 \in F(x_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ converges to y_0.

Definition 2.3. [30] A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(i) Hausdorff upper semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for each neighborhood V of the origin in Y, there exists a neighborhood U of x_0 such that $F(x) \subset F(x_0) + V$ for all $x \in U$.

(ii) Hausdorff lower semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for any neighborhood V of the origin in Y, there exists a neighborhood U of x_0 such that $F(x_0) \subset F(x) + V$ for all $x \in U$.

(iii) Hausdorff lower (upper) semicontinuous on a subset D of X if and only if F is Hausdorff lower (upper) semicontinuous at every point of D.

Remark 2.4. [31] If F is upper semicontinuous at $x_0 \in \mathrm{Dom}F$, then F is Hausdorff upper semicontinuous at x_0; the converse implication is true when $F(x_0)$ is compact.

Next, we recall concepts of Hausdorff distance and Hausdorff convergence of sequence of sets. Let S be a nonempty subset of X and $x \in X$. The distance d between x and S is defined as

$$d(x, S) = \inf_{u \in S} d(x, u).$$

Let S_1 and S_2 be two nonempty subsets of X. The Hausdorff distance between S_1 and S_2, denoted by $H(S_1, S_2)$, is defined as

$$H(S_1, S_2) = \max\{H^*(S_1, S_2), H^*(S_2, S_1)\},$$

where $H^*(S_1, S_2) = \sup_{x \in S_1} d(x, S_2)$.

Definition 2.4. [32] Let $\{A_n\}$ be a sequence of subsets of X. We say that

(i) A_n converge to $A \subset X$ in the sense of the upper Hausdorff set-convergence, denoted by $A_n \rightharpoonup A$, if and only if $H^*(A_n, A) \to 0$.

(ii) A_n converge to $A \subset X$ in the sense of the lower Hausdorff set-convergence, denoted by $A_n \rightharpoondown A$, if and only if $H^*(A, A_n) \to 0$.

(iii) A_n converge to $A \subset X$ in the sense of the Hausdorff set-convergence, denoted by $A_n \to A$, if and only if $H(A_n, A) \to 0$.

3 *B*-well-posedness for set optimization problems

In this section, two kinds of global B-well-posedness for the problem (P_α) are considered and their relationships are discussed. Moreover, we also provide characterizations and sufficient conditions of B-well-posedness for such problems.

We observe from the definitions of set order relations that $\leq^s$ is a combination of $\leq^l$ and $\leq^u$. For relationships between $\leq^l$ and $\leq^u$, they were given in Remark 2.6.10 of [4] as the following

$$A \leq^l B \Leftrightarrow -B \leq^u -A.$$

Some properties about these set order relations are demonstrated in next results.

Proposition 3.1. *The following statements are true:*

(i) *If $A \leq^{\alpha} B$, then $\lambda A \leq^{\alpha} \lambda B$, $\forall \lambda > 0$;*

(ii) *$A \leq^{l} B \Leftrightarrow \lambda A \geq^{u} \lambda B$, $\forall \lambda < 0$;*

(iii) *$A \leq^{s} B \Leftrightarrow \lambda A \geq^{s} \lambda B$, $\forall \lambda < 0$.*

Proof. (i) We give the proof of the assertion (i) for the case $\alpha = s$, proofs of this assertion for other cases $\alpha = l$ and $\alpha = u$ are similar. Since $A \leq^{s} B$, $B \subset A + K$ and $A \subset B - K$. Clearly, for any $\lambda > 0$, we get $\lambda B \subset \lambda A + K$ and $\lambda A \subset \lambda B - K$. Hence, $\lambda A \leq^{s} \lambda B$.

(ii) We have $B \subset A + K$ as $A \leq^{l} B$. For any $\lambda < 0$, this yields $\lambda B \subset \lambda A - K$, i.e., $\lambda B \leq^{u} \lambda A$.

(iii) Since $A \leq^{s} B$, $B \subset A + K$ and $A \subset B - K$. For any $\lambda < 0$, this implies that $\lambda B \subset \lambda A - K$ and $\lambda A \subset \lambda B + K$. So, $\lambda A \geq^{s} \lambda B$. □

We define a set-valued mapping $Q : K \rightrightarrows M$ as follows

$$Q(k) = \bigcup_{y \in S_{\alpha-\mathrm{MinF}}} \{x \in M \mid F(x) \leq^{\alpha} F(y) + k\}. \tag{3.1}$$

The following results provide some properties of this mapping.

Proposition 3.2. *The following assertions hold:*

(i) *If $k_1 \leq_K k_2$, then $Q(k_1) \subset Q(k_2)$;*

(ii) *$S_{\alpha\text{-}MinF} \subset Q(0)$;*

(iii) *$Q(0) = \cap_{k \in K} Q(k)$.*

Proof. (i) We only demonstrate the proof of the above assertion for the case $\alpha = s$, proofs of this assertion for other cases are proved similarly. Let $x \in Q(k_1)$ be given, then there exists $y \in S_{\alpha\text{-MinF}}$ such that $F(x) \leq^{s} F(y) + k_1$, i.e., $F(y) + k_1 \subset F(x) + K$ and $F(x) \subset F(y) + k_1 - K$. Combining this with $k_1 \leq_K k_2$, we get $F(y) + k_2 = F(y) + k_1 + (k_2 - k_1) \subset F(x) + K$ and $F(x) \subset F(y) + k_1 - K = F(y) + k_2 + (k_1 - k_2) - K \subset F(y) + k_2 - K$. This means that $F(x) \leq^{l} F(y) + k_2$ and $F(x) \leq^{u} F(y) + k_2$. Hence, $x \in Q(k_2)$.

(ii) Clearly, for every $x \in S_{\alpha\text{-MinF}}$, we have $F(x) \leq^{\alpha} F(x)$, and hence $x \in Q(0)$. Therefore, $S_{\alpha\text{-MinF}} \subset Q(0)$.

(iii) It is obvious that $Q(0) \subset Q(k)$ for all $k \in K$, and thus $Q(0) \subset \cap_{k \in K} Q(k)$. Conversely, suppose that there exists $x \in \cap_{k \in K} Q(k)$ but $x \notin Q(0)$, i.e., $x \notin \cup_{y \in S_{\alpha\text{-MinF}}} \{z \in M | F(z) \leq^{\alpha} F(y)\}$. Then, $F(x) \not\leq^{\alpha} F(y)$ for any $y \in S_{\alpha\text{-MinF}}$. On the other hand, since $x \in \cap_{k \in K} Q(k)$, $x \in Q(k)$ for all $k \in K$. So, there is $y \in S_{\alpha\text{-MinF}}$ such that $F(x) \leq^{\alpha} F(y) + k$ for all $k \in K$. Particularly, for $k = 0$, there exists $y \in S_{\alpha\text{-MinF}}$ such that $F(x) \leq^{\alpha} F(y)$ which is a contradiction. □

Next, we give two concepts related to global B-well-posedness for (P_{α}).

Definition 3.1. Problem (P_α) is said to be

(i) B-well-posed if and only if $S_{\alpha\text{-MinF}} \neq \emptyset$ and Q is upper semicontinuous at $k = 0$.

(ii) generalized B-well-posed if and only if $S_{\alpha\text{-MinF}} \neq \emptyset$ and Q is Hausdorff upper semicontinuous at $k = 0$.

Remark 3.3. Clearly, if the problem (P_α) is B-well-posedness, then it is generalized B-well-posedness. It follows from Proposition 3.2(ii) and Remark 2.4 that the converse holds if $S_{\alpha\text{-MinF}}$ is compact. In the sequel, we focus on generalized B-well-posedness.

Definition 3.2. A sequence $\{x_n\} \subset M$ is said to be a generalized B-minimizing sequence of (P_α) if and only if there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_{\alpha\text{-MinF}}$ such that $F(x_n) \leq^\alpha F(y_n) + k_n$.

Equivalently, $\{x_n\}$ is a generalized B-minimizing sequence of (P_α) if and only if there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_{\alpha\text{-MinF}}$ such that $x_n \in Q(k_n)$.

Remark 3.4. When $\alpha = u$, concepts in Definitions 3.1 and 3.2 reduce to ones in Definitions 3.1-3.3 in [24], respectively.

Characterizations of B-well-posedness for (P_α) are provided in the next result through the B-minimizing sequence.

Theorem 3.5. *Problem (P_α) is generalized B-well-posed if and only if these following conditions are satisfied*

(a) $S_{\alpha\text{-}MinF} \neq \emptyset$;

(b) *for every generalized B-minimizing sequence $\{x_n\} \subset M$ and for every neighborhood U of the origin in X, there exists $n_0 \in \mathbb{N}$ such that $x_n \in Q(0) + U$ for all $n \geq n_0$.*

Proof. Suppose that (P_α) is generalized B-well-posed. Let $\{x_n\} \subset M$ be a generalized B-minimizing sequence of (P_α), then there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_{\alpha\text{-MinF}}$ such that $x_n \in Q(k_n)$. Since (P_α) is generalized B-well-posed, Q is Hausdorff upper semicontinuous at 0. Let U be a neighborhood of the origin in X, there exists $n_0 \in \mathbb{N}$ such that $Q(k_n) \subset Q(0) + U$ for all $n \geq n_0$. Therefore, we get $x_n \in Q(0) + U$ for all $n \geq n_0$.

Conversely, suppose on the contrary that (P_α) is not generalized B-well-posed. Thus, Q is not Hausdorff upper semicontinuous at 0. Then, there exists a neighborhood U of the origin in X such that $Q(k) \not\subset Q(0) + U$ for some k belongs to a neighborhood of 0. So, we can build a sequence $\{k_n\} \subset K$ converging to 0 such that $Q(k_n) \not\subset Q(0) + U$. It leads to the existence of a sequence $\{x_n\}$ with $x_n \in Q(k_n)$ satisfying $x_n \notin Q(0) + U$ which contradicts the assumption (b). This completes the proof. □

We now give sufficient conditions for (P_α) to be generalized B-well-posed.

Theorem 3.6. *Suppose that $S_{\alpha\text{-}MinF} \neq \emptyset$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(F(M) - F(S_{\alpha\text{-}MinF})) \cap (\delta B_Y - K) \subset \varepsilon B_Y. \tag{3.2}$$

Then,

(i) *(P_l) is generalized B-well-posed if each sequence of sets $\{A_n\} \subset M$ with $F(A_n) \rightharpoondown F(S_{l\text{-}MinF})$ satisfies $A_n \rightharpoondown S_{l\text{-}MinF}$.*

(ii) *(P_u) is generalized B-well-posed if each sequence of sets $\{A_n\} \subset M$ with $F(A_n) \rightharpoonup F(S_{u\text{-}MinF})$ satisfies $A_n \rightharpoonup S_{u\text{-}MinF}$.*

(iii) *(P_s) is generalized B-well-posed if each sequence of sets $\{A_n\} \subset M$ with $F(A_n) \rightarrow F(S_{s\text{-}MinF})$ satisfies $A_n \rightarrow S_{s\text{-}MinF}$.*

Proof. (i) By contradiction, suppose that (P_l) is not generalized B-well-posed. It follows from Theorem 3.5 that there exist a generalized B-minimizing sequence $\{x_n\}$ and a neighborhood U of the origin in X such that for some $n_0 \in \mathbb{N}$, $x_n \notin Q(0) + U$ for all $n \geq n_0$. Combining this with Proposition 3.2(ii), we get

$$x_n \notin S_{l\text{-MinF}} + U, \ \forall n \geq n_0. \tag{3.3}$$

Since $\{x_n\}$ is a generalized B-minimizing sequence, there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_{l\text{-MinF}}$ such that

$$F(x_n) \leq^l F(y_n) + k_n. \tag{3.4}$$

We consider two following cases:
Case 1: If $F(x_n) \rightharpoondown F(S_{l\text{-MinF}})$, then choosing $A_n = \{x_n\}$. It implies from the hypothesis that $\{x_n\} \rightharpoondown S_{l\text{-MinF}}$, and hence $H^*(S_{l\text{-MinF}}, \{x_n\}) \to 0$. Therefore, $d(x_n, S_{l\text{-MinF}}) \to 0$ which contradicts (3.3).
Case 2: If $F(x_n) \not\rightharpoondown F(S_{l\text{-MinF}})$, then $\sup_{x \in S_{l\text{-MinF}}} d(x, F(x_n)) \not\to 0$. So, there exists $x \in S_{l\text{-MinF}}$ such that $d(x, F(x_n)) \not\to 0$, i.e., there exist $n_1 \in \mathbb{N}$ and a neighborhood V of the origin in Y such that

$$x \notin F(x_n) + V, \ \forall n \geq n_1. \tag{3.5}$$

Take ε such that $\varepsilon B_Y \subset V$. For δ satisfying (3.2), since $\{k_n\} \subset K$ converges to 0, there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$, we have

$$k_n \in \delta B_Y. \tag{3.6}$$

By (3.4), we get $F(y_n) + k_n \subset F(x_n) + K$. Therefore, $F(y_n) \subset F(x_n) - k_n + K \subset F(x_n) + \delta B_Y + K$. This implies that for an arbitrary $z_n \in F(y_n)$, there exists $\bar{z}_n \in F(x_n)$ such that $z_n \in \bar{z}_n + \delta B_Y + K$, and thus $z_n - \bar{z}_n \in \delta B_Y + K$. So, $\bar{z}_n - z_n \in \delta B_Y - K$. On the other hand, we have $\bar{z}_n - z_n \in F(x_n) - F(y_n) \subset F(M) - F(S_{l\text{-MinF}})$. It derives from (3.2) that $\bar{z}_n - z_n \in \varepsilon B_Y \subset V$. So, now we get $z_n \in \bar{z}_n + V \subset F(x_n) + V$ which contradicts (3.5).

For (ii) (iii), the proofs of these assertions are technically similar to that of the assertion (i). □

4 Pointwise B-well-posedness for set optimization problems

In this section, we consider a notion of pointwise B-well-posedness for the problem (P_α). At a reference point $x_0 \in M$, we define a corresponding set-valued mapping as follows $Q_{x_0} : K \rightrightarrows M$, $Q_{x_0}(k) = \{x \in M \mid F(x) \leq^\alpha F(x_0) + k\}$.

Definition 4.1. Problem (P_α) is said to be B-well-posed at $x_0 \in S_{\alpha\text{-MinF}}$ if and only if Q_{x_0} is upper semicontinuous at $k = 0$.

We observe that $Q_{x_0}(0) = \{x \in M \mid F(x) \sim^\alpha F(x_0)\}$ for each $x_0 \in S_{\alpha\text{-MinF}}$. The next results give some properties of the mapping Q_{x_0}.

Proposition 4.1. *The following statements are true:*

(i) *If* $k_1 \leq_K k_2$*, then* $Q_{x_0}(k_1) \subset Q_{x_0}(k_2)$;

(ii) $Q_{x_0}(0) \subset S_{\alpha\text{-}MinF}$ *with* $x_0 \in S_{\alpha\text{-}MinF}$;

(iii) $x_0 \in Q_{x_0}(0) \subset Q_{x_0}(k)$ *for every* $k \in K$;

(iv) $Q(0) = \cup_{x_0 \in S_{\alpha\text{-}MinF}} Q_{x_0}(0)$.

Proof. (i) The statement is proved by a similar argument in Proposition 3.2(i).

(ii) By the similarity, we prove the assertion for the case $\alpha = s$. Let $x \in Q_{x_0}(0)$ and $y \in M$ such that

$$F(y) \leq^s F(x), \tag{4.1}$$

we need to show that $F(x) \leq^s F(y)$. Since $x \in Q_{x_0}(0)$,

$$F(x) \leq^s F(x_0). \tag{4.2}$$

Combining (4.1) and (4.2), we get $F(y) \leq^s F(x_0)$. Because $x_0 \in S_{\alpha\text{-MinF}}$, $F(x_0) \leq^s F(y)$. Therefore, $F(x) \leq^s F(y)$.

For (iii) and (iv), these assertions are implied by definitions of mappings Q and Q_{x_0}. □

Now we discuss the converse of (ii) of the above proposition. Because the proof of this assertion is elementary, we would like to omit it.

Lemma 4.2. *If* $x_0 \in S_{\alpha\text{-}MinF}$*, then* $S_{\alpha\text{-}MinF} \subset Q_{x_0}(0)$ *if and only if* $F(x) \sim^\alpha F(y)$ *for all* $x, y \in S_{\alpha\text{-}MinF}$.

Definition 4.3. A sequence $\{x_n\} \subset M$ is said to be an x_0-minimizing sequence of (P_α) where $x_0 \in S_{\alpha\text{-MinF}}$ if and only if there exists $\{k_n\} \subset K$ converging to 0 such that

$$F(x_n) \leq^\alpha F(x_0) + k_n.$$

Definition 4.3 reduces to Definition 3.3 in [25] when $\alpha = u$.

Clearly, for each $x_0 \in S_{\alpha\text{-MinF}}$, $\{x_n\} \subset M$ is an x_0-minimizing sequence if and only if there exists $\{k_n\} \subset K$ converging to 0 such that $x_n \in Q_{x_0}(k_n)$.

The next result illustrates the relationship between the pointwise B-well-posedness and B-well-posedness for (P_α).

Theorem 4.2. *If $S_{\alpha\text{-}MinF}$ is a finite set and (P_α) is pointwise B-well-posed at every $x \in S_{\alpha\text{-}MinF}$, then it is B-well-posed.*

Proof. Let $S_{\alpha\text{-MinF}} = \{x_1, \dots, x_n\}$ and V be an open set in X such that $Q(0) \subset V$. By Proposition 4.1(iv), $Q_{x_i}(0) \subset V$ for all $i = 1, \dots, n$. Since (P_α) is pointwise B-well-posed at every x_i, Q_{x_i} is upper semicontinuous at $k = 0$, and hence for each $i \in \{1, \dots, n\}$, there exists a neighborhood U_{x_i} of 0 such that $Q_{x_i}(U_{x_i}) \subset V$. Let $U = \cap_{i=1}^n U_{x_i}$, this finite intersection of neighborhoods U_{x_i} is also a neighborhood of 0. Obviously, we have $Q_{x_i}(U) \subset V$ for all $i = 1, \dots, n$. By definitions of mappings Q_{x_i} and Q, we also get $Q(U) \subset V$. It leads to the upper semicontinuity at $k = 0$ of Q. We conclude that (P_α) is B-well-posed. □

We next investigate characterizations of pointwise B-well-posedness for (P_α).

Theorem 4.3. *Problem (P_α) is pointwise B-well-posed at $x_0 \in S_{\alpha\text{-}MinF}$ if and only if for a given $e \in intK$, the set-valued mapping $Q^+_{x_0} : \mathbb{R}_+ \rightrightarrows M$ defined as*

$$Q^+_{x_0}(t) = \{x \in M \mid F(x) \leq^\alpha F(x_0) + te\}$$

is upper semicontinuous at $t = 0$.

Proof. Assume that (P_α) is pointwise B-well-posed at $x_0 \in S_{\alpha\text{-MinF}}$, then Q_{x_0} is upper semicontinuous at $k = 0$. Let V be an open set in X such that $Q^+_{x_0}(0) \subset V$, we get $Q_{x_0}(0) \subset V$. By the upper semicontinuity of Q_{x_0}, there is a positive number r such that $Q_{x_0}(k) \subset V$ for all $k \in B(0, r) \cap K$, where $B(0, r)$ is the open ball centered at the origin in Y with radius r. Then, there exists a positive number β such that $[0, \beta e) \subset B(0, r)$, where $[0, \beta e) = \{te \mid t \in [0, \beta)\}$. For $t \in [0, \beta)$ and $x \in Q^+_{x_0}(t)$, we have $F(x) \leq^\alpha F(x_0) + te$, which implies that $x \in Q_{x_0}(te)$. This fact, together with $te \in B(0, r)$, yields $x \in V$, and so $Q^+_{x_0}(t) \subset V$. We conclude that $Q^+_{x_0}$ is upper semicontinuous at $t = 0$.

Conversely, suppose that $Q^+_{x_0}$ is upper semicontinuous at $t = 0$. Let V be an open set in X such that $Q_{x_0}(0) \subset V$, then $Q^+_{x_0}(0) \subset V$. It follows from the upper semicontinuity of $Q^+_{x_0}$ that there exists a positive number β such that $Q^+_{x_0}(t) \subset V$ for every $t \in [0, \beta)$. For convenience in writing, we only prove the assertion for case $\alpha = s$ because proofs of this assertion for other cases $\alpha = l$ and $\alpha = u$ are similar. Let $\gamma \in [0, \beta)$, there exists a positive number r such that $B(0, r) \subset \gamma e - K$ and $B(0, r) \subset -\gamma e + K$. Let $k \in B(0, r) \cap K$ and $x \in Q_{x_0}(k)$, it follows from the definition of Q_{x_0} that

$$F(x) \leq^u F(x_0) + k, \tag{4.3}$$

and

$$F(x) \leq^l F(x_0) + k. \tag{4.4}$$

It yields from (4.3) that $F(x) \subset F(x_0) + k - K \subset F(x_0) + \gamma e - K$ as $k \in \gamma e - K$. On other hand, by (4.4), we get $F(x_0) + k \subset F(x) + K$, and hence $F(x_0) \subset F(x) - k + K$. Since $-k \in B(0, r) \subset -\gamma e + K$, we have $F(x_0) \subset F(x) - \gamma e + K$, i.e., $F(x_0) + \gamma e \subset F(x) + K$. So, we now get $F(x) \leq^s F(x_0) + \gamma e$. It implies that $x \in Q^+_{x_0}(\gamma)$, and thus $x \in V$ and $Q_{x_0}(k) \subset V$. The proof is complete. □

Theorem 4.4. *If $S_{\alpha\text{-}MinF}$ is closed and (P_α) is pointwise B-well-posed at $x_0 \in S_{\alpha\text{-}MinF}$, then for every x_0-minimizing sequence $\{x_n\} \subset M \backslash S_{\alpha\text{-}MinF}$, one can extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some $\bar{x} \in S_{\alpha\text{-}MinF}$.*

Proof. Assume that (P_α) is pointwise B-well-posed at $x_0 \in S_{\alpha\text{-MinF}}$, then Q_{x_0} is upper semicontinuous at $k = 0$. By contradiction, suppose that there exists an x_0-minimizing sequence $\{x_n\} \subset M \backslash S_{\alpha\text{-MinF}}$ which admits no subsequence $\{x_{n_k}\}$ converging to some $\bar{x} \in S_{\alpha\text{-MinF}}$. By the closedness of $S_{\alpha\text{-MinF}}$, we may find an open set $V \subset X$ such that $S_{\alpha\text{-MinF}} \subset V$ and $x_n \notin V$. We have $Q_{x_0}(0) \subset V$ due to $Q_{x_0}(0) \subset S_{\alpha\text{-MinF}}$ and Proposition 4.1(ii). Since $\{x_n\}$ is an x_0-minimizing sequence, there exists $\{k_n\} \subset K$ converging to 0 such that $x_n \in Q_{x_0}(k_n)$. It follows from the upper semicontinuity of Q_{x_0} at $k = 0$ that $Q_{x_0}(k_n) \subset V$. Hence, $x_n \in V$ which is a contradiction. So, we get the desired result. □

Remark 4.5. Our results extend the corresponding results of Long and Peng [24]. More precisely,

(i) When $\alpha = l$ or $\alpha = s$, our results here are new. To the best of our knowledge, there is no paper devoted to this type of well-posedness for set optimization problem involving the set less relation $\leq^s$.

(ii) When $\alpha = u$, the corresponding set optimization problem (P_u) was studied in [24].

5 Pointwise L-well-posedness and relationship with pointwise B-well-posedness

Motivated by the studies in [22, 25], we introduce the concept of pointwise L-well-posedness for the problem (P_α).

Definition 5.1. Problem (P_α) is said to be L-well-posed at $x_0 \in S_{\alpha\text{-MinF}}$ if and only if every x_0-minimizing sequence of (P_α) has a subsequence converging to some element $\bar{x} \in S_{\alpha\text{-MinF}}$.

Remark 5.1. When $\alpha = l$, Definition 5.1 reduces to Definition 2.1 in [22].

We are going to study sufficient and necessary conditions of pointwise L-well-posedness for (P_α).

Theorem 5.2. *Let $x_0 \in S_{\alpha\text{-}MinF}$ be given.*

(i) *If* (P_α) *is* L*-well-posed at* x_0 *and* $Q_{x_0}(0) = S_{\alpha\text{-Min}\,F}$*, then* Q_{x_0} *is upper semicontinuous and compact-valued at* 0.

(ii) *If* Q_{x_0} *is upper semicontinuous and compact-valued at* 0*, then* (P_α) *is* L*-well-posed at* x_0.

Proof. (i) Assume that (P_α) is L-well-posed at x_0. First of all, we show that Q_{x_0} is upper semicontinuous at 0. By contradiction, suppose that Q_{x_0} is not upper semicontinuous at 0. Then, there exist a neighborhood U of $Q_{x_0}(0)$ and $\{k_n\} \subset K$ converging to 0 such that for each $n \in \mathbb{N}$, there exists $x_n \in Q_{x_0}(0) \setminus U$, i.e.,

$$x_n \notin U \tag{5.1}$$

and

$$F(x_n) \leq^\alpha F(x_0) + k_n. \tag{5.2}$$

It follows from (5.2) that $\{x_n\}$ is an x_0-minimizing sequence of (P_α). Because (P_α) is L-well-posed at x_0, there exists a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to some element $\bar{x} \in S_{\alpha\text{-MinF}}$, and thus we get $\bar{x} \in Q_{x_0}(0)$. Therefore, $\bar{x} \in U$ which contradicts (5.1). So, Q_{x_0} is upper semicontinuous at 0.

Next, we prove that $Q_{x_0}(0)$ is compact. Indeed, for every sequence $\{x_n\} \subset Q_{x_0}(0)$, we have $F(x_n) \leq^\alpha F(x_0) + k_n$ where $\{k_n\} \subset K$ converges to 0. This means that $\{x_n\}$ is an x_0-minimizing sequence of (P_α). By the L-well-posedness of (P_α) at x_0, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\bar{x} \in S_{\alpha\text{-Min}\,F}$. Therefore, $\bar{x} \in Q_{x_0}(0)$. This leads to the compactness of $Q_{x_0}(0)$.

(ii) Let $\{x_n\} \subset M$ be an x_0-minimizing sequence of (P_α), there exists $\{k_n\} \subset K$ converging to 0 such that $F(x_n) \leq^\alpha F(x_0) + k_n$. Hence, $x_n \in Q_{x_0}(k_n)$. Since Q_{x_0} is upper semicontinuous and compact-valued at 0, there exists a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to some $\bar{x} \in Q_{x_0}(0)$. Combining this with Proposition 4.1(ii), we get $\bar{x} \in S_{\alpha\text{-Min}\,F}$. So, (P_α) is L-well-posed at x_0. □

The next results illustrate relationships between the pointwise L-well-posedness and pointwise B-well-posedness for the problem (P_α).

Theorem 5.3. *Let* $x_0 \in S_{\alpha\text{-}MinF}$ *be given.*

(i) *If* (P_α) *is* L*-well-posed at* x_0 *and* $Q_{x_0}(0) = S_{\alpha\text{-}MinF}$*, then it is* B*-well-posed at* x_0.

(ii) *If* (P_α) *is* B*-well-posed at* x_0 *and* $S_{\alpha\text{-}MinF}$ *is compact, then it is* L*-well-posed at* x_0.

Proof. (i) By contradiction, suppose that (P_α) is not B-well-posed at x_0. We get that Q_{x_0} is not upper semicontinuous at $k = 0$. Hence, there exist a neighborhood V of $Q_{x_0}(0)$ and $\{k_n\} \subset K$ converging to 0 such that for each $n \in \mathbb{N}$, there exists $x_n \in Q_{x_0}(k_n) \setminus V$. By definition of Q_{x_0}, we have $F(x_n) \leq^\alpha F(x_0) + k_n$, i.e., $\{x_n\}$ is an x_0-minimizing sequence of (P_α). It follows from the L-well-posedness at x_0

of (P_α) that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\bar{x} \in S_{\alpha\text{-MinF}}$. Therefore, $\bar{x} \in Q_{x_0}(0)$, and hence we now get

$$\bar{x} \in Q_{x_0}(0) \subset V. \tag{5.3}$$

On the other hand, since $x_n \notin V$, $x_n \in X \setminus V$. By the closedness of $X \setminus V$, we have $\bar{x} \in X \setminus V$ which contradicts (5.3). So, (P_α) is B-well-posed at x_0.

(ii) Suppose that (P_α) is B-well-posed at x_0. Let $\{x_n\}$ be an x_0-minimizing sequence of (P_α), we consider two cases as follows:

Case 1: $\{x_n\}$ has infinite elements which belong to $S_{\alpha\text{-MinF}}$. Since $S_{\alpha\text{-MinF}}$ is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some $\bar{x} \in S_{\alpha\text{-MinF}}$. Hence, (P_α) is L-well-posed at x_0.

Case 2: $\{x_n\}$ has infinite elements which do not belong to $S_{\alpha\text{-MinF}}$. Without lost of generality, we can assume that $\{x_n\} \subset M \setminus S_{\alpha\text{-MinF}}$. By Theorem 4.4, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some $\bar{x} \in S_{\alpha\text{-MinF}}$. Therefore, (P_α) is L-well-posed at x_0. □

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References

[1] D. Kuroiwa, T. Tanaka, T.X.D. Ha, On cone convexity of set-valued maps, Nonlinear Anal. 30 (1997) 1487–1496.

[2] J. Jahn, T.X.D. Ha, New order relations in set optimization, J. Optim. Theory Appl. 148 (2011) 209–236.

[3] D. Kuroiwa, Some duality theorems of set-valued optimization with natural criteria, In: Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis, World Scientific, River Edge, NJ (1999) 221–228.

[4] A.A. Khan, C. Tammer, C. Zălinescu, Set-valued Optimization, Springer, Berlin, 2016.

[5] D. Kuroiwa, On set-valued optimization, Nonlinear Anal. 47 (2001) 1395–1400.

[6] M. Alonso, L. Rodríguez-Marín, Set-relations and optimality conditions in set-valued maps, Nonlinear Anal. 63 (2005) 1167–1179.

[7] T.X.D. Ha, Some variants of the Ekeland variational principle for a set-valued map, J. Optim. Theory Appl. 124 (2005) 187–206.

[8] Y. Han, N.J. Huang, Well-posedness and stability of solutions for set optimization problems, Optimization 66 (2017) 17–33.

[9] E. Hernández, L. Rodríguez-Marín, Existence theorems for set optimization problems, Nonlinear Anal. 67 (2007) 1276–1736.

[10] E. Hernández, L. Rodríguez-Marín, Nonconvex scalarization in set optimization with set-valed maps, J. Math. Anal. Appl. 325 (2007) 1–18.

[11] E. Hernández, L. Rodríguez-Marín, Lagrangian duality in set-valued optimization, J. Optim. Theory Appl. 134 (2007) 119–134.

[12] A.N. Tikhonov, On the stability of the functional optimization problem, USSR Comput. Math. Math. Phys. 6 (1966) 28–33.

[13] E.S. Levitin, B.T. Polyak, Convergence of minimizing sequences in conditional extremum problem, Soviet Math. Dokl. 7 (1966) 764–767.

[14] A.L. Dontchev, T. Zolezzi, Well-posed Optimization Problems, Lecture Notes in Mathematics, Springer, Berlin, 1993.

[15] R. Lucchetti, J. Revalski (eds.), Recent Development in Well-posed Variational Problems, Kluwer Academic Publishers, Dordrecht, 1995.

[16] E.M. Bednarczuck, Well-posedness of vector optimization problem, In: J. Jahn, W. Krabs (eds.), Recent Advances and Historical Development of Vector Optimization Problems, Lecture Notes in Economics and Mathematical Systems, Springer, Berlin (1987), 51–61.

[17] E. Bednarczuck, An approach to well-posedness in vector optimization: consequences to stability and parametric optimization, Control Cybern. 23 (1994) 107–122.

[18] E. Miglierina, E. Molho, M. Rocca, Well-posedness and scalarization in vector optimization, J. Optim. Theory Appl. 126 (2005) 391–409.

[19] Y.P. Fang, R. Hu, N.J. Huang, Extended B-well-posedness and property (H) for set-valued vector optimization with convexity, J. Optim. Theory Appl. 135 (2007) 445–458.

[20] L.Q. Anh, P.Q. Khanh, D.T.M. Van, J.C. Yao, Well-posedness for vector quasiequilibria, Taiwan. J. Math. 13 (2009) 713–737.

[21] R. Wangkeeree, L.Q. Anh, P. Boonman, Well-posedness for general parametric quasi-variational inclusion problems, Optimization 66 (2017) 93–111.

[22] W.Y. Zhang, S.J. Li, K.L. Teo, Well-posedness for set optimization problems, Nonlinear Anal. 71 (2009) 3769–3778.

[23] S. Khoshkhabar-amiranloo and E. Khorram, Scalarization of Levitin-Polyak well-posed set optimization problems, Optimization 66 (2017) 113–127.

[24] X.J. Long, J.W. Peng, Generalized B-well-posedness for set optimization problems, J. Optim. Theory Appl. 157 (2013) 612–623.

[25] X.J. Long, J.W. Peng, Z.Y. Peng, Scalarization and pointwise well-posedness for set optimization problems, J. Global Optim. 62 (2015) 763–773.

[26] M. Dhingra, C.S. Lalitha, Well-setness and scalarization in set optimization, Optim. Lett. 10 (2016) 1657–1667.

[27] J. Jahn, Vector Optimization: Theory, Applications, and Extensions, Springer, Berlin, 2004.

[28] D. Kuroiwa, Natural criteria of set-valued optimization, Manuscript, Shimane University, Japan, 1998.

[29] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, 2009.

[30] L.Q. Anh, P.Q. Khanh, Continuity of solution maps of parametric quasiequilibrium problems, J. Global Optim. 46 (2010) 247–259.

[31] A. Göpfert, A. Riahi, C. Tammer, C. Zălinescu, Variational Methods in Partially Ordered Spaces, Springer, New York, 2003.

[32] K. Kuratowski, Topology, Vol. 2, Academic Press, New York, 1968.

[33] E. Hernández, L. Rodríguez-Marín, Optimality conditions for set-valued maps with set optimization, Nonlinear Anal. 70 (2009) 3057–3064.

[34] X.X. Huang, Extended and strongly extended well-posedness of set-valued optimization problems, Math. Meth. Oper. Res. 53 (2001) 101–116.

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