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Weak and Strong Convergence of Hybrid Subgradient Method for Pseudomonotone Equilibrium Problem and Two Finite Families of Multivalued Nonexpansive Mappings in Hilbert Spaces

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Abstract : In this paper, we first introduce an iterative algorithm for finding a common element of the set of solutions of a class of pseudomonotone equilibrium problems and the set of fixed points of two finite families of multivalued nonexpansive mappings in Hilbert space. Moreover, we prove that the proposed iterative algorithm converges weakly and strongly to a common element of the set of solutions of a class of pseudomonotone equilibrium problems and the set of fixed points of two finite families of multivalued nonexpansive mappings under some suitable conditions.

Keywords : pseudomonotone; equilibrium problem; multivalued nonexpansive mapping; hybrid subgradient method; fixed point; weak and strong convergence

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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

An equilibrium problem in the sense of Blum and Oettli [1] is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

Problem of the form (1.1) on one hand covers many important problems in optimization as well as in nonlinear analysis such as (generalized) variational inequality, nonlinear complementary problem, nonlinear optimization problem, just to name a few. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences.

Recall that a mapping $S : K \to K$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in K.$$

A subset $K \subset H$ is called proximal if for each $x \in H$, there exists an element $y \in K$ such that

$$\text{dist}(x, K) := \|x - y\| = \inf\{\|x - z\| : z \in K\}.$$

We denote by $B(K), C(K)$, and $P(K)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets and nonempty proximal bounded subsets of K, respectively. The Hausdorff metric H on $B(H)$ is defined by

$$H(K_1, K_2) := \max\big\{\sup_{x \in K_1} \text{dist}(x, K_2), \sup_{y \in K_2} \text{dist}(y, K_1)\big\}, \quad \forall K_1, K_2 \in B(H).$$

Let $S : H \to 2^H$ be a multivalued mapping, of which the set of fixed points is denoted by $\text{Fix}(S)$, *i.e.*, $\text{Fix}(S) := \{x \in Sx : x \in K\}$. A multivalued mapping $S : K \to B(K)$ is said to be nonexpansive if

$$H(Sx, Sy) \leq \|x - y\|, \quad \forall x, y \in K. \tag{1.2}$$

Existence theorem for fixed point of multivalued contractions and nonexpansive mappings using the Hausdorff metric have been proved by several authors; see,*e.g.*, [2, 3]. Later, an interesting and rich fixed point theory for such maps and more general maps was developed which has applications incontrol theory, convex optimization, differential inclusion, and economics.

Lately, the problem of finding a common element of the set of solutions of equilibrium problems and the set of fixed points of nonlinear mappings has become an attractive subject, and various methods have been extensively examined by many authors; see,*e.g.*, [4, 5, 6, 7, 8, 9, 10]. It is noteworthy to mention that almost all the existing algorithms for this problem are based on the proximal point method applied to the equilibrium problem combining with a Mann iteration to fixed point problems of nonexpansive mappings, of which the convergence analysis has been considered if the bifunction F is monotone. The reason is that the proximal point method is not valid when the underlying operator F is pseudomonotone. Another basic idea for solving equilibrium problems is the projection method; see, *e.g.*, [11, 12]. Nevertheless, Facchinei and Pang [13] present that the projection method, in general, is not convergent for monotone variational inequality, which is a special case of monotone equilibrium problems.

In 2008, Tran *etal.* [14] introduced an extragradient method for pseudomonotone equilibrium problems, which is computationally expensive because of the two projections defined onto the constrained set. Efforts for deducing the computational costs in computing the projection have been made by using penalty function methods or relaxing the constrained convex set by polyhedral convex ones; see, *e.g.*, [15, 16, 17, 18, 19].

In 2011, Santos and Scheimberg [19] further proposed an inexact subgradient algorithm for solving a wide class of equilibrium problems that requires only one projection rather than two as in the extragradient method, and of which computational results show the efficiency of this algorithm in finite dimensional Euclidean spaces. On the other hand, iterative schemes for multivalued nonexpansive mappings are far less developed than those for nonexpansive mappings though they have more powerful applications in solving optimization problems; see. *e.g.*, [20, 21, 22] and the references therein.

In 2014, Wen [23] introduce a hybrid subgradient method for finding a common element of the set of solutions of a class of pseudomonotone equilibrium problems and the set of fixed points of a finite family of multivalued nonexpansive mappings in Hilbert space. He proposed the following iterative method:

$$\begin{cases} w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_K(x_n - \gamma_n w_n), \quad \gamma_n = \frac{\beta_n}{\max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)z_n, \quad n \geq 0, \end{cases} \tag{1.3}$$

where $\partial_\varepsilon F(x,\cdot)(x)$ stands for ε- subdifferential of the convex function $F(x,\cdot)$ at x, $T_n = T_{n(\text{mod}N)}, z_n \in T_n u_n$ such that T_i is a finite family of multivalued nonexpansive mappings for $i = 1, 2, \dots N$, and $\{\alpha_n\}, \{\beta_n\}$, and $\{\epsilon_n\}$ are nonegative sequences satisfying the suitable conditions. Moreover, he futher proved the weak and strong convergence theorems of the iterative sequences under the condition of pseudomonotone definded on a bifunction F.

Motivated by the work of D-J Wen [23], in this paper, we first introduce an iterative algorithm for finding a common element of the set of solutions of a class of pseudomonotone equilibrium problems and the set of fixed points of two finite families of multivalued nonexpansive mappings in Hilbert space. Moreover, we prove that the proposed iterative algorithm converges weakly and strongly to a common element of the set of solutions of a class of pseudomonotone equilibrium problems and the set of fixed points of two finite families of multivalued nonexpansive mappings under some suitable conditions.

2 Preliminaries

Let H be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$ and let K be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in K, denoted by $P_K(x)$, such that

$$\|x - P_K(x)\| \leq \|x - y\|, \ \ \forall y \in K.$$

P_K is called the metric projection of H onto K. In addition, $P_K(x)$ is characterized by the following properties: $P_K(x) \in K$ and

$$\langle x - P_K(x), y - P_K(x)\rangle \geq 0, \tag{2.1}$$

$$\|x - y\|^2 \geq \|x - P_K(x)\|^2 + \|y - P_K(x)\|^2, \ \ \forall x \in H, y \in K. \tag{2.2}$$

Recall that a bifunction $F : K \times K \to \mathbb{R}$ is said to be

(i) monotone on K if

$$F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in K;$$

(ii) pseudomonotone on K with respect to $x \in K$ if

$$F(x, y) \geq 0 \quad \Rightarrow \quad F(y, x) \leq 0, \quad \forall y \in K. \tag{2.3}$$

It is clear that (i)$\Rightarrow$(ii), for every $x \in K$. Moreover, F is said to be pseudomonotone on K with respect to $A \subseteq K$, if it is pseudomonotone on K with respect to every $x \in A$. When $A \equiv K, F$ is called pseudomonotone on K.

To study the equilibrium problem (1.1), we may assume that Δ is an open convex set containing K and the bifunction $F : \Delta \times \Delta \to \mathbb{R}$ satisfy the following assumptions:

(C1) $F(x, x) = 0$ for each $x \in K$ and $F(x,\cdot)$ is convex and lower semicontinuous on K;

(C2) $F(\cdot, y)$ weakly upper semicontinuous for each $y \in K$ on the open set Δ;

(C3) F pseudomonotone on K with respect to $\mathrm{EP}(F, K)$ and satises the strict paramonotonicity property,i.e., $F(y, x) = 0$ for $x \in \mathrm{EP}(F, K)$ and $y \in K$ implies $y \in \mathrm{EP}(F, K)$;

(C4) if $\{x_n\} \subseteq K$ is bounded and $\epsilon_n \to 0$ as $n \to \infty$, then the sequence $\{w_n\}$ with $w_n \in \partial_{\epsilon_n} F(x_n,\cdot)$ is bounded, where $\partial_\epsilon F(x,\cdot)x$ stands for the ϵ-subdifferential of the convex function $F(x,\cdot)$ at x.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. ([27]) *Let H be a real Hilbert space. Then, for all $x, y \in H$ and $\lambda \in \mathbb{R}$, we have*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Lemma 2.2. ([26]) *Let H be a real Hilbert space. Then for all $x, y \in H$,*

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle.$$

Lemma 2.3. ([27]Opial's condition) *Let H be a real Hilbert space. If for each sequence $\{x_n\}$ in H which converges weakly to a point $x \in H$, then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \ \forall y \in H, y \neq x.$$

Lemma 2.4. ([28]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n, \ \ n \geq 0,$$

where $\sum_{n=0}^{\infty} b_n < \infty$. Then the sequence $\{a_n\}$ is convergent.

Lemma 2.5. ([21]) *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to C(K)$ be a multivalued nonexpansive mapping. If $x_n \rightharpoonup q$ and $lim_{n\to\infty}\mathrm{dist}(x_n, Tx_n) = 0$, then $q \in Tq$.*

3 Main Results

In this section, we first introduce the following iterative algorithm.

Algorithm 3.1. *Let K be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $K \times K$ into $\mathbb{R}$. Let $\{S_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ be two finite families of multivalued nonexpansive mappings from K into $C(K)$. For a given $x_0 \in K$, arbitrarily, suppose the sequence $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{w_n\}$ are generated iteratively by*

$$\begin{cases} w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n; \\ u_n = P_K(x_n - \gamma_n w_n), \quad \gamma_n = \frac{\beta_n}{\max\{\sigma_n; \|w_n\|\}}; \\ y_n = \lambda_n x_n + (1-\lambda_n)v_n; \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)z_n, \quad n \geq 0, \end{cases} \tag{3.1}$$

where $S_n = S_{n(\mathrm{mod} N)}, v_n \in S_n u_n,$, $T_n = T_{n(\mathrm{mod} N)}, z_n \in T_n u_n$ $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{\epsilon_n\}$ are nonnegative sequences.

Next, we prove the weak convergence of Algorithm 3.1 is investigated under certain assumptions.

Theorem 3.2. *Let K be a nonempty closed convex subset of a Hilbert space H and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying* (C1)-(C4)*. Let $\{S_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ be two finite families of multivalued nonexpansive mappings from K into $C(K)$ such that $\Omega = \bigcap_{i=1}^N \Big(\mathrm{Fix}(S_i) \cap \mathrm{Fix}(T_i)\Big) \cap \mathrm{EP}(F, K) \neq \phi$ and $S_i(q) = T_i(q) = \{q\}$ for $i = 1, 2, ..., N$ and $q \in \Omega$. Assume that $0 < c < \sigma_n < \sigma, \{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{\epsilon_n\}$ are nonnegative sequences satisfying the following conditions:*

(i) $\alpha_n \in [a,b] \subset (0,1)$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty, \sum_{n=0}^{\infty} \beta_n^2 < \infty$ *and* $\sum_{n=0}^{\infty} \beta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges weakly to* $\bar{x} \in \Omega$.

Proof. We divide the proof into four steps as follows.
Step 1. For every $p \in \Omega$ and every n, we show that

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n \epsilon_n + 2(1-\alpha_n)\beta_n^2,$$

and there exists the limit

$$c := \lim_{n\to\infty} \|x_n - p\|.$$

Let $p \in \Omega$. Then by (3.1) and Lemma 2.1, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\lambda_n x_n + (1-\lambda_n)v_n - p\|^2 \\
&= \|\lambda_n(x_n - p) + (1-\lambda_n)(v_n - p)\|^2 \\
&= \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\|v_n - p\|^2 - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&= \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\mathrm{dist}(v_n, S_n p)^2 - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&\leq \lambda_n\|x_n - p\|^2 + (1-\lambda_n)H(S_n u_n, S_n p)^2 - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&\leq \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\|u_n - p\|^2 - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2. \quad (3.2)
\end{aligned}$$

Using (3.2) and Lemma 2.2, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(y_n - p) + (1-\alpha_n)(z_n - p)\|^2 \\
&= \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|y_n - z_n\|^2 \\
&= \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\mathrm{dist}(z_n, T_n p)^2 - \alpha_n(1-\alpha_n)\|y_n - z_n\|^2 \\
&\leq \alpha_n\|y_n - p\|^2 + (1-\alpha_n)H(T_n u_n, T_n p)^2 - \alpha_n(1-\alpha_n)\|y_n - z_n\|^2 \\
&\leq \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|y_n - z_n\|^2 \\
&\leq \alpha_n\big(\lambda_n\|x_n - p\|^2 + (1-\lambda_n)\|u_n - p\|^2 - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2\big) \\
&\quad +(1-\alpha_n)\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|y_n - z_n\|^2 \\
&= \alpha_n\lambda_n\|x_n - p\|^2 + (1-\alpha_n\lambda_n)\|u_n - p\|^2 - \alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|y_n - z_n\|^2 \\
&= \alpha_n\lambda_n\|x_n - p\|^2 \\
&\quad +(1-\alpha_n\lambda_n)\big(\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\langle p - u_n, x_n - u_n\rangle\big) \\
&\quad -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 - \alpha_n(1-\alpha_n)\|y_n - z_n\|^2 \\
&= \|x_n - p\|^2 + (1-\alpha_n\lambda_n)\big(2\langle x_n - u_n, p - u_n\rangle - \|x_n - u_n\|^2\big) \\
&\quad -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 - \alpha_n(1-\alpha_n)\|y_n - z_n\|^2 \\
&\leq \|x_n - p\|^2 + 2(1-\alpha_n\lambda_n)\langle x_n - u_n, p - u_n\rangle \\
&\quad -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 - \alpha_n(1-\alpha_n)\|y_n - z_n\|^2. \quad (3.3)
\end{aligned}$$

Using $u_n = P_K(x_n - \gamma_n w_n)$ and (2.1), we have

$$\begin{aligned}\langle x_n - u_n, p - u_n\rangle &= \langle x_n - P_K(x_n - \gamma_n w_n), p - u_n\rangle \\ &\leq \gamma_n \langle w_n, p - u_n\rangle. \end{aligned} \tag{3.4}$$

Using $u_n = P_K(x_n - \gamma_n w_n)$ and $x_n \in K$, we obtain

$$\begin{aligned}\|x_n - u_n\|^2 &= \langle x_n - u_n, x_n - u_n\rangle \\ &\leq \gamma_n\langle w_n, x_n - u_n\rangle \\ &\leq \gamma_n\|w_n\|\|x_n - u_n\| \\ &= \frac{\beta_n}{\max\{\sigma_n, \|w_n\|\}}\|w_n\|\|x_n - u_n\| \\ &\leq \beta_n\|x_n - u_n\|, \end{aligned} \tag{3.5}$$

which implies that $\|x_n - u_n\| \leq \beta_n$. By (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq\ & \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n\langle w_n, p - u_n\rangle \\ & -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\ =\ & \|x_n - p\|^2 + 2(1-\alpha_n)\big(\gamma_n\langle w_n, p - x_n\rangle + \gamma_n\langle w_n, x_n - u_n\rangle\big) \\ & -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\ =\ & \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n\langle w_n, p - x_n\rangle + 2(1-\alpha_n)\gamma_n\langle w_n, x_n - u_n\rangle \\ & -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\ \leq\ & \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n\langle w_n, p - x_n\rangle + 2(1-\alpha_n)\gamma_n\|w_n\|\|x_n - u_n\| \\ & -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\ \leq\ & \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n\langle w_n, p - x_n\rangle + 2(1-\alpha_n)\beta_n^2 \\ & -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \end{aligned} \tag{3.6}$$

Since $w_n \in \partial_{\epsilon_n} F(x_n, \cdot)(x_n)$ and $F(x, x) = 0$ for all $x \in K$, so we have

$$\begin{aligned}\langle w_n, p - x_n\rangle &\leq F(x_n, p) - F(x_n, x_n) + \epsilon_n \\ &\leq F(x_n, p) + \epsilon_n. \end{aligned} \tag{3.7}$$

On the other hand, since $p \in \mathrm{EP}(F, K), i.e.,$ $F(p, x) \geq 0$ for all $x \in K$, by the pseudomonotonicity of F with respect to p, we have $F(x, p) \leq 0$ for all $x \in K$. Replacing x by $x_n \in K$, we get $F(x_n, p) \leq 0$. Then from (3.6) and (3.7), it follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq\ & \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n\langle w_n, p - x_n\rangle + 2(1-\alpha_n)\beta_n^2 \\ & -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\ \leq\ & \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n(F(x_n, p) + \epsilon_n) + 2(1-\alpha_n)\beta_n^2 \\ & -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\ =\ & \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n F(x_n, p) + 2(1-\alpha_n)\gamma_n\epsilon_n + 2(1-\alpha_n)\beta_n^2 \\ & -\alpha_n\lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\ \leq\ & \|x_n - p\|^2 + 2(1-\alpha_n)\gamma_n\epsilon_n + 2(1-\alpha_n)\beta_n^2 \end{aligned} \tag{3.8}$$

Appply Lemma 2.4 to (3.8), we get the existence of

$$c := \lim_{n\to\infty} \|x_n - p\|.$$

Step 2. For every $p \in \Omega$, we show that $\limsup_{n\to\infty} F(x_n, p) = 0$.
Let $p \in \Omega$. Since F is pseudomonotone on K and $F(p, x_n) \geq 0$, we have $-F(x_n, p) \geq 0$. From (3.8), we have

$$\begin{aligned}
2(1-\alpha_n)\gamma_n\big[-F(x_n,p)\big] &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1-\alpha_n)\gamma_n\epsilon_n + 2(1-\alpha_n)\beta_n^2 \\
&= \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\alpha_n)\big(2\gamma_n\epsilon_n + 2\beta_n^2\big) \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\gamma_n\epsilon_n + 2\beta_n^2. \qquad (3.9)
\end{aligned}$$

Summing up (3.9) for every n, we obtain

$$\begin{aligned}
0 &\leq 2\sum_{n=0}^{\infty}(1-\alpha_n)\gamma_n\big[-F(x_n,p)\big] \\
&\leq \sum_{n=0}^{\infty}\|x_n - p\|^2 - \sum_{n=0}^{\infty}\|x_{n+1} - p\|^2 + 2\sum_{n=0}^{\infty}\gamma_n\epsilon_n + 2\sum_{n=0}^{\infty}\beta_n^2 \\
&= \|x_0 - p\|^2 + 2\sum_{n=0}^{\infty}\gamma_n\epsilon_n + 2\sum_{n=0}^{\infty}\beta_n^2 < +\infty \qquad (3.10)
\end{aligned}$$

By the assumption (C4), we can find a real number w such that $\|w_n\| \leq w$ for every n. Setting $L := \max\{\sigma, w\}$, where σ is a real number such that $0 < \sigma_n < \sigma$ for every n, it follows from (i) that

$$\begin{aligned}
0 &\leq 2\sum_{n=0}^{\infty}(1-\alpha_n)\gamma_n\big[-F(x_n,p)\big] \\
&\leq 2(1-b)\frac{\beta_n}{\max\{\sigma, w\}}\sum_{n=0}^{\infty}\big[-F(x_n,p)\big] \\
&= \frac{2(1-b)}{L}\sum_{n=0}^{\infty}\beta_n\big[-F(x_n,p)\big] \\
&\leq 2\sum_{n=0}^{\infty}(1-\alpha_n)\gamma_n\big[-F(x_n,p)\big] < +\infty,
\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty}\beta_n\big[-F(x_n,p)\big] < +\infty. \qquad (3.11)$$

Combining with $-F(x_n, p) \geq 0$ and $\sum_{n=0}^{\infty}\beta_n = \infty$, we can deduced that

$$\limsup_{n\to\infty} F(x_n, p) = 0.$$

Step 3. We show that any weak subsequential limit of the sequence of $\{x_n\}$ is an element of Ω. To do this, suppose that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$. For simplicity of notation, without loss of generality, we may assume that $x_{n_i} \rightharpoonup \bar{x}$ as $i \to \infty$. By convexity, K is weakly closed and hence $\bar{x} \in K$. Since $F(\cdot, p)$ is weakly upper semicontinuous for $p \in \Omega$, we have

$$F(\bar{x}, p) \geq \lim_{i\to\infty} \sup F(x_{n_i}, p) = \lim_{i\to\infty} F(x_{n_i}, p)$$
$$\lim_{n\to\infty} \sup F(x_n, p) = 0. \tag{3.12}$$

By the pseudomonotonicity of F with respect to p and $F(p, \bar{x}) \geq 0$, we obtain $F(\bar{x}, p) \leq 0$. Thus $F(\bar{x}, p) = 0$ Moreover, by assumption (C3), we can deduce that $\bar{x}$ is a solution of $\mathrm{EP}(F, K)$. On the other hand, it follows from (3.5) and condition (ii) that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.13}$$

From (3.8) and conditions (i)-(ii), we have

$$\alpha_n \lambda_n (1 - \lambda_n) \|x_n - v_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1 - \alpha_n)\gamma_n \epsilon_n + 2(1 - \alpha_n)\beta_n^2,$$

taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \|x_n - v_n\| = 0, \tag{3.14}$$

and thus

$$\lim_{n\to\infty} \mathrm{dist}(x_n, S_n u_n) \leq \lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{3.15}$$

Again from (3.3), (3.4), (3.5) and conditions (i)-(ii), we have

$$\alpha_n (1 - \lambda_n) \|y_n - z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1 - \alpha_n)\gamma_n \epsilon_n + 2(1 - \alpha_n)\beta_n^2,$$

taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \|y_n - z_n\| = 0, \tag{3.16}$$

and thus

$$\lim_{n\to\infty} \mathrm{dist}(y_n, S_n u_n) \leq \lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.17}$$

It follows that

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} (1 - \lambda_n) \|v_n - x_n)\| = 0. \tag{3.18}$$

Notice that

$$\|z_n - x_n\| \leq \|z_n - y_n\| + \|y_n - x_n\|. \tag{3.19}$$

Combining and (3.18), we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.20}$$

Using (3.1) again, we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n y_n + (1-\alpha_n)z_n - x_n\| \\ &= \|\alpha_n(y_n - z_n) + (z_n - x_n)\| \\ &\leq \alpha_n\|y_n - z_n\| + \|x_n - z_n\| \\ &\leq \alpha_n\|y_n - z_n\| + \|x_n - y_n\| + \|y_n - z_n\|. \end{aligned} \tag{3.21}$$

From (3.21), we have

$$\|x_{n+1} - x_n\| \leq \alpha_n\|y_n - z_n\| + \|x_n - y_n\| + \|y_n - z_n\|, \tag{3.22}$$

taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.23}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+i} - x_n\| = 0, \quad i = 1, 2, \ldots, N. \tag{3.24}$$

Notice that

$$\|u_{n+1} - u_n\| \leq \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - u_n\|.$$

Combining (3.13) and (3.23) we obtain

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.25}$$

This implies that

$$\lim_{n\to\infty}\|u_{n+i} - u_n\| = 0, \quad i = 1, 2, \ldots, N. \tag{3.26}$$

Notice that

$$\begin{aligned}\mathrm{dist}(u_n, S_{n+i}u_n) &\leq \|u_n - x_n\| + \|x_n - x_{n+i}\| + \mathrm{dist}(x_{n+i}, S_{n+i}u_{n+i}) \\ &\quad + H(S_{n+i}u_{n+i}, S_{n+i}u_n) \\ &\leq \|u_n - x_n\| + \|x_n - x_{n+i}\| + \mathrm{dist}(x_{n+i}, S_{n+i}u_{n+i}) \\ &\quad + \|u_{n+i} - u_n\|. \end{aligned}$$

Together with (3.13), (3.15), (3.24) and (3.26), we have

$$\lim_{n\to\infty}\mathrm{dist}(u_n, S_{n+i}u_n) = 0, \quad i = 1, 2, \ldots, N, \tag{3.27}$$

which implies that the sequence

$$\bigcup_{i=0}^{N}\{\mathrm{dist}(u_n, S_{n+i}u_n)\}_{n\geq 0} \to 0 \quad \text{as} \quad n\to\infty. \tag{3.28}$$

For $i=1,2,\ldots,N$, we note that

$$\begin{aligned}\{\mathrm{dist}(u_n, S_i u_n)\}_{n\geq 0} &= \{\mathrm{dist}(u_n, S_{n+(i-n)}u_n)\}_{n\geq 0}\\ &= \{\mathrm{dist}(u_n, S_{n+i_n}u_n)\}_{n\geq 0}\\ &\subset \bigcup_{i=0}^{N}\{\mathrm{dist}(u_n, S_{n+i}u_n)\}_{n\geq 0},\end{aligned}$$

where $i-n=i_n(\mathrm{mod}N)$ and $i_n\in\{1,2,\ldots,N\}$. Thus, we have

$$\lim_{n\to\infty}\mathrm{dist}(u_n, S_i u_n)=0, \quad i=1,2,\ldots,N. \tag{3.29}$$

Similarly, for $i=1,2,\ldots,N$, we obtain

$$\begin{aligned}\mathrm{dist}(x_n, S_i x_n) &\leq \|x_n-u_n\| + \mathrm{dist}(u_n, S_i u_n) + H(S_i u_n, S_i x_n)\\ &\leq 2\|x_n-u_n\| + \mathrm{dist}(u_n, S_i u_n).\end{aligned}$$

It follows from (3.13) and (3.29) that

$$\lim_{n\to\infty}\mathrm{dist}(x_n, S_i x_n)=0, \quad i=1,2,\ldots,N. \tag{3.30}$$

Applying Lemma 2.5 to (3.30), we can deduce that $\bar{x}\in\mathrm{Fix}(S_i)$ for $i=1,2,\ldots,N$. In a Similar way, we can show that $\bar{x}\in\mathrm{Fix}(T_i)$ for $i=1,2,\ldots,N$. So, we get $\bar{x}\in\Omega$.

Step 4. Finally, we prove that $\{x_n\}$ converges weakly to an element of Ω. Indeed to verify that the claim is valid it is sufficient to show that $\omega_w(x_n)$ is a single point set, where $\omega_w(x_n)=\{x\in H : x_{n_i}\rightharpoonup x\}$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Indeed since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Taking $p_1,p_2\in\omega_w(x_n)$ arbitrarily, let $\{x_{n_k}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_k}\rightharpoonup p_1$ and $x_{n_j}\rightharpoonup p_2$ respectively. Since $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in\Omega$ and $p_1,p_2\in\Omega$, so we get $\lim_{n\to\infty}\|x_n-p_1\|$ and $\lim_{n\to\infty}\|x_n-p_2\|$ exist. Now let $p_1\neq p_2$, then by Opial's condition this yields

$$\begin{aligned}\lim_{n\to\infty}\|x_n-p_1\| &= \lim_{k\to\infty}\|x_{n_k}-p_1\|\\ &< \lim_{k\to\infty}\|x_{n_k}-p_2\|\\ &= \lim_{n\to\infty}\|x_n-p_2\|\\ &= \lim_{j\to\infty}\|x_{n_j}-p_2\|\\ &< \lim_{n\to\infty}\|x_{n-j}-p_1\|\\ &= \lim_{n\to\infty}\|x_n-p_1\|\end{aligned}$$

which is a contradiction. Thus, $p_1 = p_2$. This shows that $\omega_w(x_n)$ is a single point set, *i.e.*, $x_n \rightharpoonup \bar{x}$. This completes the proof. □

Putting $\lambda_n = 1$ for all $n \in \mathbb{N}$ and $S_i = I$ which is the identity operator in Theorem 3.2 for $i = 1, 2, \dots, N$, we obtain the following results.

Corollary 3.3. (Dao-Jun, Wen [23, Theorem 3.1]) *Let K be a nonempty closed convex subset of a Hilbert space H and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying* (C1)-(C4). *Let $\{T_i\}_{i=1}^N$ be a finite family of multivalued nonexpansive mappings from K into $C(K)$ such that $\Omega = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \cap \mathrm{EP}(F, K) \neq \phi$ and $T_i(q) = \{q\}$ for $i = 1, 2, \dots, N$ and $q \in \Omega$. For a given point $x_0 \in K, 0 < c < \sigma_n < \sigma$, let $\{x_n\}$ be defined by*

$$\begin{cases} w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_K(x_n - \gamma_n w_n), \quad \gamma_n = \frac{\beta_n}{\max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \quad n \geq 0, \end{cases} \tag{3.31}$$

where $T_n = T_{n(\mathrm{mod} N)}, z_n \in T_n u_n$, $\{\alpha_n\}, \{\beta_n\}$, and $\{\epsilon_n\}$ are nonegative sequences satisfying the following conditions:

(1) $\alpha_n \in [a, b] \subset (0, 1)$ *;*

(2) $\sum_{n=0}^{\infty} \beta_n = \infty, \sum_{n=0}^{\infty} \beta_n^2 < \infty$ *and* $\sum_{n=0}^{\infty} \beta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ generated by (3.31) *converges weakly to $\bar{x} \in \Omega$.*

Corollary 3.4. *Let K be a nonempty closed convex subset of a Hilbert space H and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying* (C1)-(C4). *Let S and T be two multivalued nonexpansive mappings from K into $C(K)$ such that $\Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \cap \mathrm{EP}(F, K) \neq \phi$ and $S(q) = T(q) = \{q\}$ for all $q \in \Omega$. Assume that $0 < c < \sigma_n < \sigma, \{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{\epsilon_n\}$ are nonnegative sequences satisfying the following conditions:*

(i) $\alpha_n \in [a, b] \subset (0, 1)$ *;*

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty, \sum_{n=0}^{\infty} \beta_n^2 < \infty$ *and* $\sum_{n=0}^{\infty} \beta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ generated by (3.1) *converges weakly to $\bar{x} \in \Omega$.*

Proof. Putting $N = 1$ in Theorem 3.2. □

Next, we prove the strong convergence of proposed algorithms is investigated under certain assumptions.

Theorem 3.5. *Let K be a nonempty closed convex subset of a Hilbert space H and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying* (C1)-(C4). *Let $\{S_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ be two finite families of multivalued nonexpansive mappings from K into $C(K)$ such that $\Omega = \bigcap_{i=1}^N \Big(\mathrm{Fix}(S_i) \cap \mathrm{Fix}(T_i)\Big) \cap \mathrm{EP}(F, K) \neq \phi$ and $S_i(q) = T_i(q) = \{q\}$ for $i = 1, 2, \dots, N$ and $q \in \Omega$. Assume that $0 < c < \sigma_n < \sigma, \{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{\epsilon_n\}$ are nonnegative sequences satisfying the following conditions:*

(i) $\alpha_n \in [a,b] \subset (0,1)$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty, \sum_{n=0}^{\infty} \beta_n^2 < \infty$ *and* $\sum_{n=0}^{\infty} \beta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $\bar{x} \in \Omega$.

Proof. By a similar argument to the proof of Theorem 3.2 and (2.2), we have

$$\begin{aligned}\|y_n - P_\Omega(x_n)\|^2 &\le \|y_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2 \\ &\le \|y_n - x_n\|^2 \end{aligned} \tag{3.32}$$

and

$$\|z_n - P_\Omega(x_n)\|^2 \le \|z_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2. \tag{3.33}$$

It follows from (3.32) and (3.33) that

$$\begin{aligned}\|x_{n+1} - P_\Omega(x_{n+1})\|^2 &= \|\alpha_n y_n + (1-\alpha_n) z_n - P_\Omega(x_n))\|^2 \\ &\le \|\alpha_n (y_n - P_\Omega(x_n)) + (1-\alpha_n)(z_n - P_\Omega(x_n))\|^2 \\ &\le \alpha_n \|y_n - P_\Omega(x_n)\|^2 + (1-\alpha_n)\|z_n - P_\Omega(x_n)\|^2 \\ &\le \alpha_n \|y_n - x_n\|^2 \\ &\quad + (1-\alpha_n)\big(\|z_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2\big) \\ &\le \alpha_n \|y_n - x_n\|^2 + (1-\alpha_n)\|z_n - x_n\|^2 \\ &\quad -(1-\alpha_n)\|x_n - P_\Omega(x_n)\|^2. \end{aligned} \tag{3.34}$$

Combining (3.18), (3.20) and the boundedness of the sequence $\{x_n - P_\Omega(x_n)\}$, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - P_\Omega(x_{n+1})\| = 0. \tag{3.35}$$

By the assumptions (C1) and (C2), the set Ω is convex. For the simplicity of notation, let $s_n := P_\Omega(x_n)$ for each $n \ge 1$. Then, for all $m > n$ we have

$$\frac{1}{2}(s_m + s_n) \in \Omega,$$

and thus

$$\begin{aligned}\|s_m - s_n\|^2 &= 2\|x_m - s_m\|^2 + 2\|x_m - s_n\|^2 - 4\big\|x_m - \frac{1}{2}(s_m + s_n)\big\|^2 \\ &\le 2\|x_m - s_m\|^2 + 2\|x_m - s_n\|^2 - 4\|x_m - s_m\|^2 \\ &= 2\|x_m - s_n\|^2 - 2\|x_m - s_m\|^2. \end{aligned} \tag{3.36}$$

Using (3.8) with $p = s_n$, we have

$$\begin{aligned}\|x_m - s_n\|^2 &\le \|x_{m-1} - s_n\|^2 + 2(1-\alpha_{m-1})\gamma_{m-1}\epsilon_{m-1} + 2(1-\alpha_{m-1})\beta_{m-1}^2 \\ &\le \|x_{m-2} - s_n\|^2 + \xi_{m-1} + \xi_{m-2} \\ &\le \cdots \\ &\le \|x_n - s_n\|^2 + \sum_{j=m}^{m-1} \xi_j, \end{aligned} \tag{3.37}$$

where $\xi_j = 2(1-\alpha_j)\gamma_j\epsilon_j + 2(1-\alpha_j)\beta_j^2$. It follows from (3.36) and (3.36) that

$$\|s_m - s_n\|^2 \leq 2\|x_n - s_n\|^2 + 2\sum_{j=n}^{m-1}\xi_j - 2\|x_m - s_m\|^2. \tag{3.38}$$

Together with (3.35) and $\sum_{j=n}^{m-1}\xi_j < \infty$ this implies that $\{s_n\}$ is a Cauchy sequence. Hence $\{s_n\}$ strongly converges to some point $x^* \in \Omega$. However, since $s_{n_i} := P_\Omega(x_{n_i})$, letting $i \to \infty$, we obtain in the limit that

$$x^* = \lim_{i\to\infty} P_\Omega(x_{n_i}) = P_\Omega(\bar{x}) = \bar{x}, \tag{3.39}$$

which implies that $P_\Omega(x_n) \to x^* = \bar{x} \in \Omega$. Then, from (3.30), (3.35) and (3.39) we can conclude that $x_n \to x^*$. This completes the proof. □

Putting $\lambda_n = 1$ and $S_i = I$ which is the identity operator for $i = 1, 2, \ldots, N$ and $\lim_{n\to\infty}\alpha_n = \dfrac{1}{2}$ in Theorem 3.5, we obtain the following results.

Corollary 3.6. (Dao-Jun, Wen [23, Theorem 4.1]) *Let K be a nonempty closed convex subset of a Hilbert space H and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying* (C1)-(C4). *Let $\{T_i\}_{i=1}^N$ be a finite family of multivalued nonexpansive mappings from K into $C(K)$ such that $\Omega = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \cap \mathrm{EP}(F,K) \neq \phi$ and $T_i(q) = \{q\}$ for $i = 1, 2, \ldots, N$ and $q \in \Omega$. For a given point $x_0 \in K, 0 < c < \sigma_n < \sigma$, let $\{x_n\}$ be defined by*

$$\begin{cases} w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_K(x_n - \gamma_n w_n), \quad \gamma_n = \frac{\beta_n}{\max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)z_n, \quad n \geq 0, \end{cases} \tag{3.40}$$

where $T_n = T_{n(\mathrm{mod} N)}, z_n \in T_n u_n$, $\{\alpha_n\}, \{\beta_n\}$, and $\{\epsilon_n\}$ are nonegative real sequences satisfying the following conditions:

(1) $\alpha_n \in [a,b] \subset (0,1)$ *and* $\lim_{n\to\infty}\alpha_n = \dfrac{1}{2}$*;*

(2) $\sum_{n=0}^\infty \beta_n = \infty, \sum_{n=0}^\infty \beta_n^2 < \infty$ *and* $\sum_{n=0}^\infty \beta_n\epsilon_n < \infty$.

Then the sequence $\{x_n\}$ generated by (3.40) *converges strongly to $x^* \in \Omega$.*

Theorem 3.7. *Let K be a nonempty closed convex subset of a Hilbert space H and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying* (C1)-(C4). *Let $\{S_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ be two finite families of multivalued nonexpansive mappings from K into $C(K)$ such that $P_{S_i} := \{y \in S_i x : \mathrm{dist}(x, S_i x) = \|x-y\|\}$, $P_{T_i} := \{y \in T_i x : \mathrm{dist}(x, T_i x) = \|x-y\|\}$ and $\Omega = \bigcap_{i=1}^N \Big(\mathrm{Fix}(S_i) \cap \mathrm{Fix}(T_i)\Big) \cap \mathrm{EP}(F,K) \neq \phi$. Assume that $0 < c < \sigma_n < \sigma, \{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{\epsilon_n\}$ are nonnegative sequences satisfying the following conditions:*

(i) $\alpha_n \in [a,b] \subset (0,1)$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty, \sum_{n=0}^{\infty} \beta_n^2 < \infty$ *and* $\sum_{n=0}^{\infty} \beta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $\bar{x} \in \Omega$.

Proof. Taking $p \in \Omega$, then $P_{S_n}(p) = P_{T_n}(p) = \{p\}$. By substituting P_S instead of S and similar argument as (3.30) in the proof of Theorem 3.2 we obtain

$$\lim_{n\to\infty} \mathrm{dist}(x_n, S_i(x_n)) \le \lim_{n\to\infty} \mathrm{dist}(x_n, P_{S_i}(x_n)) = 0. \tag{3.41}$$

By compactness of K there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x^*$ for some $x^* \in K$. Since P_{S_i} is nonexpansive for $i = 1, 2, ..., N$, we have

$$\begin{aligned} \mathrm{dist}(x^*, S_i(x^*)) &\le \mathrm{dist}(x^*, P_{S_i}(x^*)) \\ &\le \|x^* - x_{n_k}\| + \mathrm{dist}(x_{n_k}, P_{S_i}(x_{n_k})) + H(P_{S_i}(x_{n_k}), P_{S_i}(x^*)) \\ &\le 2\|x^* - x_{n_k}\| + \mathrm{dist}(x_{n_k}, P_{S_i}(x_{n_k})). \end{aligned} \tag{3.42}$$

It follows from (3.41) and (3.42) that

$$\lim_{k\to\infty} \mathrm{dist}(x^*, S_i(x^*)) = 0, \tag{3.43}$$

which implies that $x^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(S_i)$. In a similar way, we can show that $x^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. Since $\{x_{n_k}\}$ converges strongly to x^* and $\lim_{n\to\infty} \|x_n - x^*\|$ exists (as in the proof of Theorem 3.2), we find that $\{x_n\}$ converges strongly to x^*. This completes the proof. □

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