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Fixed Points of Set-Valued Mappings under Generalized Contractive Conditions in Ordered Metric Spaces

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Abstract : In the present work we establish some new fixed point theorems for a set-valued operator satisfying generalized contractive condition in a partially ordered complete metric space. Thereafter we present common fixed point results for a pair of weakly isotone increasing set-valued mappings. Finally, some examples are given to illustrate the usability of our results, also showing that the use of order can be crucial.

Keywords : common fixed point; Hausdorff distance; set-valued mapping; control function; weakly isotone increasing mappings.
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1 Introduction

Fixed point theory for set-valued mappings was originally initiated by von Neumann in his study on Game Theory. Fixed point theorems for set-valued

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mappings are quite useful in Control Theory and have been frequently used for solving problems in Economics and Game Theory.

The study of fixed points for set-valued contraction mappings was an active topic, as well. The development of geometric fixed point theory for multifunctions was initiated with the work of Nadler [1] in 1969. He used the concept of Hausdorff metric to establish the set-valued contraction principle containing the Banach contraction principle as a special case, as follows.

Theorem 1.1. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T}$ be a mapping from $\mathcal{X}$ into $CB(\mathcal{X})$ such that for all $x, y \in \mathcal{X}$,*

$$H(\mathcal{T}x, \mathcal{T}y) \leq \lambda d(x, y)$$

where, $0 \leq \lambda < 1$. Then $\mathcal{T}$ has a fixed point.

Since then, this discipline has been developed further, and many profound concepts and results have been established with considerable generality; see, for example, the work of Itoh and Takahashi [2], Mizoguchi and Takahashi [3], Rhoades [4], and the references cited therein. Very recently, results on common fixed points for a pair of set-valued operators have been obtained by applying various types of contractive conditions; we refer the reader to [5–13]. Shen and Hong [14] proved fixed point theorems for a pair of set-valued operators which satisfy generalized contractive condition and gave analogy of the results for single-valued operators of Zhang [15].

On the other hand, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [16, Theorem 2.1] who presented its applications to matrix equations. Subsequently, Nieto and Rodríguez-López [17] extended the result of [16] for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed point theorems in ordered metric spaces. Hong [6] proved new hybrid fixed point theorems involving set-valued operators which satisfy weakly generalized contractive conditions in a complete ordered metric space and presented application for hyperbolic differential inclusion. Beg and Butt [18–20] worked on set-valued mappings and proved common fixed point results for mappings satisfying implicit relation in partially ordered metric space. Recently, Choudhury and Metiya [21] also proved fixed point theorems for set-valued mappings in the framework of a partially ordered metric space.

In the present paper an attempt is made, first, to prove fixed point theorems for a set-valued operator which satisfies generalized contractive condition in a partially ordered complete metric space. Secondly, we extend these results for a pair of set-valued operators. We will do this using the concept of weakly isotone increasing mappings introduced by Dhage, O'Regan and Agarwal [22]. Our results are ordered version generalization of the results of Shen and Hong [14]. They generalize Theorem 2.1–Theorem 2.5 of Choudhury and Metiya [21] by considering a more general contractive condition and a pair of set-valued mappings. Finally, some examples are given to illustrate the usability of our results.

2 Preliminaries

For more details on the following definitions, we refer the reader to [14, 21].

Recall that a function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

1. ψ is increasing and continuous,
2. $\psi(t) = 0$ if and only if $t = 0$.

Weaker forms of the above conditions have also been used to establish fixed point results in a number of subsequent works, some of which are noted in [23–25] and the references cited therein.

Definition 2.1. Let $\mathcal{N} \in (0, +\infty]$. Denote by $\mathcal{F}$ the set of functions $f : [0, \mathcal{N}) \to \mathbb{R}$ satisfying:

(i) $f(0) = 0$ and $f(t) > 0$ for each $t \in (0, \mathcal{N})$;

(ii) f is continuous;

(iii) f is nondecreasing on $[0, \mathcal{N})$;

(iv) $f(t_1 + t_2) \leq f(t_1) + f(t_2)$, whenever $t_1, t_2, t_1 + t_2 \in (0, \mathcal{N})$.

Definition 2.2. Let $\mathcal{N} \in (0, +\infty]$. Denote by Ψ the set of functions $\psi : [0, \mathcal{N}) \to [0, +\infty)$ satisfying:

(i) $\psi(t) < t$ for each $t \in (0, \mathcal{N})$;

(ii) for each $t \in (0, \mathcal{N})$, $\sum_{n=1}^{\infty} \psi^n(t) < \infty$.

If $\mathcal{X}$ is a nonempty set, the set $2^{\mathcal{X}} \setminus \{\emptyset\}$ of nonempty subsets of $\mathcal{X}$ will be denoted by $N(\mathcal{X})$.

Definition 2.3. For arbitrary nonempty subsets $\mathcal{A}, \mathcal{B}$ of a metric space $(\mathcal{X}, d)$, the expression

$$H(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{a \in \mathcal{A}} D(a, \mathcal{B}), \sup_{b \in \mathcal{B}} D(b, \mathcal{A})\right\}$$

is called the *Hausdorff distance* of $\mathcal{A}$ and $\mathcal{B}$, where $D(a, \mathcal{B}) = D(\mathcal{B}, a) = \inf_{b \in \mathcal{B}} d(a, b)$.

Definition 2.4. A point $x^* \in \mathcal{X}$ is called a *fixed point* of a set-valued operator $\mathcal{T} : \mathcal{X} \to N(\mathcal{X})$ if $x^* \in \mathcal{T}x^*$.

Definition 2.5. A set $\mathcal{B} \subset \mathcal{X}$ is said to be an *approximation* if for each given $y \in \mathcal{X}$, there exists $z \in \mathcal{B}$ such that $D(\mathcal{B}, y) = d(z, y)$.

A set-valued operator $\mathcal{T}$ is said to have approximate values in $\mathcal{X}$ if $\mathcal{T}x$ is an approximation for each $x \in \mathcal{X}$.

Throughout this paper we always assume that all set-valued operators have approximate values.

Definition 2.6. Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called an *ordered metric space* if:

(i) $(\mathcal{X}, d)$ is a metric space,

(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

Let $(\mathcal{X}, \preceq)$ be a partially ordered set. Then $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.7 ([6])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a partially ordered set $(\mathcal{X}, \preceq)$. The relation $\preceq_2$ between two nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$ is defined as follows:

$$\mathcal{A} \preceq_2 \mathcal{B}, \text{ if } a \preceq b \text{ for every } a \in \mathcal{A} \text{ and every } b \in \mathcal{B}.$$

3 Results for a Single Set-Valued Mapping

In this section, we prove fixed point theorems for a set-valued mapping in ordered complete metric space. The first result is the following

Theorem 3.1. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space and let $\mathcal{Y} = \sup\{ d(x, y) : x, y \in \mathcal{X} \}$. Set $\mathcal{N} = \mathcal{Y}$ if $\mathcal{Y} = \infty$, and $\mathcal{N} > \mathcal{Y}$ if $\mathcal{Y} < \infty$. Suppose that $\mathcal{T} : \mathcal{X} \to N(\mathcal{X})$ is a set-valued mapping such that the following conditions are satisfied:*

(i) there exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \preceq_2 \mathcal{T}x_0$,

(ii) for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \preceq_2 \mathcal{T}y$,

(iii) $f(H(\mathcal{T}x, \mathcal{T}y)) \leq \psi(f(M(x, y)))$ for all comparable $x, y \in \mathcal{X}$, where $f \in \mathcal{F}$, $\psi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), D(\mathcal{T}x, x), D(\mathcal{T}y, y), \frac{1}{2}(D(\mathcal{T}x, y) + D(\mathcal{T}y, x))\right\}. \tag{3.1}$$

If the condition

$$\begin{cases} \textit{if } \{x_n\} \subset \mathcal{X} \textit{ is an increasing sequence with } x_n \to z \textit{ in } \mathcal{X}, \\ \textit{then } x_n \preceq z \textit{ for all } n \end{cases} \tag{3.2}$$

holds, then $\mathcal{T}$ has a fixed point.

Proof. In view of the property of approximation, for $x_0 \in \mathcal{X}$ (given by (i)), there exists $x_1 \in \mathcal{T}x_0$ such that $D(\mathcal{T}x_0, x_0) = d(x_1, x_0)$. Property (i) implies that $x_0 \preceq x_1$. Now, there exists $x_2 \in \mathcal{T}x_1$ such that $D(\mathcal{T}x_1, x_1) = d(x_2, x_1)$ and, by (ii), $x_1 \preceq x_2$. Continuing this process we construct a nondecreasing sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_{n+1} \in \mathcal{T}x_n$, for all $n \geq 0$,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots,$$

and $D(\mathcal{T}x_n, x_n) = d(x_{n+1}, x_n)$. If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of $\mathcal{T}$. Hence we shall assume that $x_n \neq x_{n+1}$, for all $n \geq 0$.

Using (3.1), we have for all $n \geq 0$,

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), D(\mathcal{T}x_n, x_n), D(\mathcal{T}x_{n+1}, x_{n+1}), \\ &\qquad \tfrac{1}{2}(D(\mathcal{T}x_n, x_{n+1}) + D(\mathcal{T}x_{n+1}, x_n))\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}), \\ &\qquad \tfrac{1}{2}(d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n))\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+2}, x_{n+1}), \tfrac{1}{2}d(x_{n+2}, x_n)\}. \end{aligned}$$

Since $\frac{1}{2}d(x_n, x_{n+2}) \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$, it follows that

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+2}, x_{n+1})\}. \tag{3.3}$$

Suppose that $d(x_{n+2}, x_{n+1}) \geq d(x_n, x_{n+1})$ for some positive integer n. Then from condition (iii), (3.3) and the properties of functions $f \in \mathcal{F}$, $\psi \in \Psi$, we have

$$\begin{aligned} f(d(x_{n+2}, x_{n+1})) &= f(D(\mathcal{T}x_{n+1}, x_{n+1})) \\ &\leq f(H(\mathcal{T}x_n, \mathcal{T}x_{n+1})) \leq \psi(f(M(x_n, x_{n+1}))) \\ &< f(M(x_n, x_{n+1})) = f(d(x_{n+2}, x_{n+1})), \end{aligned}$$

a contradiction. So we have $d(x_{n+2}, x_{n+1}) < d(x_n, x_{n+1})$. This yields

$$\begin{aligned} f(d(x_{n+2}, x_{n+1})) &= f(D(\mathcal{T}x_{n+1}, x_{n+1})) \leq f(H(\mathcal{T}x_n, \mathcal{T}x_{n+1})) \\ &\leq \psi(f(M(x_n, x_{n+1}))) = \psi(f(d(x_{n+1}, x_n))). \end{aligned}$$

Using the obtained inequality several times, we get

$$f(d(x_{n+1}, x_n)) \leq \psi(f(d(x_n, x_{n-1}))) \leq \cdots \leq \psi^n(f(d(x_1, x_0))).$$

Let $m, n \in N$, $n > m$; then in virtue of the triangular inequality, we have

$$d(x_n, x_m) \leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}).$$

This implies, by the properties of $f \in \mathcal{F}$,

$$\begin{aligned} f(d(x_n, x_m)) &\leq f(d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)) \\ &\leq \sum_{i=m}^{n-1} \psi^i(f(d(x_1, x_0))). \end{aligned}$$

Let $m, n \to \infty$; by the above inequality, using the condition of $\sum_{i=m}^{\infty} \psi^n(t) < \infty$ of $\psi \in \Psi$, it follows that $\{x_n\}$ is a Cauchy sequence.

From the completeness of $\mathcal{X}$, there exists a $z \in \mathcal{X}$ such that

$$x_n \longrightarrow z \text{ as } n \longrightarrow \infty.$$

By the assumption (3.2), $x_n \preceq z$, for all n.

Now we prove $D(\mathcal{T}z, z) = 0$. Suppose that this is not true; then $D(\mathcal{T}z, z) > 0$. For large enough n, using (3.1) for $x = z$ and $y = x_{2n+1}$, we claim that

$$\begin{aligned} M(z, x_{2n+1}) &= \max\{d(z, x_{2n+1}), D(\mathcal{T}z, z), D(\mathcal{T}x_{2n+1}, x_{2n+1}), \\ &\qquad \tfrac{1}{2}(D(\mathcal{T}z, x_{2n+1}) + D(\mathcal{T}x_{2n+1}, z))\} \\ &= D(\mathcal{T}z, z). \end{aligned}$$

Indeed, since $\lim_{n\to\infty} d(z, x_{2n+1}) = 0$ and $\lim_{n\to\infty} D(\mathcal{T}x_{2n+1}, x_{2n+1}) = 0$, it follows that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{2}(D(\mathcal{T}z, x_{2n+1}) + D(\mathcal{T}x_{2n+1}, z)) \\ &\leq \lim_{n\to\infty} \frac{1}{2}(D(\mathcal{T}z, z) + d(z, x_{2n+1}) + D(\mathcal{T}x_{2n+1}, x_{2n+1}) + d(x_{2n+1}, z)) \\ &= \tfrac{1}{2} D(\mathcal{T}z, z). \end{aligned}$$

Therefore, there exists n_1 such that $M(z, x_{2n+1}) = D(\mathcal{T}z, z)$ for all $n > n_1$. Note that

$$f(D(\mathcal{T}z, x_{2n+2})) \leq f(H(\mathcal{T}z, \mathcal{T}x_{2n+1})) \leq \psi(f(M(z, x_{2n+1}))).$$

Letting $n \to \infty$ and applying (i) of Definition 2.2, we get

$$f(D(\mathcal{T}z, z)) \leq \psi(f(D(\mathcal{T}z, z))) < f(D(\mathcal{T}z, z)),$$

a contradiction. Hence $D(\mathcal{T}z, z) = 0$; in virtue of the approximation property of $\mathcal{T}z$, we have $z \in \mathcal{T}z$. This completes the proof of the theorem. □

We illustrate Theorem 3.1 by the following example. It also shows that the use of order is crucial.

Example 3.2. Let $\mathcal{X} = \{A, B, C\}$, where $A = (0, 0)$, $B = (1, 1)$, $C = (2, 0) \in \mathbb{R}^2$. Metric d is defined as $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ so that $d(A, B) = 1$, $d(A, C) = 2$ and $d(B, C) = 1$. Order $\preceq$ is introduced by $(x_1, y_1) \preceq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$, so that $A \preceq B$ and $A \preceq C$, while B and C are incomparable.

Consider the mapping $\mathcal{T} : \mathcal{X} \to N(\mathcal{X})$ given by

$$\mathcal{T} = \begin{pmatrix} A & B & C \\ \{A\} & \{A\} & \{A, B\} \end{pmatrix},$$

and functions $f \in \mathcal{F}$, $\psi \in \Psi$ given by $f(t) = \frac{1}{2}t$, $\psi(t) = \frac{2}{3}t$. Conditions (i) and (ii), as well as (3.2) of Theorem 3.1 are satisfied. To prove that (iii) holds, it is enough to check that it is satisfied for $x = A$, $y = B$ and for $x = A$, $y = C$ (in the case when $x = y$, (iii) is trivially satisfied).

If $x = A$, $y = B$, then $\mathcal{T}x = \mathcal{T}y = \{A\}$ and $H(\mathcal{T}x, \mathcal{T}y) = 0$, so (iii) holds. If $x = A$, $y = C$, then

$$H(\mathcal{T}x, \mathcal{T}y) = \max\{D(A, \{A, B\}), \max\{D(A, \{A\}), D(B, \{A\})\}\} = d(A, B) = 1,$$

and

$$\begin{aligned} M(x, y) &= \max\{d(A, C), D(A, \{A\}), D(C, \{A, B\}), \tfrac{1}{2}(D(A, \{A, B\}) + D(C, \{A\}))\} \\ &= \max\{2, 0, 1, \tfrac{1}{2}(0 + 2)\} = 2. \end{aligned}$$

Hence, $f(H(\mathcal{T}x, \mathcal{T}y)) = \frac{1}{2} < \frac{2}{3} = \psi(f(M(x, y))$. All the conditions of Theorem 3.1 are fulfilled and $\mathcal{T}$ has a fixed point A.

Note that for (incomparable) points $x = B$, $y = C$ condition (iii) is not satisfied, and so Theorem 1 of [14] (with $T = S$) cannot be applied to reach the conclusion. Indeed, in this case, $\mathcal{T}x = \{A\}$, $\mathcal{T}y = \{A, B\}$,

$$H(\mathcal{T}x, \mathcal{T}y) = d(A, B) = 1, \quad M(x, y) = \max\{1, 1, 1, \tfrac{1}{2}(0 + 2)\} = 1,$$

and $f(H(\mathcal{T}x, \mathcal{T}y)) = \frac{1}{2} > \frac{1}{3}\psi(f(M(x, y))$.

The following corollary is a special case of Theorem 3.1 when $\mathcal{T}$ is a single-valued mapping.

Corollary 3.3. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space and let $\mathcal{Y} = \sup\{\, d(x, y) : x, y \in \mathcal{X}\,\}$. Set $\mathcal{N} = \mathcal{Y}$ if $\mathcal{Y} = \infty$, and $\mathcal{N} > \mathcal{Y}$ if $\mathcal{Y} < \infty$. Suppose that a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfies the following conditions:*

1. *there exists $x_0 \in \mathcal{X}$ such that $x_0 \preceq \mathcal{T}x_0$,*
2. *for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$,*
3. *$f(d(\mathcal{T}x, \mathcal{T}y)) \le \psi(f(M(x, y)))$ for all comparable $x, y \in \mathcal{X}$, where $f \in \mathcal{F}$, $\psi \in \Psi$ and*

$$M(x, y) = \max\left\{d(x, y), d(\mathcal{T}x, x), d(\mathcal{T}y, y), \frac{1}{2}(d(\mathcal{T}x, y) + d(\mathcal{T}y, x))\right\}.$$

If the condition (3.2) holds, then $\mathcal{T}$ has a fixed point.

In the following theorem we replace condition (3.2) of the above corollary by requiring $\mathcal{T}$ to be continuous.

Theorem 3.4. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space and let $\mathcal{Y} = \sup\{\, d(x, y) : x, y \in \mathcal{X}\,\}$. Set $\mathcal{N} = \mathcal{Y}$ if $\mathcal{Y} = \infty$, and $\mathcal{N} > \mathcal{Y}$ if $\mathcal{Y} < \infty$. Suppose that a continuous mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfies the following conditions:*

1. *there exists $x_0 \in \mathcal{X}$ such that $x_0 \preceq \mathcal{T}x_0$,*

2. *for* $x, y \in \mathcal{X}$, $x \preceq y$ *implies* $\mathcal{T}x \preceq \mathcal{T}y$,

3. $f(d(\mathcal{T}x, \mathcal{T}y)) \le \psi(f(M(x,y)))$ *for all comparable* $x, y \in \mathcal{X}$, *where* $f \in \mathcal{F}$, $\psi \in \Psi$, *and*

$$M(x,y) = \max\left\{d(x,y), d(\mathcal{T}x,x), d(\mathcal{T}y,y), \frac{1}{2}(d(\mathcal{T}x,y) + d(\mathcal{T}y,x))\right\}.$$

Then $\mathcal{T}$ *has a fixed point.*

Proof. Consider $\mathcal{T}$ as a set-valued mapping for which $\mathcal{T}x$ is a singleton set for every $x \in \mathcal{X}$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 3.1 and following the line of its proof, we have that $\{x_n\}$ is a Cauchy sequence and

$$\lim_{n\to\infty} x_n = z.$$

Then, if $\mathcal{T}$ is continuous, we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}x_n = \mathcal{T}z$$

and this proves that z is a fixed point of $\mathcal{T}$ and we have the result. □

Recall that a subset $\mathcal{K}$ of a partially ordered set $\mathcal{X}$ is said to be totally ordered if every two elements of $\mathcal{K}$ are comparable.

Theorem 3.5. *Under the assumptions of Corollary 3.3 or Theorem 3.4, the set* $F(\mathcal{T})$ *of fixed points of* $\mathcal{T}$ *is a singleton if and only if it is totally ordered.*

Proof. Suppose that the set of fixed points of $\mathcal{T}$ is totally ordered. We claim that the fixed point of $\mathcal{T}$ is unique. Assume to the contrary, that $u \in \mathcal{T}u$ and $v \in \mathcal{T}v$ but $u \ne v$, and hence $d(u,v) > 0$. By supposition, we can replace x by u and y by v in condition (3) of Corollary 3.3 or Theorem 3.4 to obtain

$$f(d(u,v)) \le f(d(\mathcal{T}u, \mathcal{T}v) \le \psi(f(M(u,v)))$$

where

$$\begin{aligned} M(u,v) &= \max\left\{d(u,v), d(\mathcal{T}u,v), d(\mathcal{T}v,v), \tfrac{1}{2}(d(\mathcal{T}u,v) + d(\mathcal{T}v,u))\right\} \\ &\le \max\left\{d(u,v), d(u,v), 0, \tfrac{1}{2}(d(u,v) + d(v,u))\right\} = d(u,v) \end{aligned}$$

and

$$f(d(u,v) \le \psi(f(d(u,v))) < f(d(u,v)),$$

a contradiction. Hence $d(u,v) = 0$, that is, $u = v$. The converse is trivial. □

4 Results for a Pair of Set-valued Mappings

Next we prove common fixed point theorems for a pair of set-valued mappings in a complete ordered metric space. To complete the result, we extend to set-valued mappings the notion of weakly isotone increasing mappings given by Dhage, O'Regan and Agarwal [22].

Definition 4.1. Let $(\mathcal{X}, \preceq)$ be a partially ordered set. Two maps $\mathcal{S}, \mathcal{T} : \mathcal{X} \to N(\mathcal{X})$ are said to be weakly isotone increasing if for any $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq_2 \mathcal{T}y$ for all $y \in \mathcal{S}x$ and $\mathcal{T}x \preceq_2 \mathcal{S}y$ for all $y \in \mathcal{T}x$.

Note that, in particular, single-valued mappings $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are weakly isotone increasing [22] if $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x$ and $\mathcal{T}x \preceq \mathcal{S}\mathcal{T}x$ hold for each $x \in \mathcal{X}$.

Theorem 4.2. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space and let $\mathcal{Y} = \sup\{ d(x, y) : x, y \in \mathcal{X} \}$. Set $\mathcal{N} = \mathcal{Y}$ if $\mathcal{Y} = \infty$, and $\mathcal{N} > \mathcal{Y}$ if $\mathcal{Y} < \infty$. Suppose that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to N(\mathcal{X})$ are two set-valued mappings such that the following condition is satisfied:*

$$f(H(\mathcal{T}x, \mathcal{S}y)) \leq \psi(f(M(x, y))) \tag{4.1}$$

for all comparable $x, y \in \mathcal{X}$, where $f \in \mathcal{F}$, $\psi \in \Psi$ and

$$M(x, y) = \max \left\{ d(x, y), D(\mathcal{T}x, x), D(\mathcal{S}y, y), \frac{1}{2}(D(\mathcal{T}x, y) + D(\mathcal{S}y, x)) \right\}. \tag{4.2}$$

Also suppose that $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing and there exists an $x_0 \in \mathcal{X}$ such that $\{x_0\} \preceq_2 \mathcal{T}x_0$. If the condition

$$\begin{cases} \text{if } \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to z \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq z \text{ for all } n \end{cases} \tag{4.3}$$

holds, then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.

Proof. First of all we show that, if $\mathcal{S}$ or $\mathcal{T}$ has a fixed point, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Indeed, let z be a fixed point of $\mathcal{T}$, that is $z \in \mathcal{T}z$, but $z \notin \mathcal{S}z$. Since $\mathcal{S}z$ is an approximation, $D(\mathcal{S}z, z) > 0$. If we use the inequality (4.2), for $x = y = z$, we have

$$M(z, z) = \max \left\{ d(z, z), D(\mathcal{T}z, z), D(\mathcal{S}z, z), \tfrac{1}{2}(D(\mathcal{T}z, z) + D(\mathcal{S}z, z)) \right\} = D(\mathcal{S}z, z),$$

and it follows that

$$\begin{aligned} f(D(\mathcal{S}z, z)) &\leq f(\mathcal{H}(\mathcal{T}z, \mathcal{S}z)) \leq \psi(f(M(z, z))) \\ &= \psi(f(D(\mathcal{S}z, z))) < f(D(\mathcal{S}z, z)). \end{aligned}$$

This is a contradiction, so $z \in \mathcal{S}z$. Analogously, one can observe that if $z \in \mathcal{S}z$, then $z \in \mathcal{T}z$.

Let us start with the given x_0. In view of the property of approximation, we can define a sequence $\{x_n\}\subset\mathcal{X}$ as follows

$$\begin{cases} x_{2n+1}\in\mathcal{T}x_{2n},\ D(\mathcal{T}x_{2n},x_{2n})=d(x_{2n+1},x_{2n}),\\ x_{2n+2}\in\mathcal{S}x_{2n+1},\ D(\mathcal{S}x_{2n+1},x_{2n+1})=d(x_{2n+2},\ x_{2n+1}),\quad \text{for } n\in\{0,1,\dots\}.\end{cases}$$

If $x_{n_0}\in\mathcal{S}x_{n_0}$ or $x_{n_0}\in\mathcal{T}x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n\neq x_{n+1}$ for all n.

Now we use that $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing. Note that $x_1\in\mathcal{T}x_0$, so that $x_0\preceq x_1$ and since $\mathcal{T}x_0\preceq_2\mathcal{S}y$ for all $y\in\mathcal{T}x_0$ we have $\mathcal{T}x_0\preceq_2\mathcal{S}x_1$. In particular, $x_1\preceq x_2$. Continuing this process we construct a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$x_1\preceq x_2\preceq\cdots\preceq x_n\preceq x_{n+1}\preceq\cdots.$$

Now we claim that

$$d(x_{n+1},x_n)<d(x_n,x_{n-1}).$$

Setting $x=x_{2n}$ and $y=x_{2n+1}$ in (4.2), we have for all $n\geq 0$,

$$\begin{aligned} M(x_{2n},x_{2n+1}) &= \max\{d(x_{2n},x_{2n+1}),D(\mathcal{T}x_{2n},x_{2n}),D(\mathcal{S}x_{2n+1},x_{2n+1}),\\ &\qquad \tfrac12(D(\mathcal{T}x_{2n},x_{2n+1})+D(\mathcal{S}x_{2n+1},x_{2n}))\}\\ &= \max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n}),d(x_{2n+2},x_{2n+1}),\\ &\qquad \tfrac12(d(x_{2n+1},x_{2n+1})+d(x_{2n+2},x_{2n}))\}\\ &= \max\{d(x_{2n},x_{2n+1}),d(x_{2n+2},x_{2n+1}),\tfrac12 d(x_{2n+2},x_{2n})\}.\end{aligned}$$

Since $\frac12 d(x_n,x_{n+2})\leq\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}$, it follows that

$$M(x_{2n},x_{2n+1})=\max\{d(x_{2n},x_{2n+1}),d(x_{2n+2},x_{2n+1})\}. \tag{4.4}$$

Suppose that $d(x_{2n+2},x_{2n+1})\geq d(x_{2n},x_{2n+1})$ for some positive integer n. Then from (4.4), we have

$$\begin{aligned} f(d(x_{2n+2},x_{2n+1})) &= f(D(\mathcal{S}x_{2n+1},x_{2n+1}))\\ &\leq f(H(\mathcal{T}x_{2n},\mathcal{S}x_{2n+1}))\leq\psi(f(M(x_{2n},x_{2n+1})))\\ &< f(M(x_{2n},x_{2n+1}))=f(d(x_{2n+2},x_{2n+1})),\end{aligned}$$

a contradiction. So we have $d(x_{2n+2},x_{2n+1})<d(x_{2n},x_{2n+1})$. This yields

$$\begin{aligned} f(d(x_{2n+2},x_{2n+1})) &= f(D(\mathcal{T}x_{2n+1},x_{2n+1}))\leq f(H(\mathcal{T}x_{2n},\mathcal{T}x_{2n+1}))\\ &\leq\psi(f(M(x_{2n},x_{2n+1})))=\psi(f(d(x_{2n+1},x_{2n}))).\end{aligned}$$

Proceeding in the same way, we have

$$d(x_{2n+1},x_{2n})<d(x_{2n},x_{2n-1}),$$

and

$$\begin{aligned} f(d(x_{2n+1},x_{2n})) &= f(D(\mathcal{T}x_{2n},x_{2n}))\leq f(H(\mathcal{T}x_{2n},\mathcal{S}x_{2n-1}))\\ &\leq f(M(x_{2n},x_{2n-1}))=\psi(f(d(x_{2n},x_{2n-1}))).\end{aligned}$$

So for each n, we have

$$f(d(x_{n+1},x_n)) \leq \psi(f(d(x_n,x_{n-1}))).$$

Using the obtained inequality several times, we get

$$f(d(x_{n+1},x_n)) \leq \psi(f(d(x_n,x_{n-1}))) \leq \cdots \leq \psi^n(f(d(x_1,x_0))).$$

Let $m,n \in N$, $n > m$. Then in virtue of the triangular inequality, we have

$$d(x_n,x_m) \leq \sum_{i=m}^{n-1} d(x_i,x_{i+1}).$$

This implies, using properties of $f \in \mathcal{F}$,

$$\begin{aligned} f(d(x_n,x_m)) &\leq f(d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2}) + \cdots + d(x_{m+1},x_m)) \\ &\leq \sum_{i=m}^{n-1} \psi^i(f(d(x_1,x_0))). \end{aligned}$$

Letting $m,n \to \infty$, by the above inequality, using the condition $\sum_{i=m}^{\infty} \psi^n(t) < \infty$, it follows that $\{x_n\}$ is a Cauchy sequence.

From the completeness of $\mathcal{X}$, there exists a $z \in \mathcal{X}$ such that

$$x_n \longrightarrow z \quad \text{as} \quad n \longrightarrow \infty.$$

By the assumption (4.3), $x_n \preceq z$, for all n.

Now we prove $D(\mathcal{T}z,z) = 0$. Suppose that this is not true, i.e., $D(\mathcal{T}z,z) > 0$. For large enough n, we use that the condition (4.1) holds for $x = z$ and $y = x_{2n+1}$, where

$$\begin{aligned} M(z,x_{2n+1}) &= \max\{d(z,x_{2n+1}), D(\mathcal{T}z,z), D(\mathcal{S}x_{2n+1},x_{2n+1}), \\ &\qquad \tfrac{1}{2}(D(\mathcal{T}z,x_{2n+1}) + D(\mathcal{S}x_{2n+1},z))\} \\ &= D(\mathcal{T}z,z). \end{aligned}$$

Indeed, since $\lim_{n\to\infty} d(z,x_{2n+1}) = 0$ and $\lim_{n\to\infty} D(\mathcal{S}x_{2n+1},x_{2n+1}) = 0$, it follows that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{2}(D(\mathcal{T}z,x_{2n+1}) + D(\mathcal{S}x_{2n+1},z)) \\ &\quad \leq \lim_{n\to\infty} \frac{1}{2}(D(\mathcal{T}z,z) + d(z,x_{2n+1}) + D(\mathcal{S}x_{2n+1},x_{2n+1}) + d(x_{2n+1},z)) \\ &\quad = \frac{1}{2}D(\mathcal{T}z,z). \end{aligned}$$

Therefore, there exists n_1 such that $M(z,x_{2n+1}) = D(\mathcal{T}z,z)$ for all $n > n_1$. Note that

$$f(D(\mathcal{T}z,x_{2n+2})) \leq f(H(\mathcal{T}z,\mathcal{S}x_{2n+1})) \leq \psi(f(M(z,x_{2n+1}))).$$

Letting $n \to \infty$ and applying (i) of Definition 2.2, we get

$$f(D(\mathcal{T}z, z)) \le \psi(f(D(\mathcal{T}z, z))) < f(D(\mathcal{T}z, z)),$$

a contradiction. Hence $D(\mathcal{T}z, z) = 0$, and in virtue of the approximation of $\mathcal{T}z$, we have $z \in \mathcal{T}z$. Using the conclusion from the beginning of the proof, we get that z is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. This completes the proof of the theorem. $\square$

Example 4.3. Consider the space $\mathcal{X} = C[a, b]$ of continuous real functions with the standard metric $d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|$ and the order $\preceq$ defined by

$$x \preceq y \iff x(t) \ge y(t) \text{ for all } t \in [a, b]$$

(note the reverse ordering). Let $f \in \mathcal{F}$ and $\psi \in \Psi$ be given by $f(t) = \frac{2}{3}t$ and $\psi(t) = \frac{1}{2}t$. Consider the following mappings $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$:

$$\begin{aligned}
\mathcal{T}x &= \left[\frac{1}{4}x, \frac{1}{3}x\right] = \left\{z \in \mathcal{X} : \frac{1}{4}x(t) \le z(t) \le \frac{1}{3}x(t),\ t \in [a, b]\right\} \\
\mathcal{S}x &= \left[\frac{1}{5}x, \frac{3}{10}x\right] = \left\{z \in \mathcal{X} : \frac{1}{5}x(t) \le z(t) \le \frac{3}{10}x(t),\ t \in [a, b]\right\}.
\end{aligned}$$

Check first that $\mathcal{T}$ and $\mathcal{S}$ are weakly isotone increasing. Suppose that $y \in \mathcal{S}x = [\frac{1}{5}x, \frac{3}{10}x]$ and $z \in \mathcal{S}x = [\frac{1}{5}x, \frac{3}{10}x]$. Then $u \in \mathcal{T}y = [\frac{1}{4}y, \frac{1}{3}y]$ implies that $u(t) \le \frac{1}{3} \cdot \frac{3}{10}x(t) = \frac{1}{10}x(t) < \frac{1}{5}x(t) \le z(t)$ for $t \in [a, b]$ and so $z \preceq u$. This means that for any $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq_2 \mathcal{T}y$ for all $y \in \mathcal{S}x$. Similarly, one can prove that for each $x \in \mathcal{X}$ we have $\mathcal{T}x \preceq_2 \mathcal{S}y$ for all $y \in \mathcal{T}x$.

Take now arbitrary compatible functions $x, y \in \mathcal{X}$. Then, for each $t \in [a, b]$,

$$\begin{aligned}
f(H(\mathcal{T}x, \mathcal{S}y)) &\le \frac{2}{3} \max\left\{\frac{1}{3}x(t), \frac{3}{10}x(t)\right\} \le \frac{1}{3} \max\left\{\frac{2}{3}x(t), \frac{7}{10}x(t)\right\} \\
&\le \frac{1}{3} \max\{D(x, Tx), D(y, Sy)\} = \frac{1}{3}M(x, y) = \psi(f(M(x, y))).
\end{aligned}$$

Hence, condition (4.1) of Theorem 4.2 is fulfilled. The other conditions of this theorem are easy to check, and so there is a fixed point z of $\mathcal{T}$ and $\mathcal{S}$ (which is $z = 0$).

In Theorem 4.2, if $\mathcal{T}, \mathcal{S}$ are single-valued mappings and condition (4.3) is replaced by requiring that one of $\mathcal{T}$, $\mathcal{S}$ is continuous, then we have the following result.

Theorem 4.4. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space and let $\mathcal{Y} = \sup\{\, d(x, y) : x, y \in \mathcal{X}\,\}$. Set $\mathcal{N} = \mathcal{Y}$ if $\mathcal{Y} = \infty$, and $\mathcal{N} > \mathcal{Y}$ if $\mathcal{Y} < \infty$. Suppose that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are single-valued operators satisfying*

$$f(d(\mathcal{T}x, \mathcal{S}y)) \le \psi(f(M(x, y))) \tag{4.5}$$

for all comparable $x, y \in \mathcal{X}$, where $f \in \mathcal{F}$, $\psi \in \Psi$ and

$$M(x,y) = \max\left\{d(x,y), d(\mathcal{T}x,x), d(\mathcal{S}y,y), \frac{1}{2}(d(\mathcal{T}x,y) + d(\mathcal{S}y,x))\right\}.$$

Also suppose that $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing and there exists an $x_0 \in \mathcal{X}$ such that $x_0 \preceq \mathcal{T}x_0$. If one of $\mathcal{S}$ and $\mathcal{T}$ is continuous, then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.

Proof. Consider $\mathcal{T}$ and $\mathcal{S}$ as set-valued mappings for which $\mathcal{T}x$ and $\mathcal{S}x$ are singletons for every $x \in \mathcal{X}$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 4.2 and following the line of its proof, we have that $\{x_n\}$ is a Cauchy sequence and

$$\lim_{n\to\infty} x_n = z.$$

Then, if $\mathcal{T}$ is continuous, we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}x_n = \mathcal{T}z$$

and this proves that z is a fixed point of $\mathcal{T}$ and so z is also a fixed point of $\mathcal{S}$. Similarly, if $\mathcal{S}$ is continuous, we have the result. Thus it is immediate to conclude that $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. $\square$

Theorem 4.5. *Under the assumptions of Theorem 4.4, the set of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is totally ordered if and only if $\mathcal{T}$ and $\mathcal{S}$ have one and only one common fixed point.*

Proof. Suppose that the set of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is totally ordered. We claim that the common fixed point of $\mathcal{T}$ and $\mathcal{S}$ is unique. Assume to the contrary that $u \in \mathcal{S}u$, $u \in \mathcal{T}u$ and $v \in \mathcal{S}v$, $v \in \mathcal{T}v$ but $u \neq v$, then $d(u,v) > 0$. By supposition, we can replace x by u and y by v in (4.5) to obtain

$$f(d(u,v)) \leq f(d(\mathcal{T}u, \mathcal{S}v)) \leq \psi(f(M(u,v)))$$

where

$$\begin{aligned} M(u,v) &= \max\left\{d(u,v), d(\mathcal{T}u,v), d(\mathcal{S}v,v), \tfrac{1}{2}(d(\mathcal{T}u,v) + d(\mathcal{S}v,u))\right\} \\ &\leq \max\left\{d(u,v), d(u,v), 0, \tfrac{1}{2}(d(u,v) + d(v,u))\right\} = d(u,v) \end{aligned}$$

and

$$f(d(u,v) \leq \psi(f(d(u,v))) < f(d(u,v)),$$

a contradiction. Hence $d(u,v) = 0$, that is, $u = v$. Conversely, if $\mathcal{T}$ and $\mathcal{S}$ have only one common fixed point then the set of common fixed point of $\mathcal{T}$ and $\mathcal{S}$, being singleton, is totally ordered. $\square$

Putting $\mathcal{S} = \mathcal{T}$ in Theorem 4.2, we obtain the following

Corollary 4.6. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space and let $\mathcal{Y} = \sup\{ d(x,y) : x, y \in \mathcal{X} \}$. Set $\mathcal{N} = \mathcal{Y}$ if $\mathcal{Y} = \infty$, and $\mathcal{N} > \mathcal{Y}$ if $\mathcal{Y} < \infty$. Suppose that $\mathcal{T} : \mathcal{X} \to N(\mathcal{X})$ is a set-valued mapping such that*

$$f(H(\mathcal{T}x, \mathcal{T}y)) \leq \psi(f(M(x,y)))$$

for all comparable $x, y \in \mathcal{X}$, where $f \in \mathcal{F}$, $\psi \in \Psi$, and

$$M(x,y) = \max\left\{ d(x,y), D(\mathcal{T}x, x), D(\mathcal{T}y, y), \frac{1}{2}(D(\mathcal{T}x, y) + D(\mathcal{T}y, x)) \right\}.$$

Also suppose that $\mathcal{T}x \preceq_2 \mathcal{T}(\mathcal{T}x)$ for all $x \in \mathcal{X}$ and that $\{x_0\} \preceq_2 \mathcal{T}x_0$ for some $x_0 \in \mathcal{X}$. If the condition (4.3) holds, then $\mathcal{T}$ has a fixed point.

If $\mathcal{T}$ is a single-valued mapping in Corollary 4.6, then we have the following consequence:

Corollary 4.7. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space and let $\mathcal{Y} = \sup\{ d(x,y) : x, y \in \mathcal{X} \}$. Set $\mathcal{N} = \mathcal{Y}$ if $\mathcal{Y} = \infty$, and $\mathcal{N} > \mathcal{Y}$ if $\mathcal{Y} < \infty$. Suppose that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$f(d(\mathcal{T}x, \mathcal{T}y)) \leq \psi(f(M(x,y)))$$

for all comparable $x, y \in \mathcal{X}$, where $f \in \mathcal{F}$, $\psi \in \Psi$ and

$$M(x,y) = \max\left\{ d(x,y), d(\mathcal{T}x, x), d(\mathcal{T}y, y), \frac{1}{2}(d(\mathcal{T}x, y) + d(\mathcal{T}y, x)) \right\}.$$

Also suppose that $\mathcal{T}x \preceq \mathcal{T}(\mathcal{T}x)$ for all $x \in \mathcal{X}$. If the condition (4.3) holds, then $\mathcal{T}$ has a fixed point.

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