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On Non-Absolute Integrals

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Abstract : The Riemann integral, McShane integral and Henstock integral are well-known. They are the Riemann approach integrals. In this note we introduce the Riemann $\mathcal{B}$-integral using Riemann approach, which are non-absolute integrals. The Fundamental Theorem of Calculus is proved. The "local Character" plays very important role in the prove of the Fundamental Theorem of Calculus. Some convergence theorems are also proved.

Keywords : non-absolute integral; Riemann $\mathcal{B}$-integral; Henstock integral; McShane integral; fundamental theorem of calculus; double Lusin condition.
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1 Introduction

The usual way to define the integrals using Riemann approach, is through the Riemann sums

$$S(f, D) = (D)\sum f(\xi)|I|,$$

where $D = \{(\xi, I)\}$ is the division of domain $[a, b]$, with some control condition on D. The well-known integrals using Riemann approach are the Riemann integral, the McShane integral and the Henstock integral, see, for example, [1]. The Henstock integral, also known as the Kurzweil-Henstock integral, is non-absolute integral, while the McShane integral, equivalent to the Lebesgue integral, is an absolute integral.

The point-interval pairs (ξ, I) in the division D are chosen by taking the tag point ξ first. After that we always use some control condition to choose an associated interval I. For example, for the Henstock case, for each point ξ, an associate interval I must satisfy the condition

$$\xi \in I \subseteq (\xi - \delta(\xi), \xi + \delta(\xi)),$$

where $\delta(\xi)$ is a positive number depend on the tag point ξ. For the Riemann integral, $\delta(\xi)$ is the positive constant, i.e., $\delta(\xi) = \delta$ for all $\xi \in [a, b]$. The McShane integral relaxes the condition of the Henstock such that ξ may or may not be contained in I.

In this paper, we change the way to define the division D of $[a, b]$. We shall choose an interval I first, then choose tag point ξ under some condition. We shall use the interval-point pair (I, ξ) instead of point-interval pair (ξ, I).

2 Riemann $\mathcal{B}$-integral on $[a, b]$

In this note, let $\mathbb{R}$ denote the set of all real numbers and $|I|$ denote the length of interval $I \subseteq \mathbb{R}$.

A set-valued function v defined on the set of all non-degenerate closed subintervals of $[a, b]$ is called a ***tag function*** on $[a, b]$ if $v(I) \subseteq [a, b]$ for all $I \subseteq [a, b]$. An interval-point pair (I, ξ), where $I \subseteq [a, b]$ and $\xi \in [a, b]$, is said to be v-**fine** if $\xi \in v(I) \neq \emptyset$.

A finite collection of interval-point pair, $D = \{(I, \xi)\}$, is called to be ***v-fine partial division*** of $[a, b]$ if $\{I\}$ is a partial partition of $[a, b]$, that is, their union is a subset of $[a, b]$, and each (I, ξ) is v-fine. In addition, if $\{I\}$ is a partition of $[a, b]$, then D is said to be a ***v-fine division*** of $[a, b]$.

A tag function v is said to have the ***division property*** if for every subinterval $[c, d]$ of $[a, b]$, there exists a v-fine division of $[c, d]$.

Let $\mathcal{B}$ be a collection of tag functions on $[a, b]$, with the division property. Let $v_1, v_2 \in \mathcal{B}$, we say that v_2 is ***finer*** than v_1, denoted by $v_1 \leq v_2$, if $v_2(I) \subseteq v_1(I)$ for all non-degenerate closed subintervals I of $[a, b]$. Clearly, the collection $\mathcal{B}$ together with the relation $\leq$ on $\mathcal{B}$, $(\mathcal{B}, \leq)$, is a partially ordered set. $\mathcal{B}$ is said to be a ***filtered set*** on $[a, b]$ if the partially ordered set $(\mathcal{B}, \leq)$ is directed, i.e., for any $v_1, v_2 \in \mathcal{B}$, there exists $v \in \mathcal{B}$ such that $v_1 \leq v$ and $v_2 \leq v$. We note that the division and filtering properties play very important role in the proofs.

Definition 2.1 (Riemann $\mathcal{B}$-integral)**.** Let $\mathcal{B}$ be a filtered set on $[a, b]$. A real valued function f defined on $[a, b]$ is said to be ***Riemann $\mathcal{B}$-integrable*** to A on $[a, b]$ if for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$ such that for every v-fine division $D = \{(I, \xi)\}$ of $[a, b]$, we have

$$|S(f, v, D) - A| < \epsilon,$$

where $S(f, v, D) = (D) \sum f(\xi)|I|$. We denote A by $(\mathcal{B}) \int_a^b f$.

Note that, ideas of Riemann $\mathcal{B}$-integrals are given in [2,3]. We can prove easily, by using Cauchy's Criterion, that the Definition 2.1 and the definition using Upper and Lower Riemann sums in [2, 3] are equivalent. The Definition 2.1 we use here is the usual way to define the integral using Riemann approach. The proof of all basic properties such as uniqueness, linearity, etc., are straightforward.

Theorem 2.2 (Cauchy's Criterion)**.** *Let $\mathcal{B}$ be a filtered set on $[a,b]$. A function f is Riemann $\mathcal{B}$-integrable on $[a,b]$ if and only if for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$ such that for every v-fine two divisions D and D' of $[a,b]$, we have*

$$|S(f,v,D) - S(f,v,D')| < \epsilon.$$

Proof. The proof is standard (see [1]). □

Theorem 2.1 (Henstock's Lemma)**.** *Let $\mathcal{B}$ be a filtered set on $[a,b]$. If f is Riemann $\mathcal{B}$-integrable to F on $[a,b]$, for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$ such that for any v-fine partial division D of $[a,b]$, we have*

$$(D)\sum |f(\xi)|I| - F(I)| < \epsilon.$$

Proof. The proof is standard (see [1]). □

Let η be a positive constant. Let v be a tag function on $[a,b]$ defined by $v_\eta(I) = I$ if $|I| < \eta$; and $v_\eta(I) = \emptyset$ otherwise. Let β be a positive real values function defined on $[a,b]$. Let $v_\beta(I) = \{\xi \in [a,b] : I \subseteq (\xi - \beta(\xi), \xi + \beta(\xi))\}$. Let δ be a positive real-value function defined on $[a,b]$. Let $v_\delta(I) = \{\xi \in [a,b] : \xi \in I \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))\}$. By Heine-Borel theorem, v_η, v_β and v_δ have the division property. Let $\mathcal{B}_R = \{v_\eta : \eta > 0\}$, $\mathcal{B}_M = \{v_\beta : \beta(\xi) > 0\}$ and $\mathcal{B}_H = \{v_\delta : \delta(\xi) > 0\}$. Clearly that, $\mathcal{B}_R$, $\mathcal{B}_M$ and $\mathcal{B}_H$ are filtered sets on $[a,b]$.

Example 2.3. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ if x is rational; and $f(x) = 0$ otherwise, i.e., $f(x) = \chi_{\mathbb{Q}}$, where $\mathbb{Q}$ is the set of all rational number.

Let $0 < \epsilon < 1$. Let $v_\eta \in \mathcal{B}_R$. Let $D = \{(I,\xi)\}$ and $D' = \{(\xi', I')\}$ be two v_η-fine divisions of $[0,1]$. Note that, if we take an interval I from interval-point pair $(I,\xi) \in D$, $v_\eta(I) \neq \emptyset$, whenever D is a v_η-fine division, that is, by definition of v_η, we have $v_\eta(I) = I$ for such I. Thus $v_\eta(I)$ is an non-degenerate closed subintervals of $[0,1]$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we may assume that $\xi \in \mathbb{Q}$ for all $(I,\xi) \in D$ and $\xi' \in \mathbb{Q}^c$ for all $(\xi', I') \in D'$. Hence

$$|S(f,v,D) - S(f,v,D')| = \left|(D)\sum |I| - (D')\sum 0|I'|\right| = |1-0| = 1 > \epsilon.$$

Hence, by Cauchy's Criterion, f is not $\mathcal{B}_R$-integrable on $[0,1]$.

Now we shall consider the collection $\mathcal{B}_M$. Write $\mathbb{Q} \cap [0,1] = \{r_1, r_2, \ldots\}$. Let β be a positive function defined on $[0,1]$ defined by $\beta(r_j) = \frac{\epsilon}{2^j}$ for all $r_j \in \mathbb{Q} \cap [0,1]$ and $\beta(\xi) = 1$ for all $x \in \mathbb{Q}^c \cap [0,1]$. Let $v_\beta(I) = \{\xi \in [0,1] : I \subseteq (\xi - \beta(\xi), \xi + \beta(\xi))\}$. Clearly, $v_\beta \in \mathcal{B}_M$.

Notice that for any $I \subseteq [0,1]$,

$$\begin{aligned} v_\eta(I) =& \{\xi \in [0,1] : I \subseteq (\xi - \beta(\xi), \xi + \beta(\xi))\} \\ =& \{\xi \in \mathbb{Q} \cap [0,1] : I \subseteq (\xi - \beta(\xi), \xi + \beta(\xi))\} \cup \{\xi \in \mathbb{Q}^c \cap [0,1] : I \subseteq (\xi - 1, \xi + 1)\} \\ =& \left\{ r_j : I \subseteq \left(\xi - \frac{\epsilon}{2^j}, \xi + \frac{\epsilon}{2^j}\right) \right\} \cup (\mathbb{Q}^c \cap [0,1]). \end{aligned}$$

Thus, there are two kinds of points in $v_\eta(I)$. The rational point r_j such that $I \subseteq \left(\xi - \frac{\epsilon}{2^j}, \xi + \frac{\epsilon}{2^j}\right)$ and the irrational points. However, for the irrational points the function f vanishes there.

Let $D = \{(I, \xi)\}$ be a v_β-fine division of $[0,1]$. Hence,

$$\left|(D) \sum f(\xi)|I|\right| \leq \left|(D) \sum_{\xi \in \mathbb{Q} \cap [0,1]} f(\xi)|I|\right| + \left|(D) \sum_{\xi \in \mathbb{Q}^c \cap [0,1]} f(\xi)|I|\right| < \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j-1}} = 2\epsilon.$$

Therefore, f is $\mathcal{B}_M$-integrable on $[0,1]$.

From the above example, we can see that the Riemann $\mathcal{B}$-integrability of f does not only depend on the function f itself but also depend on the choice of the collection of tag functions $\mathcal{B}$.

Notice that for every two filtered sets $\mathcal{B}$ and $\mathcal{B}'$ on $[a,b]$ if $\mathcal{B} \subseteq \mathcal{B}'$, then every tag function v in $\mathcal{B}$ is again a tag function in $\mathcal{B}'$. Therefore we get the following theorem.

Theorem 2.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two filtered sets on $[a,b]$ such that $\mathcal{B} \subseteq \mathcal{B}'$ and $f : [a,b] \to \mathbb{R}$. If f is Riemann $\mathcal{B}'$-integrable on $[a,b]$, then f is Riemann $\mathcal{B}$-integrable on $[a,b]$ and*

$$(\mathcal{B}) \int_a^b f = (\mathcal{B}') \int_a^b f.$$

It is easy to see that division induced by the tag functions v_η, v_β and v_δ above are equivalent to divisions induced by Riemann, McShane and Henstock integrals, respectively. Hence, we have the following theorems.

Theorem 2.5. *A real valued function f define on $[a,b]$ is a Riemann integrable on $[a,b]$ if and only if f is Riemann $\mathcal{B}_R$-integrable on $[a,b]$.*

Theorem 2.6. *A real valued function f define on $[a,b]$ is a McShane integrable on $[a,b]$ if and only if f is Riemann $\mathcal{B}_M$-integrable on $[a,b]$.*

Theorem 2.7. *A real valued function f define on $[a,b]$ is a Henstock integrable on $[a,b]$ if and only if f is Riemann $\mathcal{B}_H$-integrable on $[a,b]$.*

For the definitions of Riemann, McShane and Henstock integral, see [1].

3 Fundamental Theorem of Calculus

A collection $\mathcal{B}$ of tag functions on $[a,b]$ is said to has a ***local character*** if for each $\xi \in [a,b]$, a tag function $v_\xi \in \mathcal{B}$ is given, then there exists a common $v \in \mathcal{B}$ such that $v(I) \subseteq v_\xi(I)$ whenever $\xi \in v(I)$.

We remark here that the filtered set $\mathcal{B}_M$ and $\mathcal{B}_H$ have a local character, but $\mathcal{B}_R$ does not. For example, for each $\xi \in [0,1]$, let $v_\xi(I) = I$ if $|I| < \xi$ and $v_\xi(I) = \emptyset$ otherwise. Clearly that $v_\xi \in \mathcal{B}_R$ for all $\xi \in [0,1]$. We shall point out that $\mathcal{B}_R$ does not have local character. Suppose that $\mathcal{B}_R$ has a local character, i.e., there exist a positive constant η such that $v(I) = I$ if $|I| < \eta$ and $v(I) = \emptyset$ otherwise, and $v(I) \subseteq v_\xi(I)$ whenever $\xi \in v(I)$. Choose $x = \frac{\eta}{4}$ and $I = [0, \frac{\eta}{2}]$. So, $|I| = \frac{\eta}{2} > x$, that is, $v_x(I) = \emptyset$. Hence $x \in v(I)$ but $v(I) \nsubseteq v_x(I)$. Contradict to the assumption that v is a local character. Therefor $\mathcal{B}_R$ does not have a local character.

A local character plays very important role in the proofs of the Fundamental Theorem of Calculus.

Example 3.1. Let $\mathcal{B}_H = \{v_\delta : \delta(\xi) > 0\}$ be a filtered set on $[a,b]$ defined as above. For each $\xi \in [a,b]$, choose $v_{\delta_\xi} \in \mathcal{B}_H$. Let $\delta : [a,b] \to (0,\infty)$ be defined by $\delta(\xi) = \delta_\xi(\xi)$. Clearly,

$$\begin{aligned} v_\delta(I) &= \{\xi \in [a,b] : \xi \in I \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))\} \\ &= \{\xi \in [a,b] : \xi \in I \subseteq (\xi - \delta_\xi(\xi), \xi + \delta_\xi(\xi))\} \\ &= v_{\delta_\xi}(I) \end{aligned}$$

for all $I \in \mathcal{I}$. Thus $v_\delta(I) \subseteq v_{\delta_\xi}(I)$ whenever $\xi \in v_\delta(I)$. Hence $\mathcal{B}_H$ has a local character.

Let $\mathcal{I}$ be the collection of all subinterval of $[a,b]$. An interval function $F : \mathcal{I} \to \mathbb{R}$ is said to be ***additive*** if $F(I \cup J) = F(I) + F(J)$ for all $I, J \in \mathcal{I}$ with $I \cap J = \emptyset$.

Definition 3.2. Let $\mathcal{B}$ be a filtered set on $[a,b]$. An interval function F defined on $\mathcal{I}$ is said to be ***$\mathcal{B}$-differentiable at*** $\xi \in [a,b]$ with ***$\mathcal{B}$-derivative*** $f(\xi)$ if for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$, such that for every non-degenerate closed subinterval I of $[a,b]$ with $\xi \in v(I)$, we have

$$|F(I) - f(\xi)|I|| < \epsilon |I|.$$

We write $F'_{\mathcal{B}}[I] = f(\xi)$.

Theorem 3.3. *Let $\mathcal{B}$ be a filtered set on $[a,b]$ and F an additive $\mathcal{B}$-differentiable on $[a,b]$. Suppose $\mathcal{B}$ has local character. Then $f(\xi) = F'_{\mathcal{B}}[I]$ is Riemann $\mathcal{B}$-integrable on $[a,b]$ and*

$$(\mathcal{B}) \int_a^b f = F([a,b]).$$

Proof. Let $\epsilon > 0$ be given. For each $\xi \in [a,b]$, there exists $v_\xi \in \mathcal{B}$, such that for every non-degenerate closed subinterval I of $[a,b]$ with $\xi \in v_\xi(I)$, we have

$$|F(I) - f(\xi)|I|| < \epsilon |I|.$$

Since $\mathcal{B}$ has local character, there exists $v \in \mathcal{B}$ such that $v(I) \subseteq v_\xi(I)$ whenever $\xi \in v(I)$. Let $D = \{(I, \xi)\}$ be a v-fine division of $[a, b]$. Then, for every $(I, \xi) \in D$, we have $\xi \in v(I) \subseteq v_\xi(I)$. Hence,

$$\left|(D)\sum f(\xi)|I| - F([a,b])\right| \leq (D)\sum |f(\xi)|I| - F(I)| < (D)\sum \epsilon|I| = \epsilon(b-a).$$

Therefore, $f(\xi) = F'_{\mathcal{B}}[I]$ is Riemann $\mathcal{B}$-integrable to $F([a, b])$ on $[a, b]$. □

Example 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ with antiderivative $F : \mathbb{R} \to \mathbb{R}$, that is $F'(\xi) = f(\xi)$ for all $\xi \in \mathbb{R}$. Let $\mathbf{F}([x, y]) = F(y) - F(x)$ for all subinterval $[x, y]$ in $\mathbb{R}$.

Let $\xi \in \mathbb{R}$. Since $F'(\xi) = f(\xi)$, then for each $\epsilon > 0$, there exists $\delta_\xi > 0$ such that for any $I = [x, y] \in (\xi - \delta_\xi, \xi + \delta_\xi)$ with $\xi \in I$, we have

$$|\mathbf{F}(I) - f(\xi)|I|| = |F(y) - F(x) - f(\xi)|y - x|| < \epsilon|y - x| = \epsilon|I|.$$

Let $[a, b] \subseteq \mathbb{R}$. Let $\delta : [a, b] \to \mathbb{R}$ defined by $\delta(\xi) = \delta_\xi$. Let $v_\delta(I) = \{\xi \in [a, b] : \xi \in I \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))\}$. Then $v_\delta \in \mathcal{B}_H$ and for any subinterval I of $[a, b]$ with $\xi \in v_\delta(I)$, we have

$$|\mathbf{F}(I) - f(\xi)|I|| < \epsilon|I|.$$

Thus, for each $\xi \in [a, b]$, $\mathbf{F}$ is $\mathcal{B}$-differentiable at ξ with $\mathcal{B}$-derivative $f(\xi)$. Example 3.1 shows that $\mathcal{B}_H$ has a local character. Hence, by Theorem 3.3, f is Riemann $\mathcal{B}_H$-integrable on $[a, b]$ and

$$(\mathcal{B}_H)\int_a^b f = \mathbf{F}([a, b]) = F(b) - F(a).$$

By Example 2.3, f is Henstock integrable on $[a, b]$ and

$$(\mathcal{H})\int_a^b f = F(b) - F(a).$$

The property discussed in this example is also known as Fundamental Theorem of Calculus for the Henstock integral, see [1].

Given a pair of functions f and F on $[a, b]$ and $\epsilon > 0$, define

$$\Gamma_\epsilon = \{(I, \xi) : |F(I) - f(\xi)|I|| \geq \epsilon|I|\}.$$

Lemma 3.5. *Let $\mathcal{B}$ be a filtered set on $[a, b]$ and f be $\mathcal{B}$-integrable on $[a, b]$ with primitive F, i.e., $F(I) = (\mathcal{B})\int_I f$ for all subinterval I of $[a, b]$. If $\mathcal{B}$ has local character, then for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$, such that for every v-fine partial division $D = \{(I, \xi)\} \subseteq \Gamma_\epsilon$ of $[a, b]$, we have*

$$(D)\sum |f(\xi)|I|| < \epsilon \text{ and } (D)\sum |F(I)| < \epsilon. \tag{3.1}$$

Proof. Let $0 < \epsilon < 1$ be given. By Henstock's Lemma, for every $n \in \mathbb{N}$, there exists a tag function $v_n \in \mathcal{B}$ such that for any v_n-fine partial division D of $[a,b]$, we have

$$(D)\sum |F(I) - f(\xi)|I|| < \frac{\epsilon^2}{n2^{n+1}}.$$

For each $n \in \mathbb{N}$, let

$$X_n = \{x \in [a,b] : n-1 \leq |f(x)| < n\}.$$

For each $\xi \in X_n$, let $v_\xi = v_n$. Since $\mathcal{B}$ has a local character, there exists $v \in \mathcal{B}$ such that $v(I) \subseteq v_\xi(I)$ whenever $\xi \in v_\xi(I)$. Let $D = \{(I,\xi)\}$ be a v-fine partial division of $[a,b]$ such that $D \subseteq \Gamma_\epsilon$. Note that for every $(I,\xi) \in D$, $\xi \in v(I) \subseteq v_\xi(I) = v_n(I)$ for some $n \in \mathbb{N}$. For each n, let $D_n = \{(I,\xi) \in D : \xi \in X_n\}$. Hence,

$$(D)\sum |f(\xi)|I|| \leq \sum_{n=1}^{\infty}(D_n)\sum |f(\xi)||I| < \sum_{n=1}^{\infty} n \cdot (D_n)\sum |I|.$$

Since $D_n \subseteq D \subseteq \Gamma_\epsilon$, we have

$$(D_n)\sum |I| < (D_n)\sum \frac{|F(I) - f(\xi)|I||}{\epsilon} < \frac{\epsilon}{n2^{n+1}}.$$

Thus,

$$(D)\sum |f(\xi)|I|| < \sum_{n=1}^{\infty} n \cdot (D_n)\sum |I| < \sum_{n=1}^{\infty} n \cdot \frac{\epsilon}{n2^{n+1}} = \frac{\epsilon}{2}.$$

Notice that

$$(D)\sum |F(I)| \leq (D)\sum |F(I) - f(\xi)|I|| + (D)\sum |f(\xi)|I||\,.$$

Consider the first sum on the right hand side, we have

$$(D)\sum |F(I) - f(\xi)|I|| < \sum_{n=1}^{\infty}(D_n)\sum |F(I) - f(\xi)|I|| < \sum_{n=1}^{\infty} \frac{\epsilon^2}{n2^{n+1}} < \frac{\epsilon}{2}.$$

Hence

$$(D)\sum |F(I)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

The inequalities (3.1) are also known as the ***double Lusin condition***, see [4].

Lemma 3.6. *Let $\mathcal{B}$ be a filtered set on $[a,b]$. If for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$, such that*

$$(D)\sum |f(\xi)|I|| < \epsilon \text{ and } (D)\sum |F(I)| < \epsilon$$

whenever v-fine partial division $D = \{(I,\xi)\} \subseteq \Gamma_\epsilon$ of $[a,b]$, then f is $\mathcal{B}$-integrable on $[a,b]$ with primitive F.

Proof. Let $\epsilon > 0$ and $v \in \mathcal{B}$ satisfying the double Lusin condition above. Let $D = \{(I,\xi)\}$ be a v-fine division of $[a,b]$. Let $D_{\Gamma_\epsilon} = \{(I,\xi) \in D \cap \Gamma_\epsilon\} \subseteq \Gamma_\epsilon$ and $D^c_{\Gamma_\epsilon} = D \setminus D_{\Gamma_\epsilon}$. Hence

$$\begin{aligned}\left|(D)\sum f(\xi)|I| - F([a,b])\right| &\leq (D)\sum |f(\xi)|I| - F(I)| \\ &\leq (D_{\Gamma_\epsilon})\sum |f(\xi)|I|| + (D_{\Gamma_\epsilon})\sum |F(I)| \\ &\quad + (D^c_{\Gamma_\epsilon})\sum |f(\xi)|I| - F(I)| \\ &< \epsilon + \epsilon + (D^c_{\Gamma_\epsilon})\sum \epsilon|I| = (2+|b-a|)\epsilon.\end{aligned}$$

Therefore, we get the required result. □

We may rewrite Lemma 3.5 together with Lemma 3.6 as follows.

Theorem 3.7. *Let $\mathcal{B}$ be a filtered set on $[a,b]$ with a local character. Then f is $\mathcal{B}$-integrable on $[a,b]$ with primitive F if and only if for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$, such that*

$$(D)\sum |f(\xi)|I|| < \epsilon \text{ and } (D)\sum |F(I)| < \epsilon$$

whenever v-fine partial division $D = \{(I,\xi)\} \subseteq \Gamma_\epsilon$ of $[a,b]$.

Given a pair of functions f and F on $[a,b]$ and a tag function $v \in \mathcal{B}$, for each $n \in \mathbb{N}$, define

$$IV(f,n,v) = \sup_D \left\{(D)\sum |f(\xi)|I|| : D \subseteq \Gamma_{1/n} \text{ is a } v\text{-fine partial division of } [a,b]\right\}$$

and

$$IV(F,n,v) = \sup_D \left\{(D)\sum |F(I)| : D \subseteq \Gamma_{1/n} \text{ is a } v\text{-fine partial division of } [a,b]\right\}.$$

Define

$$IV(f,n) = \inf_{v\in\mathcal{B}} IV(f,n,v) \text{ and } IV(F,n) = \inf_{v\in\mathcal{B}} IV(F,n,v).$$

The following theorem is a consequence of Lemma 3.5.

Theorem 3.8. *Let $\mathcal{B}$ be a filtered set on $[a,b]$ and f be $\mathcal{B}$-integrable on $[a,b]$ with primitive F. Suppose $\mathcal{B}$ has a local character. Then for each $n \in \mathbb{N}$, we have $IV(f,n) = 0$ and $IV(F,n) = 0$.*

An interval function F defined on $\mathcal{I}$ is said to be $\mathcal{B}$-differentiable with $\mathcal{B}$-derivative $f(\xi)$ almost all interval-point pairs $(I,\xi) \in \mathcal{I} \times [a,b]$, abbreviated a.a. on $\mathcal{I} \times [a,b]$, if $IV(f,n) = 0$ and $IV(F,n) = 0$ for all $n \in \mathbb{N}$.

Theorem 3.9. *Let $\mathcal{B}$ be a filtered set on $[a,b]$ and f be $\mathcal{B}$-integrable on $[a,b]$ with primitive F. Suppose $\mathcal{B}$ has local character. Then $F'_{\mathcal{B}}(I) = f(\xi)$ a.a. on $\mathcal{I} \times [a,b]$.*

4 Convergence Theorems

Definition 4.1. Let $\mathcal{B}$ be a filtered set on $[a,b]$. A sequence of function $\{f_n\}$ defined on $[a,b]$ is said to be ***Riemann $\mathcal{B}$-equiintegrable*** to sequence $\{F_n\}$ on $[a,b]$ if for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$ such that for every v-fine division $D = \{(I,\xi)\}$ of $[a,b]$, for every $n \in \mathbb{N}$, we have

$$|S(f_n, v, D) - F_n| < \epsilon,$$

where $S(f_n, v, D) = (D)\sum f_n(\xi)|I|$.

Example 4.2. Let $\mathbb{Q}$ be the set of all rational number. Write $\mathbb{Q} \cap [0,1] = \{r_1, r_2, \ldots\}$. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = 1$ if $x \in \{r_1, r_2, \ldots, r_n\}$ and $f_n(x) = 0$ otherwise, i.e., $f_n(x) = \chi_{\{r_1, r_2, \ldots, r_n\}}(x)$.

Let $\epsilon > 0$ be given. Let

$$v_\epsilon(I) = \left\{r_j : I \subseteq \left(\xi - \frac{\epsilon}{2^j}, \xi + \frac{\epsilon}{2^j}\right)\right\} \cup (\mathbb{Q}^c \cap [0,1]).$$

Let $\mathcal{B} = \{v_\epsilon : \epsilon > 0\}$. Note that $\mathcal{B} \subseteq \mathcal{B}_M$. It is easy to see that the sequence of function $\{f_n\}$ is Riemann $\mathcal{B}$-equiintegrable to sequence $\{F_n \equiv 0\}$ on $[0,1]$, because

$$\left|(D)\sum f_n(\xi)|I|\right| \leq \left|(D)\sum_{\xi \in \mathbb{Q}\cap[0,1]} f_n(\xi)|I|\right| + \left|(D)\sum_{\xi \in \mathbb{Q}^c\cap[0,1]} f_n(\xi)|I|\right| < \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j-1}} = 2\epsilon$$

whenever D is a v-fine division of $[0,1]$. We also note that $f_n \to \chi_{\mathbb{Q}}$ on $[0,1]$. The next theorem shall guarantee that the $\chi_{\mathbb{Q}}$ is Riemann $\mathcal{B}$-integrable on $[0,1]$ and

$$(\mathcal{B})\int_0^1 \chi_{\mathbb{Q}} = \lim_{n\to\infty} (\mathcal{B})\int_0^1 f_n = 0.$$

Theorem 4.3. *Let $\mathcal{B}$ be a filtered set on $[a,b]$. If $\{f_n\}$ is Riemann $\mathcal{B}$-equiintegrable to $\{F_n\}$ on $[a,b]$ and $f_n \to f$ on $[a,b]$, then f is Riemann $\mathcal{B}$-integrable on $[a,b]$ and*

$$(\mathcal{B})\int_a^b f = \lim_{n\to\infty} (\mathcal{B})\int_a^b f_n.$$

Proof. Let $\epsilon > 0$ be given. There exists a tag function $v \in \mathcal{B}$ such that for every v-fine division $D = \{(I,\xi)\}$ of $[a,b]$, for every $n \in \mathbb{N}$, we have

$$|S(f_n, v, D) - F_n| < \epsilon.$$

Let $D = \{(I,\xi)\}$ be a v-fine division of $[a,b]$. Note that the division $D = \{(I,\xi)\}$ is fixed. So, the set of tax point $\{\xi : (I,\xi) \in D\}$ is finite. Since the sequence of function $\{f_n\}$ is pointwise convergent and $\{\xi : (I, \xi) \in D\}$ is finite, there exists $N(D) \in \mathbb{N}$ such that for every $m, n > N(D)$,

$$|S(f_m, v, D) - S(f_n, v, D)| = \left|(D)\sum(f_m(\xi) - f_n(\xi))|I|\right| < \epsilon.$$

Thus, for every $m, n > N(D)$, we have

$$\begin{aligned}|F_m - F_n| \leq\ & |S(f_m, v, D) - F_m| + |S(f_m, v, D) - S(f_n, v, D)| + |S(f_n, v, D) - F_n| \\ <\ & \epsilon + |S(f_m, v, D) - S(f_n, v, D)| + \epsilon < 3\epsilon.\end{aligned}$$

Note that $N(D)$ depend on D. However, $|F_m - F_n|$ does not depend on D. We use D to find N. Hence the sequence $\{F_n\}$ is a Cauchy sequence in $\mathbb{R}$. There exists a real number F such that $F_n \to F$ as $n \to \infty$.

Again, let $D = \{(I, \xi)\}$ be a v-fine division of $[a, b]$. There exists $\bar{N} \in \mathbb{N}$ such that for every $n > \bar{N}$,

$$|S(f_m, v, D) - S(f, v, D)| < \epsilon \text{ and } |F_n - F| < \epsilon.$$

Hence,

$$|S(f, v, D) - F| \leq |S(f, v, D) - S(f_n, v, D)| + |S(f_n, v, D) - F_n| + |F_n - F| < 3\epsilon,$$

that is, f is Riemann $\mathcal{B}$-integrable to F on $[a, b]$ and

$$(\mathcal{B})\int_a^b f = F = \lim_{n\to\infty} F_n = \lim_{n\to\infty} (\mathcal{B})\int_a^b f_n. \qquad \square$$

Given a sequences of functions $\{f_n\}$ and $\{F_n\}$ on $[a, b]$, define

$$\Gamma_{\epsilon,n} = \{(I, \xi) : |F_n(I) - f_n(\xi)|I|| \geq \epsilon|I|\}.$$

Definition 4.4. Let $\mathcal{B}$ be a filtered set on $[a, b]$. Sequences of function $\{f_n\}$ and $\{F_n\}$ on $[a, b]$ are said to satisfy the ***uniform double Lusin condition***, introduced in [4], if for every $\epsilon > 0$, there exists a tag function $v \in \mathcal{B}$, such that for every v-fine partial division $D = \{(I, \xi)\} \subseteq \Gamma_{\epsilon,n}$ of $[a, b]$, we have

$$(D)\sum |f_n(\xi)|I|| < \epsilon \text{ and } (D)\sum |F_n(I)| < \epsilon.$$

Example 4.5. Let $\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}$. For each $n \in \mathbb{N}$, let $g_n : [0, 1] \to \mathbb{R}$ be defined by $g_n(r_n) = 1$ and $g_n(x) = 0$ otherwise, i.e., $g_n(x) = \chi_{\{r_n\}}(x)$. Let $G_n : \mathcal{I} \to \mathbb{R}$ be defined by $G_n(I) = 0$ for all $I \subseteq [0, 1]$. Then

$$\Gamma_{\epsilon,n} = \{(I, \xi) : |G_n(I) - g_n(\xi)|I|| \geq \epsilon|I|\} = \{(I, \xi) : |g_n(\xi)|I|| \geq \epsilon|I|\} = \{(I, r_n)\}.$$

if $0 < \epsilon < 1$ and $\Gamma_{\epsilon,n}$ is empty set if $\epsilon > 1$.

Let $\mathcal{B}$ be a filtered set on $[0,1]$ defined in Example 4.2, that is, $\mathcal{B}$ is the collection of all function $v_\epsilon(I) = \left\{r_j : I \subseteq \left(\xi - \frac{\epsilon}{2^j}, \xi + \frac{\epsilon}{2^j}\right)\right\} \cup (\mathbb{Q}^c \cap I)$.

Let $\epsilon > 0$. We only consider the case when $\epsilon < 1$, because if $\epsilon > 1$, then $\Gamma_{\epsilon,n} = \emptyset$. Let $D = \{(I,\xi)\}$ be a partial division of $[0,1]$ such that $D \subseteq \Gamma_{\epsilon,n}$. Thus

$$(D)\sum |g_n(\xi)|I|| = |g_n(r_n)|I|| = |I| < \frac{\epsilon}{2^{n-1}} \leq \epsilon$$

and

$$(D)\sum |G_n(I)| = 0 < \epsilon.$$

So, the sequences of function $\{g_n\}$ and $\{G_n\}$ satisfy the uniform double Lusin condition.

Lemma 4.6. *Let $\mathcal{B}$ be a filtered set on $[a,b]$. If sequences of function $\{f_n\}$ and $\{F_n\}$ satisfy the uniform double Lusin condition on $[a,b]$, then $\{f_n\}$ is Riemann $\mathcal{B}$-equiintegrable to $\{F_n\}$ on $[a,b]$.*

Proof. Follow the proof of Lemma 3.6, replace f and F by f_n and F_n, respectively. □

The following convergence theorem is a consequence of Theorem 4.3 and Lemma 4.6.

Theorem 4.7. *Let $\mathcal{B}$ be a filtered set on $[a,b]$. If sequences $\{f_n\}$ and $\{F_n\}$ satisfy the uniform double Lusin condition on $[a,b]$, then f is Riemann $\mathcal{B}$-integrable on $[a,b]$ and*

$$(\mathcal{B})\int_a^b f = \lim_{n\to\infty} (\mathcal{B})\int_a^b f_n.$$

Lemma 4.8. *Let $\mathcal{B}$ be a filtered set on $[a,b]$. If sequence of Riemann $\mathcal{B}$-integrable function $\{f_n\}$ to $\{F_n\}$ on $[a,b]$ converge to f uniformly on $[a,b]$, then $\{f_n\}$ is Riemann $\mathcal{B}$-equiintegrable to $\{F_n\}$ on $[a,b]$.*

Proof. Let $\{f_n\}$ be a sequence of Riemann $\mathcal{B}$-integrable to $\{F_n\}$ on $[a,b]$ and $f_n \to f$ uniformly on $[a,b]$. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$, such that for every $n \geq N$, we have

$$\|f_N - f_n\|_\infty < \epsilon.$$

So, for any division D of $[a,b]$, we have

$$\left|(D)\sum f(\xi)|I| - f_N(\xi)|I|\right| \leq \|f_N - f_n\|_\infty \cdot (D)\sum |I| < \epsilon \cdot |b-a|.$$

For $n \geq N$, we also have

$$|F_N - F_n| = \left|(\mathcal{B})\int_a^b f_N - (\mathcal{B})\int_a^b f_n\right| \leq (\mathcal{B})\int_a^b \|f_N - f_n\|_\infty < \epsilon \cdot |b-a|.$$

For each $n \in \mathbb{N}$, f_n is Riemann $\mathcal{B}$-integrable to F_n on $[a, b]$, then there exists a tag function $v_n \in \mathcal{B}$ such that for every v_n-fine two divisions D of $[a, b]$, we have

$$|S(f_n, v_n, D) - F_n| < \epsilon.$$

Let $v \in \mathcal{B}$ such that v finer than v and v_n, for all $n \leq N$. Let D be v-fine division of $[a, b]$. Thus, for $n \leq N$, we have

$$|S(f_n, v, D) - F_n| < \epsilon$$

and, for $n > N$, we have

$$\begin{aligned} |S(f_n, v, D) - F_n| &\leq |S(f_n, v, D) - S(f_N, v, D)| + |S(f_N, v, D) - F_N| + |F_N - F_n| \\ &= \left|(D)\sum f(\xi)|I| - f_N(\xi)|I|\right| + \epsilon + \epsilon \cdot |b - a| \\ &< \epsilon \cdot |b - a| + \epsilon + \epsilon \cdot |b - a| = \epsilon(2|b - a| + 1). \end{aligned}$$

Hence, the sequence $\{f_n\}$ is Riemann $\mathcal{B}$-equiintegrable to sequence $\{F_n\}$ on $[a, b]$. □

The following convergence theorem is a consequence of Theorem 4.3 and Lemma 4.8.

Theorem 4.9. *Let $\mathcal{B}$ be a filtered set on $[a, b]$. If a sequence of Riemann $\mathcal{B}$-integrable function $\{f_n\}$ on $[a, b]$ converges to f uniformly on $[a, b]$, then f is Riemann $\mathcal{B}$-integrable on $[a, b]$ and*

$$(\mathcal{B})\int_a^b f = \lim_{n\to\infty}(\mathcal{B})\int_a^b f_n.$$

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