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Refinement on the Constants in the Non-Uniform Version of the Berry-Esseen Theorem

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Abstract : We improve the constant in non-uniform version of the Berry-Esseen theorem by using Stein's method with the concentration inequality on which the random variables are not necessarily identically distributed and the existence of the absolute third moment is not required.

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1 Introduction

Let $X_1, X_2, ..., X_n$ be independent and not necessarily identically distributed random varibles with zero mean and finite variance. Define

$$W_n = X_1 + X_2 + ... + X_n$$

and $VarW_n = 1$. Let F_n be the distribution function of W_n and Φ the standard normal distribution function. The central limit theorem in probability theory and statistics states that

$$F_n(x) \to \Phi(x) \text{ as } n \to \infty.$$

The Berry-Esseen theorem, also known as the Berry-Esseen inequality, attempts to quantify the rate of this convergence. Statements of the theorem vary, as it was independently discovered by two mathematicians, Andrew C.Berry (1941,[2]) and Carl-Gustav Esseen (1945,[5]), who then, along with other authors, refined it repeatly over subsequent decades.

Suppose that $E|X_i|^3 < \infty$ for $i = 1, 2, ..., n$, then we have uniform Berry-Esseen theorem

$$\sup_{x\in\mathbb{R}} |F_n(x) - \Phi(x)| \le C_0 \sum_{i=1}^{n} E|X_i|^3 \tag{1.1}$$

and the non-uniform version

$$|F_n(x) - \Phi(x)| \le \frac{C_1}{1+|x|^3} \sum_{i=1}^{n} E|X_i|^3 \tag{1.2}$$

where both C_0 ans C_1 are absolute constants.

In case of uniform bound, Berry [2] and Esseen [5] are the first two persons who obtained (1.1) in case of $X_i^\prime s$ are identically distributed. Later, Siganov [11] improved the constant down to 0.7655 in 1986 and 0.7164 by Chen [4] in 2002. Without assuming the identically distributed of $X_i^\prime s$, Beek [14] sharpened the constant to 0.7975 in 1972 and improved the constant down to 0.7915 by Siganov [11] in 1986.

For non-uniform bound, Nagaev [7] is the first one who obtained (1.2) in case of $X_i^\prime s$ are identically distributed random variables and Bikelis [1] generalized Nagaev's result to the case that $X_i^\prime s$ are not necessarily identically distributed. Paditz [9] calculated C_1 to be 114.7 in 1977 and improved his bound to be 31.395 in 1989.

Michel [6] reduced the constant to 30.84 for the independent and identically distributed case.

In 2001, Chen and Shao [3] give the new versions of (1.1) and (1.2) without assuming the existence of third moments. Their results stated as follows.

$$\sup_{x\in\mathbb{R}} |F_n(x) - \Phi(x)| \le 4.1 \sum_{i=1}^{n} \{E|X_i|^2 I(|X_i| \ge 1|) + E|X_i|^3 I(|X_i| < 1)\} \tag{1.3}$$

and

$$|F_n(x) - \Phi(x)| \le C_2 \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1+|x|)}{(1+|x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|x|)}{(1+|x|)^3} \right\} \tag{1.4}$$

where C_2 is a positive constant and $I(A)$ is the indicator random variable that is

$$I(A) = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

Neammanee [8] combined the concentration inequality in [3] with coupling approach to calculate the constant in (1.4). Here is his result.

Theorem1.1 Let $X_1, X_2, ..., X_n$ be independent ramdom variables with zero means and $\sum_{i=1}^{n} EX_i^2 = 1$. Let $W_n = X_1 + X_2 + ... + X_n$ and F_n the distribution function of W_n. Then

$$|F_n(x) - \Phi(x)| \le C_3 \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1+|\frac{x}{4}|)}{(1+|\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|\frac{x}{4}|)}{(1+|\frac{x}{4}|)^3} \right\} \tag{1.5}$$

where

$$C_3 = \begin{cases} 21.44 & \text{if } |x| \le 3 \text{ or } |x| \ge 14, \\ 32 & \text{if } 3 < |x| \le 3.99, \\ 60 & \text{if } 3.99 < |x| \le 7.98, \\ 32 & \text{if } 7.98 < |x| < 14. \end{cases}$$

We observe that the bounds in (1.5) are given in term of truncated moments and the constant obtained is 21.44 for most values. In this paper, the authors improve the concentration inequality which is used in [8] and get better constants, i.e., 9.7. for almost x. Our main result is Theorem 1.2

Theorem 1.2 Under the assumptions of Theorem 1.1, we have

$$|F_n(x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^3} \right\}$$

where

$$C = \begin{cases} 21.44 & \text{if } |x| \le 3, \\ 32 & \text{if } 3 < |x| \le 3.99, \\ 49.18 & \text{if } 3.99 < |x| \le 7.98, \\ 14.69 & \text{if } 7.98 < |x| < 14, \\ 9.7 & \text{if } |x| \ge 14. \end{cases}$$

Observe that the constants in Theorem 1.2 are sharper than that in Theorem 1.1.

2 Auxiliary results

In order to improve Theorem 1.1, we needs the auxiliary results, namely the improved concentration inequality and Proposition 2.3. Let

$$Y_{i,x} = X_i I(|X_i| < 1 + x) \ , \ S_x = \sum_{i=1}^{n} Y_{i,x} \ ,$$

$$\alpha_x = \sum_{i=1}^{n} EX_j^2 I(|X_j| \ge 1 + x), \ \beta_x = \sum_{i=1}^{n} E|X_j|^3 I(|X_j| < 1 + x),$$

$$\gamma_x = \frac{\beta_x}{2} \ \text{ and } \delta_x = \frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3} \ \text{ for } x > 0.$$

Proposition 2.1 *Let Λ be a nonempty subset of $\{1,2,...,n\}$ and $S_{\Lambda,x}=\sum_{i\in\Lambda}Y_{i,x}$. Then*

$$ES^4_{\Lambda,x}\leq\ (1+x)\beta_x+1+\frac{\alpha_x\beta_x}{1+x}+(\frac{\alpha_x}{1+x})^2+(\frac{\alpha_x}{1+x})^4.$$

Proof Note that

$$|\sum_{i\in\Lambda}EY_{i,x}|\ \leq\ \sum_{i=1}^{n}E|X_i|I(|X_i|\geq 1+x)\leq\ \sum_{i=1}^{n}\frac{E|X_i|^2I(|X_i|\geq 1+x)}{1+x}\ =\ \frac{\alpha_x}{1+x}.$$

From this inequalities and the fact that $|Y_{i,x}|<1+x$ and $\sum_{i\in\Lambda}EY^2_{i,x}\leq 1$, we have

$$\begin{aligned}ES^4_x\ &=\ E[\sum_{i\in\Lambda}Y^4_{i,x}+\sum_{i\in\Lambda}\sum_{\substack{j\in\Lambda\\ j\neq i}}Y^2_{i,x}Y^2_{j,x}+\sum_{i\in\Lambda}\sum_{\substack{j\in\Lambda\\ j\neq i}}Y^3_{i,x}Y_{j,x}\\ &\quad+\sum_{i\in\Lambda}\sum_{\substack{j\in\Lambda\\ j\neq i}}\sum_{\substack{k\in\Lambda\\ k\neq i,j}}Y^2_{i,x}Y_{j,x}Y_{k,x}+\sum_{i\in\Lambda}\sum_{\substack{j\in\Lambda\\ j\neq i}}\sum_{\substack{k\in\Lambda\\ k\neq i,j}}\sum_{\substack{l\in\Lambda\\ l\neq i,j,k}}Y_{i,x}Y_{j,x}Y_{k,x}Y_{l,x}]\\ &\leq\ \sum_{i\in\Lambda}E|Y_{i,x}|^3|Y_{i,x}|+|\sum_{i\in\Lambda}EY^2_{i,x}||\sum_{j\in\Lambda}EY^2_{j,x}|\\ &\quad+\sum_{i\in\Lambda}E|Y_{i,x}|^3|\sum_{\substack{j\in\Lambda\\ j\neq i}}EY_{j,x}|+|\sum_{i\in\Lambda}EY^2_{i,x}||\sum_{\substack{j\in\Lambda\\ j\neq i}}EY_{j,x}||\sum_{\substack{k\in\Lambda\\ k\neq i,j}}EY_{k,x}|\\ &\quad+|\sum_{i\in\Lambda}EY_{i,x}||\sum_{\substack{j\in\Lambda\\ i\neq j}}EY_{j,x}||\sum_{\substack{k\in\Lambda\\ k\neq i,j}}EY_{k,x}||\sum_{\substack{l\in\Lambda\\ l\neq i,j,k}}EY_{l,x}|\\ &\leq\ (1+x)\beta_x+1+\frac{\alpha_x\beta_x}{1+x}+(\frac{\alpha_x}{1+x})^2+(\frac{\alpha_x}{1+x})^4.\end{aligned}$$

□

Proposition 2.2 *(Concentration Inequality)*
Let $i\in\{1,2,\ldots,n\}$, $W_n^{(i)}=W_n-X_i$, and $S_{i,a}=S_{\Lambda,a}$ where $\Lambda=\{1,2,...,n\}-\{i\}$. Then for $1\leq a<b<\infty$ and $(1+a)^2\alpha_a+(1+a)\beta_a<\frac{1}{80}$, we have

$$\begin{aligned}P(a\leq W_n^{(i)}\leq b)\leq\ &\frac{(b-a+2\gamma_a)}{C(1+a)^3}\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3}+\frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2}+\frac{3(1+\gamma_a)}{(a-\gamma_a)}+1\Big)ES^4_{i,a}\\ &+\frac{1.465\times 10^{-7}\beta_a}{[0.5-(\beta_a)^{\frac{2}{3}}-2\alpha_a-C]^4(1+a)^3}+\frac{\alpha_a}{(1+a)^2}\end{aligned}$$

for any positive constant C such that $C<0.5-(\beta_a)^{\frac{2}{3}}-2\alpha_a$. Furthermore,

1. $P(a \leq W_n^{(i)} \leq b) \leq \dfrac{7.417}{(1+a)^3}(b-a) + 8.125\delta_a \quad \textit{for} \quad a \geq 2,$

2. $P(a \leq W_n^{(i)} \leq b) \leq \dfrac{5.264}{(1+a)^3}(b-a) + 7.018\delta_a \quad \textit{for} \quad a \geq 3,$

3. $P(a \leq W_n^{(i)} \leq b) \leq \dfrac{3.522}{(1+a)^3}(b-a) + 3.916\delta_a \quad \textit{for} \quad a \geq 6.$

Proof. Since $(1+a)^2\alpha_a + (1+a)\beta_a < \dfrac{1}{80}$, we note that

$$a - \gamma_a > 0 \quad \text{and} \quad 0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a > 0.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & \text{for} \quad t < a - \gamma_a, \\ (1+t+\gamma_a)^3(t-a+\gamma_a) & \text{for} \quad a-\gamma_a \leq t \leq b+\gamma_a, \\ (1+t+\gamma_a)^3(b-a+2\gamma_a) & \text{for} \quad t > b+\gamma_a. \end{cases}$$

From equations (2.19) and (2.23) in Neammanee [8] p.1958-1959, for every positive constant C.

$$P(a \leq W_n^{(i)} \leq b) \leq \frac{1}{C(1+a)^3} ES_{i,a}f(S_{i,a}) + P(U^i_{\Lambda,a} \leq C) + \frac{\alpha_a}{(1+a)^2}, \quad (2.1)$$

where $U^i_{\Lambda,a} = \sum\limits_{\substack{j\in\Lambda \\ j\neq i}} |Y_{j,x}| \min(\gamma_x, |Y_{j,x}|)$.

To bound the right hand side of (2.1), we divide the proof into two steps.
Step 1. We will prove that

$$ES_{i,a}f(S_{i,a}) \;\leq (b-a+2\gamma_a)\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3} + \frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2} + \frac{3(1+\gamma_a)}{(a-\gamma_a)} + 1\Big) ES^4_{i,a}. \quad (2.2)$$

First, we will show that

$$ES_{i,a}f(S_{i,a}) \leq ES_{i,a}I(S_{i,a} \geq a-\gamma_a)(1+S_{i,a}+\gamma_a)^3(b-a+2\gamma_a). \quad (2.3)$$

It is obvious that (2.3) holds in case of $S_{i,a} < a - \gamma_a$ and $S_{i,a} > b + \gamma_a$.
Assume that $a - \gamma_a \leq S_{i,a} \leq b + \gamma_a$. Then

$$\begin{aligned} ES_{i,a}f(S_{i,a}) &= ES_{i,a}(1+S_{i,a}+\gamma_a)^3(S_{i,a}-a+\gamma_a) \\ &= ES_{i,a}I(S_{i,a} \geq a-\gamma_a)(1+S_{i,a}+\gamma_a)^3(S_{i,a}-a+\gamma_a) \\ &\leq ES_{i,a}I(S_{i,a} \geq a-\gamma_a)(1+S_{i,a}+\gamma_a)^3((b+\gamma_a)-a+\gamma_a) \\ &= ES_{i,a}I(S_{i,a} \geq a-\gamma_a)(1+S_{i,a}+\gamma_a)^3(b-a+2\gamma_a). \end{aligned}$$

Hence, (2.3) holds. Thus

$$\begin{aligned}
&ES_{i,a}f(S_{i,a})\\
&\quad\leq\ (b-a+2\gamma_a)|ES_{i,a}I(S_{i,a}\geq a-\gamma_a)(1+S_{i,a}+\gamma_a)^3|\\
&\quad=\ (b-a+2\gamma_a)|ES_{i,a}I(S_{i,a}\geq a-\gamma_a)\{(1+\gamma_a)^3+3(1+\gamma_a)^2S_{i,a}\\
&\qquad+3(1+\gamma_a)S_{i,a}^2+S_{i,a}^3\}|\\
&\quad\leq\ (b-a+2\gamma_a)\Big\{(1+\gamma_a)^3|ES_{i,a}I(S_{i,a}\geq a-\gamma_a)|\\
&\qquad+3(1+\gamma_a)^2ES_{i,a}^2I(S_{i,a}\geq a-\gamma_a)+3(1+\gamma_a)|ES_{i,a}^3I(S_{i,a}\geq a-\gamma_a)|\\
&\qquad+ES_{i,a}^4\Big\}.
\end{aligned}$$

From this fact and the following results :

$$|ES_{i,a}I(S_{i,a}\geq a-\gamma_a)|\ \leq\ \frac{ES_{i,a}^4I(S_{i,a}\geq a-\gamma_a)}{(a-\gamma_a)^3}\ \leq\ \frac{ES_{i,a}^4}{(a-\gamma_a)^3},$$

$$|ES_{i,a}^2I(S_{i,a}\geq a-\gamma_a)|\ \leq\ \frac{ES_{i,a}^4I(S_{i,a}\geq a-\gamma_a)}{(a-\gamma_a)^2}\ \leq\ \frac{ES_{i,a}^4}{(a-\gamma_a)^2},$$

$$|ES_{i,a}^3I(S_{i,a}\geq a-\gamma_a)|\ \leq\ \frac{ES_{i,a}^4I(S_{i,a}\geq a-\gamma_a)}{(a-\gamma_a)}\ \leq\ \frac{ES_{i,a}^4}{(a-\gamma_a)},$$

we have

$$ES_{i,a}f(S_{i,a})\ \leq (b-a+2\gamma_a)\Big(\frac{(1+\gamma_a)^3}{(a-\gamma_a)^3}+\frac{3(1+\gamma_a)^2}{(a-\gamma_a)^2}+\frac{3(1+\gamma_a)}{(a-\gamma_a)}+1\Big)ES_{i,a}^4.$$

Step 2. We will show that

$$P(U_{\Lambda,a}^i\leq C)\leq\frac{1.464\times10^{-7}\beta_a}{[0.5-(\beta_a)^{\frac{2}{3}}-2\alpha_a-C]^4(1+a)^3}. \tag{2.4}$$

To bound $P(U_{\Lambda,a}^i\leq C)$, we note that

$$EU_{\Lambda,a}^i\geq0.5-(\beta_a)^{\frac{2}{3}}-2\alpha_a$$

(see Neammanee [8], p.1959). By the same argument of Proposition 2.1, we can show that

$$E|U_{\Lambda,a}^i-EU_{\Lambda,a}^i|^4\leq(16(\frac{1}{80})+1)\gamma_a^4=1.2\gamma_a^4\leq1.465\times10^{-7}\frac{\beta_a}{(1+a)^3}.$$

Then for $C < 0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a$,

$$\begin{aligned} P(U^i_{\Lambda,a} \le C) &\le P(EU^i_{\Lambda,a} - U^i_{\Lambda,a} \ge 0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C) \\ &\le \frac{E|U^i_{\Lambda,a} - EU^i_{\Lambda,a}|^4}{[0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C]^4} \\ &\le \frac{1.465 \times 10^{-7}\beta_a}{[0.5 - (\beta_a)^{\frac{2}{3}} - 2\alpha_a - C]^4(1+a)^3}. \end{aligned}$$

By (2.1),(2.2), and (2.4), we finish the proof.
The others results obtained by choosing $C = 0.43$,0.46 and 0.46 in case $a \ge 2, a \ge 3$ and $a \ge 6$, respectively. □

Proposition 2.3 *Let x be a positive real number and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(w) = (wf_x(w))'$ where f_x is the unique solution of Stein's equation*

$$f'(w) - wf(w) = I(w \le x) - \Phi(x).$$

If $(1+x)^2\alpha_x + (1+x)\beta_x < \frac{1}{80}$, then for $|u| \le 1 + \frac{x}{4}$, we have

1. $Eg(W_n^{(i)} + u) \le \dfrac{0.458}{(1+\frac{x}{4})^3} + 0.903\delta_{\frac{x}{4}}(1+x)$ *for* $x \ge 14$,

2. $Eg(W_n^{(i)} + u) \le \dfrac{1.344}{(1+\frac{x}{4})^3} + 2.534\delta_{\frac{x}{4}}(1+x)$ *for* $7.98 \le x < 14$,

3. $Eg(W_n^{(i)} + u) \le \dfrac{20.319}{(1+\frac{x}{4})^3} + 19.828\delta_{\frac{x}{4}}(1+x)$ *for* $3.99 \le x < 7.98$.

Proof. We will prove the proposition in case of $x \ge 14$ and for the other cases, we can use the same argument.
From eq.(2.44) and (2.45) of Proposition 2.4 in [8], we have

$$\begin{aligned} Eg(W_n^{(i)} + u) \ &\le \frac{2.517}{(1+x)^3} + g(x-1)P(x-1 < W_n^{(i)} + u < x) \\ &\quad + \int_{x-1}^{x} g'(w)P(w < W_n^{(i)} < x)dw \end{aligned} \tag{2.5}$$

and

$$g(x-1) \le \frac{0.056}{(1+x)^3}. \tag{2.6}$$

Since $(x-1)-u \geq (x-1)-(1+\frac{x}{4}) \geq 8.4$ for $x \geq 14$,

$$\begin{aligned} P(x-1 < W_n^{(i)} + u < x) &\leq P(W_n^{(i)} > x-1-u) \\ &\leq P(W_n^{(i)} > 8.4) \\ &\leq \frac{EW_n^2}{70.56} \\ &\leq 0.0142. \end{aligned} \tag{2.7}$$

So, by (2.5)-(2.7) and Proposition 2.2(3),

$$\begin{aligned} Eg(W^{(i)}+u) &\leq \frac{2.518}{(1+x)^3} + \int_{x-1}^{x} g'(w)P(w < W^{(i)}+u < x)dw \\ &\leq \frac{2.518}{(1+x)^3} + \int_{x-1}^{x} g'(w)[\frac{3.522}{(1+w-u)^3}(x-w) + 3.916\delta_{w-u}]dw. \end{aligned} \tag{2.8}$$

Since δ_x is decreasing in x, g is non-negative and increasing on $[0,x)$, and $|g(x)| \leq 1+|x|$, (2.8) can be bounded by

$$\begin{aligned} Eg(W^{(i)}+u) &\leq \frac{2.518}{(1+x)^3} + \frac{3.522}{(1+\frac{3x}{5})^3}\int_{x-1}^{x} g'(w)(x-w)dw + 3.916\delta_{\frac{3x}{5}}g(x) \\ &\leq \frac{2.518}{(1+x)^3} + \frac{3.522}{(1+\frac{3x}{5})^3}\int_{x-1}^{x} (x-w)dg(w) + 3.916(1+x)\delta_{\frac{3x}{5}} \\ &\leq \frac{2.518}{(1+x)^3} + \frac{3.522}{(1+\frac{3x}{5})^3}\int_{x-1}^{x} g(w)dw + 3.916(1+x)\delta_{\frac{3x}{5}} \\ &= \frac{2.518}{(1+x)^3} + \frac{3.522}{(1+\frac{3x}{5})^3}(xf_x(x)) + 3.916(1+x)\delta_{\frac{3x}{5}} \\ &\leq \frac{0.458}{(1+\frac{x}{4})^3} + 0.903\delta_{\frac{x}{4}}(1+x) \end{aligned}$$

where we have applied the fact that $|xf_x(x)| \leq 1$([13], p.23), $\frac{1+\frac{x}{4}}{1+x} \leq 0.3$, $\frac{1+\frac{x}{4}}{1+\frac{3x}{5}} \leq 0.48$ and $\delta_{\frac{3x}{5}} \leq \frac{(1+\frac{x}{4})^2}{(1+\frac{3x}{5})^2}\delta_{\frac{x}{4}} \leq 0.23\delta_{\frac{x}{4}}$ for $x \geq 14$ in the last inequality. □

We note that Proposition 2.2 and Proposition 2.3 are the improvement of the following results of [8].

Proposition 2.4 *Let $i \in \{1,2,\ldots,n\}$ and $W^{(i)} = W - X_i$. Then for $3 \leq a < b < \infty$ and $(1+a)^2\alpha_a + (1+a)\beta_a < \frac{1}{80}$, we have*

$$P(a \leq W^{(i)} \leq b) \leq \frac{40.98}{(1+a)^3}(b-a) + 46.38\delta_a.$$

Proposition 2.5 *Let $x \geq 14$. If $(1+x)^2\alpha_x + (1+x)\beta_x < \frac{1}{80}$, then for $|u| \leq 1 + \frac{x}{4}$, we have*

$$Eg(W^{(i)} + u) \leq \frac{4.60}{(1+\frac{x}{4})^3} + 5.13\delta_{\frac{x}{4}}(1+x).$$

We are now ready to prove our main result.

3 Main Result

Our main result is an improvement of the constant in Theorem 1.1. The techniques and tools used in proving the result are the improved concentration inequality and Proposition 2.3.

Proof of Theorem 1.2
It suffices to assume that $x \geq 0$ as we can simply apply the result to $-W$ when $x < 0$. Since $W_n = S_x$ if $\max\limits_{1 \leq i \leq n} |X_i| < 1 + x$,

$$\begin{aligned}
|P(W_n \leq x) - \Phi(x)| &\leq P(W_n \neq S_x) + |P(S_x \leq x) - \Phi(x)| \\
&\leq P(\max_{1 \leq j \leq n} |X_i| \geq 1 + x) + |P(S_x \leq x) - \Phi(x)| \\
&\leq \sum_{i=1}^{n} P(|X_i| \geq 1 + x) + |P(S_x \leq x) - \Phi(x)| \\
&\leq \sum_{i=1}^{n} \frac{E|X_i|}{1+x} + |P(S_x \leq x) - \Phi(x)| \\
&\leq \sum_{i=1}^{n} \frac{EX_i^2 I(|X_i| \geq 1+x)}{(1+x)^2} + |P(S_x \leq x) - \Phi(x)| \\
&\leq \frac{\alpha_x}{(1+x)^2} + |P(S_x \leq x) - \Phi(x)|. \qquad (3.1)
\end{aligned}$$

From the fact that

$$\begin{aligned}
|P(S_x \leq x) - \Phi(x)| &\leq P(S_x > x) + (1 - \Phi(x)) \\
&\leq \frac{ES_x^4}{x^4} + (1 - \Phi(x))
\end{aligned}$$

and $1 - \Phi(x) \leq \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}$ when $x > 0$ we have

$$\begin{aligned}
|P(W_n \leq x) - \Phi(x)| &\leq \frac{\alpha_x}{(1+x)^2} + \frac{ES_x^4}{x^4} + (1 - \Phi(x)) \\
&\leq \frac{\alpha_x}{(1+x)^2} + \frac{ES_x^4}{x^4} + \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}. \qquad (3.2)
\end{aligned}$$

Case1. $x \geq 14$.

Subcase 1.1 $(1+x)^2\alpha_x + (1+x)\beta_x \geq \dfrac{1}{80}$.

Since $e^{\frac{x^2}{2}} \geq 60x^3$ and $\dfrac{1+x}{x} \leq 1.072$ for $x \geq 14$, by Porposition 2.1 and (3.2), we have

$$
\begin{aligned}
|P(W_n \leq x) - \Phi(x)| &\leq \frac{\alpha_x}{(1+x)^2} + \frac{ES_x^4}{x^4} + \frac{1}{60\sqrt{2\pi}x^4} \\
&\leq \frac{\alpha_x}{(1+x)^2} + \frac{1}{x^4}\Big[(1+x)\beta_x + \frac{\alpha_x\beta_x}{1+x} + (\frac{\alpha_x}{1+x})^2 + (\frac{\alpha_x}{1+x})^4\Big] + \frac{1.0066}{x^4} \\
&\leq 1.327(\frac{\alpha_x}{(1+x)^2}) + \frac{\beta_x}{(1+x)^3}) + \frac{1.329}{(1+x)^4} \\
&\leq 1.327\ \delta_x + \frac{1.329(80)}{(1+x)^4}\{(1+x)^2\alpha_x + (1+x)\beta_x\} \\
&= 107.73\delta_x \\
&\leq 9.7\delta_{\frac{x}{4}}
\end{aligned}
$$

where the fact that $\delta_x \leq (\dfrac{1+\frac{x}{4}}{1+x})^2\delta_{\frac{x}{4}} \leq (0.3)^2\delta_{\frac{x}{4}}$ is used in the last inequality.

Subcase 1.2 $(1+x)^2\alpha_x + (1+x)\beta_x < \dfrac{1}{80}$.

Let $K_{i,\frac{x}{4}}(t) = EY_{i,\frac{x}{4}}\{I(0 < t \leq Y_{i,\frac{x}{4}}) - I(Y_{i,\frac{x}{4}} \leq t < 0)\}$. From pp.250-251 of [3], we set

$$F(x) - \Phi(x) = R_1 + R_2 + R_3 + R_4 \tag{3.3}$$

where

$$
\begin{aligned}
R_1 &= \sum_{i=1}^{n} E\{I(|X_i| < 1 + \frac{x}{4})\int_{|t|\leq 1+\frac{x}{4}} (f'_x(W_n^{(i)} + X_i) - f'_x(W_n^{(i)} + t))K_{i,\frac{x}{4}}(t)dt\}, \\
R_2 &= \sum_{i=1}^{n} E\{I(|X_i| \geq 1 + \frac{x}{4})\int_{|t|\leq 1+\frac{x}{4}} (f'_x(W_n^{(i)} + X_i) - f'_x(W_n^{(i)} + t))K_{i\frac{x}{4}}(t)dt\}, \\
R_3 &= \alpha_{\frac{x}{4}} Ef'_x(W_n), \\
R_4 &= -\sum_{i=1}^{n} E\{X_i I(|X_i| \geq 1 + \frac{x}{4})(f_x(W_n) - f_x(W_n^{(i)}))\}.
\end{aligned}
$$

By the facts that

$$E|f'_x(W_n)| \leq \frac{15}{(1+x)^2} \text{ for } x \geq 2 \text{ ([8] p.1960)},$$

$$0 \leq f_x(w) \leq \min(\frac{\sqrt{2\pi}}{4}, \frac{1}{|x|}) \text{ for } x \in \mathbb{R} \text{ ([13] p.23-24)}$$

and

$$|f'_x(s) - f'_x(t)| \leq 1 \text{ for all } x, s, t \in \mathbb{R} \text{ ([3] p.246)}$$

we have

$$
\begin{aligned}
&|R_2+R_3+R_4| \\
&\le \sum_{i=1}^{n} P(|X_i| \ge 1+\frac{x}{4}) + \frac{15\alpha_{\frac{x}{4}}}{(1+x)^2} + \sum_{i=1}^{n} \frac{E|X_i|I(|X_i| \ge 1+\frac{x}{4})}{x} \\
&\le \sum_{i=1}^{n} \frac{EX_i^2 I(|X_i| \ge 1+\frac{x}{4})}{(1+\frac{x}{4})^2} + \frac{1.35\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} + \sum_{i=1}^{n} \frac{EX_i^2 I(|X_i| \ge 1+\frac{x}{4})}{(1+\frac{x}{4})x} \\
&= \frac{\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} + \frac{1.35\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} + \frac{0.32\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2} \\
&\le \frac{2.67\alpha_{\frac{x}{4}}}{(1+\frac{x}{4})^2}. \qquad (3.4)
\end{aligned}
$$

Note that we use the fact that $\frac{1+\frac{x}{4}}{1+x} \le 0.3$ and $\frac{1+\frac{x}{4}}{x} \le 0.32$ for $x \ge 14$ in the third inequality.

Note that $|R_1| \le R_{11} + R_{12}$ where

$R_{11} = \sum_{i=1}^{n} |E\{I(|X_i| < 1+\frac{x}{4}) \int_{|t| \le 1+\frac{x}{4}} K_{i,\frac{x}{4}}(t) \int_{t}^{X_i} Eg(W^{(i)}+u)dudt\}|$ and

$R_{12} = \sum_{i=1}^{n} E\{I(|X_i| < 1+\frac{x}{4}) \int_{|t| \le 1+\frac{x}{4}} P(x - \max(t,X_i) \le W^{(i)} \le x - \min(t,X_i)|X_i) K_{i,\frac{x}{4}}(t)dt\}$

([3] p.251).

By Proposition 2.3(1), we have

$$
\begin{aligned}
R_{11} &\le 2\Big[\frac{0.458}{(1+\frac{x}{4})^3} + 0.903(1+x)\delta_{\frac{x}{4}}\Big] \sum_{i=1}^{n} E|Y_{i,\frac{x}{4}}|^3 \\
&\le \frac{0.916\beta_{\frac{x}{4}}}{(1+\frac{x}{4})^3} + 1.806(1+x)\delta_{\frac{x}{4}}\beta_{\frac{x}{4}} \\
&\le \frac{0.916\beta_{\frac{x}{4}}}{(1+\frac{x}{4})^3} + 0.023\delta_{\frac{x}{4}} \qquad (3.5)
\end{aligned}
$$

where we have used the result that

$$\beta_{\frac{x}{4}} = \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1+\frac{x}{4}) \le \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < 1+x) = \beta_x \le \frac{1}{80(1+x)}$$

in the last inequality.

By Proposition 2.2(3) and the inequalities

$$x - \max(t,X_i) \;\ge\; x - (1+\frac{x}{4}) \;=\; \frac{3x}{4} - 1 \;\ge\; \frac{2x}{3} \ge 9.3 \qquad (3.6)$$

for $|X_i|,\ |t| \leq 1 + \dfrac{x}{4}$, we have

$$\begin{aligned}
|R_{12}| &\leq \sum_{i=1}^{n} E\Big\{I(|X_i| \leq 1 + \frac{x}{4}) \int_{|t|\leq 1+\frac{x}{4}} \Big(\frac{3.522}{(1+\frac{2x}{3})^3}(|t| + |X_i|) + 3.916\delta_{\frac{2x}{3}}\Big) K_{i,\frac{x}{4}}(t)dt\Big\} \\
&\leq \frac{7.044}{(1+\frac{2x}{3})^3}\beta_{\frac{x}{4}} + 3.916\delta_{\frac{2x}{3}} \\
&\leq \frac{0.584}{(1+\frac{x}{4})^3}\beta_{\frac{x}{4}} + 0.325\delta_{\frac{x}{4}} \qquad (3.7)
\end{aligned}$$

where we have used the fact that $\dfrac{1+\frac{x}{4}}{1+\frac{2x}{3}} \leq 0.436$ and $\delta_{\frac{2x}{3}} \leq \dfrac{(1+\frac{x}{4})^2}{(1+\frac{2x}{3})^2}\delta_{\frac{x}{4}} \leq 0.190\delta_{\frac{x}{4}}$ for $x \geq 14$ in the last inequality.
Hence, by (3.3)-(3.7) we have

$$|F(x) - \Phi(x)| \leq |R_1 + R_2 + R_3 + R_4| \leq 3.02\delta_{\frac{x}{4}}.$$

Case2. $7.98 < x < 14$.
We use the same argument as in case 1. by using Proposition 2.3(2) and Proposition 2.2(2) to bound (3.5) and (3.7), respectively.
Case3. $3.99 < x \leq 7.98$.
We use the same argument as in case 1. by using Proposition 2.3(3) and Proposition 2.2(1) to bound (3.5) and (3.7), respectively, and replacing inequality (3.6) by the following inequality

$$x - \max(t, X_i) \geq x - (1 + \frac{x}{4}) = \frac{3x}{4} - 1 = 2.$$

□

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