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Some Common Fixed Point Theorems For Two Weakly Compatible Mappings in Complex Valued Metric Spaces

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Abstract : In this paper we prove some common fixed point theorems for a pair of weakly compatible mappings satisfying (CLRg) property in complex valued metric spaces. Here we also prove some common fixed point results using property (E.A).

Keywords : complex valued metric space; weakly compatible mappings; (CLRg) property; property (E.A); common fixed point.
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1 Introduction, Definitions and Notations

In 2011, A. Azam, B. Fisher and M. Khan [1] introduced the notion of complex valued metric spaces and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. Complex valued metric space is a generalization of classical metric space. S. Bhatt, S. Chaukiyal and R.C. Dimri [2] have proved a theorem of common fixed point for weakly compatible mappings in a complex valued metric space. Recently, R.K. Verma and H.K. Pathak [3] introduced the concept of the property (E.A) in complex valued metric spaces to prove a common fixed point theorem for two pairs of weakly compatible mappings with property (E.A) and a common fixed point theorem using (CLRg) property which was introduced by Sintunavarat and Kumam [4] . The aim of this paper is to obtain some common fixed point theorems under a contractive condition of two weakly compatible self-maps satisfying the (CLRg) property.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \le Re(z_2) \text{ and } Im(z_1) \le Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$

(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of $(2), (3)$ and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1.1. *We can easily check the following:*

(i) $a, b \in \mathbb{R}, a \le b \Rightarrow az \precsim bz \quad \forall \ z \in \mathbb{C}.$

(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|.$

(iii) $z_1 \precsim z_2$ *and* $z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$

Azam et al. [1] defined the complex valued metric space in the following way:

Definition 1.2 ([1])**.** Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(C1) $0 \precsim d(x, y)$, for all $x,\ y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C3) $d(x, y) \precsim d(x, z) + d(z, y), \quad$ for all $x, y, z \in X$.

Then d is called a *complex valued metric* on X and (X, d) is called a *complex valued metric space.*

Example 1.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = i\,|z_1 - z_2| \quad \forall \ z_1, z_2 \in X.$$

One can easily check that (X, d) is a complex valued metric space.

Definition 1.4 ([1])**.** Let (X, d) be a complex valued metric space. Then

(i) A point $x \in X$ is called an *interior point* of a set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

A subset $A \subseteq X$ is called *open* if each element of A is an interior point of A.

(ii) A point $x \in X$ is called a limit point of $A \subseteq X$ if for every $0 \prec r \in \mathbb{C}$,

$$B(x,r) \cap (A - \{x\}) \neq \phi.$$

A subset $A \subseteq X$ is called *closed* if each element of $X - A$ is not a limit point of A.

(iii) The family

$$F = \{B(x,r) : x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on X.

Definition 1.5 ([1])**.** Let (X,d) be a complex valued metric space. Then

- (i) A sequence $\{x_n\}$ in X is said to *converge* to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \prec r$ for all $n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
- (ii) If for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec r$ for all $n > N$, $m \in \mathbb{N}$, then $\{x_n\}$ is called a *Cauchy sequence* in (X,d).
- (iii) If every Cauchy sequence in X is convergent in X then (X,d) is called a *complete complex valued metric space*.

Jungck introduced commuting mappings in [5], compatible mappings in [6] and weakly compatible mappings in [7]. Here similarly one can define these concepts in complex valued metric space.

Definition 1.6. Let (X,d) be a complex valued metric space. The self-maps S and T are said to be *commuting* if $STx = TSx$ for all $x \in X$.

Definition 1.7. Let (X,d) be a complex valued metric space. The self-maps S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 1.8. Let (X,d) be a complex valued metric space. The self-maps S and T are said to be *weakly compatible* if $STx = TSx$ whenever $Sx = Tx$, that is they commute at their coincidence points.

Definition 1.9 ([3])**.** Let (X,d) be a complex valued metric space. The self-maps S and T are said to satisfy the *property (E.A)* if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

The (CLRg) property is more powerful than property (E.A), which was defined by Sintunavarat and Kumam [4] in a metric space for a pair of self-mappings.

Definition 1.10 (The (CLRg) property [4])**.** Suppose that (X, d) is a metric space and f, $g : X \to X$. Then f and g are said to satisfy the *(CLRg) property* if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$, for some $x \in X$.

In a similar manner if (X, d) is a complex valued metric space then two self mappings f and g of X are said to satisfy the *(CLRg) property* if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$, for some $x \in X$.

Example 1.11. Let $X = \mathbb{C}$ and d be any complex valued metric on X. Define f, $g : X \to X$ by $fz = 2z + i$ and $gz = 3z - 1$, for all $z \in X$. Consider a sequence $\{z_n\} = \left\{i + 1 + \frac{1}{n}\right\}$ in X

Then $\lim_{n\to\infty} fz_n = \lim_{n\to\infty} \left(3i + 2 + \frac{2}{n}\right) = 3i + 2$ and

$\lim_{n\to\infty} gz_n = \lim_{n\to\infty} \left(3i + 2 + \frac{3}{n}\right) = 3i + 2 = g\,(i + 1)$

Thus f and g satisfy the (CLRg) property. Here this pair also satisfy the (CLR$_f$) property.

Definition 1.12 ([3])**.** The *'max' function* for the partial order $\precsim$ is defined as follows:

(1) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.

(2) $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.

(3) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$.

Lemma 1.13 ([1])**.** *Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.14 ([1])**.** *Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Lemma 1.15 ([8])**.** *Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then for any $a \in X$, $\lim_{n\to\infty} d(x_n, a) = d(x, a)$.*

2 Main Results

Theorem 2.1. *Let (X, d) be a complex valued metric space and $S, T : X \to X$ be weakly compatible mappings such that*

(*i*) *S and T satisfy (CLR$_S$) property and*

(*ii*) $d\,(Tx, Ty) \precsim \lambda d(Sx, Sy) + \frac{\mu d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}$
for all $x, y \in X$ and λ, μ are nonnegative reals with $\lambda + \mu < 1$.

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy (CLR$_S$) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Su, \text{ for some } u \in X. \tag{2.1}$$

From condition (ii)

$$d(Tx_n, Tu) \precsim \lambda d(Sx_n, Su) + \frac{\mu d(Tx_n, Su) d(Tu, Sx_n)}{1 + d(Sx_n, Su)}.$$

Hence

$$|d(Tx_n, Tu)| \leq \lambda\, |d(Sx_n, Su)| + \frac{\mu\, |d(Tx_n, Su)|\, |d(Tu, Sx_n)|}{|1 + d(Sx_n, Su)|}.$$

Now from Lemma 1.13 and equation (2.1) we have $|d(Sx_n, Su)| \to 0$, $|d(Tx_n, Su)| \to 0$ as $n \to \infty$. Thus $|d(Tx_n, Tu)| \to 0$ as $n \to \infty$ e.i., $\lim_{n\to\infty} Tx_n = Tu$ and so $Su = Tu$. Since S and T are weakly compatible

$$TTu = TSu = STu = SSu. \tag{2.2}$$

Again from condition (ii)

$$d(Tx_n, TTu) \precsim \lambda d(Sx_n, STu) + \frac{\mu d(Tx_n, STu) d(TTu, Sx_n)}{1 + d(Sx_n, STu)}.$$

Letting $n \to \infty$ and using Lemma 1.15 and equation (2.2) we have

$$\begin{aligned} d(Tu, TTu) &\precsim \lambda d(Tu, TTu) + \frac{\mu d(Tu, TTu) d(TTu, Tu)}{1 + d(Tu, TTu)} \\ &\precsim (\lambda + \mu) d(Tu, TTu). \end{aligned}$$

Thus $(1 - \lambda - \mu)\, |d(Tu, TTu)| \leq 0$.

Since $0 \leq \lambda + \mu < 1$, we have $|d(Tu, TTu)| = 0$ and hence $TTu = Tu$.

Therefore $STu = TTu = Tu$.

Thus Tu is a common fixed point of S and T.

Finally for the uniqueness part of the result suppose that $Sw = Tw = w$ for some $w \in X$.

From condition (ii)

$$\begin{aligned} d(Tu, w) &= d(Tu, Tw) \\ &\precsim \lambda d(Su, Sw) + \frac{\mu d(Tu, Sw) d(Tw, Su)}{1 + d(Su, Sw)} \\ &= \lambda d(Tu, w) + \frac{\mu d(Tu, w) d(w, Tu)}{1 + d(Tu, w)} \\ &\precsim (\lambda + \mu) d(Tu, w). \end{aligned}$$

Thus

$$(1-\lambda-\mu)\,|d(Tu,w)| \le 0.$$

Since $0 \le \lambda+\mu < 1$, we have $|d(Tu,w)| = 0$ and so $Tu = w$.
Therefore Tu is the unique common fixed point of S and T. □

Corollary 2.2. *Let (X,d) be a complex valued metric space and $S,T : X \to X$ be weakly compatible mappings such that*

(i) S and T satisfy (CLR$_T$)property,

(ii) $TX \subset SX$ and

(iii) $d\,(Tx,Ty) \precsim \lambda d(Sx,Sy) + \frac{\mu d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}$ for all $x,y \in X$ and λ,μ are nonnegative reals with $\lambda+\mu<1$.

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy (CLR$_T$) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tv, \quad \text{for some} \quad v \in X.$$

Since $TX \subset SX,\ Tv = Su$ for some $u \in X$. Thus S and T satisfy (CLR$_S$)property. Hence by Theorem 2.1, S and T have a unique common fixed point. □

Corollary 2.3. *Let (X,d) be a complex valued metric space and $S,T : X \to X$ be weakly compatible mappings such that*
(i) S and T satisfy property (E.A),
(ii) SX is a complete subspace of X and

$$(iii)\ d\,(Tx,Ty) \precsim \lambda d(Sx,Sy) + \frac{\mu d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}$$

for all $x,y \in X$ and λ,μ are nonnegative reals with $\lambda+\mu<1$.
Then S and T have a unique common fixed point.

Proof. Since S and T satisfy property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \ \text{for some } t \in X.$$

Now since SX is a complete subspace of X, there exists $u \in X$ such that $Su = t$. Thus S and T satisfy (CLR$_S$)property. Hence by Theorem 2.1, S and T have a unique common fixed point. □

We may now give an example in support of Theorem 2.1.

Example 2.4. Let $X = [0, \infty)$ with the complex valued metric

$$d(x, y) = i\,|x - y| \ \text{, for all } x, y \in X.$$

Define $S, T : X \to X$ by

$$\begin{aligned} Sx &= 2x \text{ , for all } \ x \in X \\ \text{and} \quad Tx &= 0 \text{ , for all } \ x \in X \end{aligned}$$

Then we note that

(1) S and T are weakly compatible since at the coincidence point 0, $ST(0) = TS(0)$.

(2) S and T satisfy (CLR_S) property.

(3) $d\,(Tx, Ty) \precsim \lambda d(Sx, Sy) + \frac{\mu d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}$
for all $x, y \in X$ and λ, μ are nonnegative reals with $\lambda + \mu < 1$.

Thus all the hypothesis of the Theorem 2.1 hold. The unique common fixed point of S and T is 0.

Theorem 2.5. *Let (X, d) be a complex valued metric space and $S, T : X \to X$ be weakly compatible mappings such that*

(i) *S and T satisfy (CLR_S) property and*

(ii) $d\,(Tx, Ty) \precsim k.\max\left\{d(Sx, Sy), \frac{d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}\right\}$
for all $x, y \in X$ and k is a real with $0 < k < 1$.

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy (CLR_S) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sa, \ \text{ for some } a \in X. \tag{2.3}$$

From condition (ii)

$$d(Tx_n, Ta) \precsim k.\max\left\{d(Sx_n, Sa), \frac{d(Tx_n, Sa)d(Ta, Sx_n)}{1 + d(Sx_n, Sa)}\right\}.$$

Letting $n \to \infty$ and using Lemma 1.15 and equation (2.3) we have

$$\begin{aligned} d(Sa, Ta) &\precsim k.\max\left\{d(Sa, Sa), \frac{d(Sa, Sa)d(Ta, Sa)}{1 + d(Sa, Sa)}\right\} \\ &= 0. \end{aligned}$$

Thus $Sa = Ta$.

Since S and T are weakly compatible we have

$$TTa = TSa = STa = SSa. \tag{2.4}$$

Again from condition (ii)

$$d(TTa, Tx_n) \precsim k.\max\left\{d(STa, Sx_n), \frac{d(TTa, Sx_n)d(Tx_n, STa)}{1 + d(STa, Sx_n)}\right\}.$$

Letting $n \to \infty$ and using Lemma 1.15 and equation (2.4) we have

$$\begin{aligned} d(TTa, Ta) &\precsim k.\max\left\{d(TTa, Ta), \frac{d(TTa, Ta)d(Ta, TTa)}{1 + d(TTa, Ta)}\right\} \\ &= k.d(TTa, Ta). \end{aligned}$$

Thus

$$(1 - k)\,|d(TTa, Ta)| \leq 0.$$

Since $0 < k < 1$, we must have $d(TTa, Ta) = 0$.
Thus $TTa = Ta$ and so $STa = TTa = Ta$.
Hence Ta is common fixed point of S and T.
To prove the uniqueness part let us suppose that $Sw = Tw = w$, for some $w \in X$.
Then

$$\begin{aligned} d(Ta, w) &= d(Ta, Tw) \\ &\precsim k.\max\left\{d(Sa, Sw), \frac{d(Ta, Sw)d(Tw, Sa)}{1 + d(Sa, Sw)}\right\} \\ &= k.\max\left\{d(Ta, w), \frac{d(Ta, w)d(w, Ta)}{1 + d(Ta, w)}\right\} \\ &= k.d(Ta, w). \end{aligned}$$

Thus

$$(1 - k)\,|d(Ta, w)| \leq 0.$$

Since $0 < k < 1$, we have $|d(Ta, w)| = 0$ and so $Ta = w$. Hence Ta is the unique common fixed point of S and T. □

Corollary 2.6. *Let (X, d) be a complex valued metric space and $S, T : X \to X$ be weakly compatible mappings such that*

(i) S and T satisfy (CLR_T) property ,

(ii) $TX \subset SX$,

(iii) $d(Tx, Ty) \precsim k.\max\left\{d(Sx, Sy), \frac{d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}\right\}$ for all $x, y \in X$ and k is a real with $0 < k < 1$.

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy (CLR_T) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tb, \quad \text{for some} \quad b \in X.$$

Since $TX \subset SX,\ Tb = Sa$ for some $a \in X$. Thus S and T satisfy (CLR_S)property. Hence by Theorem 2.5, S and T have a unique common fixed point. □

Corollary 2.7. *Let (X, d) be a complex valued metric space and $S, T : X \to X$ be weakly compatible mappings such that*

(i) S and T satisfy property (E.A),

(ii) SX is a complete subspace of X and

$$(iii)\ d(Tx, Ty) \precsim k.\max\left\{d(Sx, Sy), \frac{d(Tx, Sy)d(Ty, Sx)}{1 + d(Sx, Sy)}\right\}$$

for all $x, y \in X$ and k is a real with $0 < k < 1$.

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \ \text{ for some } t \in X.$$

Now since SX is a complete subspace of X, there exists $a \in X$ such that $Sa = t$.

Thus S and T satisfy (CLR_S)property.

Hence by Theorem 2.5, S and T have a unique common fixed point. □

Here also we may give an example in support of Theorem 2.5.

Example 2.8. Let $X = [2, \infty)$ with the complex valued metric

$$d(x, y) = i\,|x - y| \ \text{, for all } x, y \in X.$$

Define $S, T : X \to X$ by

$$\begin{aligned} Tx &= 2 \text{ , for all } x \in X \\ \text{and} \quad Sx &= x \text{ if } 2 \le x < 4 \text{ , } Sx = 5 \text{ if } x \ge 4. \end{aligned}$$

Then clearly

(1) S and T are weakly compatible since at the coincidence point 2, $ST(2) = TS(2)$,

(2) S and T satisfy (CLR_S) property.

(3) $d(Tx, Ty) \precsim k.\max\left\{d(Sx, Sy), \frac{d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}\right\}$ for all $x, y \in X$ and k is a real with $0 < k < 1$.

(4) 2 is the unique common fixed point of S and T.

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