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Using Recurrence Relation to Count a Number of Perfect Matching in Linear Chain and Snake Chain Graphs

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Abstract : This paper presents the recurrence relation using to count a number of perfect matchings in linear chain and snake chain graphs. These graphs are offen found in the chemical structure. A perfect matching graph M is a subgraph of G where there are no edges in M adjacent to each other and $V(M) = V(G)$. $\phi(G)$ is a number of perfect matching of G which leads to important chemical properties.

The results show that a number of perfect matching of a linear chain graph depends on parity of faces and number of edges in each face. A number of perfect matching of a snake chain graph depends on parity of the chain.

Keywords : perfect matching; recurrence relation; linear chain graph; snake chain graph.

2010 Mathematics Subject Classification : Primary 05C70; Secondary 05C62; 11B37.

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1 Introduction

Graphs are mathematical structures consisting of vertices and edges. We let $G(V, E)$ be a graph such that V is a vertex set and E is an edge set. Graph Theory is one of the most popular mathematical model in study, research and use to solve problems, which include logistics, electronics, industry and business management, biochemistry (genomics), electrical engineering (communications networks and coding theory), computer science (algorithms and computations) and operations research (scheduling) [1]. There are many applications of graph theory, but they remain scattered in the literature [2,3]. Some interesting application of graph theory was used to study speech patterns of both manics and schizophrenics in hopes of creating a less objective and more quantitative means of patient diagnosis [4,5].

In this paper, we present a method to count a number of perfect matching in linear chain and snake chain graphs. A matching graph M is a subgraph of a graph G where there are no edges adjacent to each other(See Figure 1). If $V(M) = V(G)$, we will call M a *perfect matching*. Let $\phi(G)$ be a number of all perfect matchings of G. Does G always have a perfect matching? How many perfect matchings are there? We want to answer these questions.

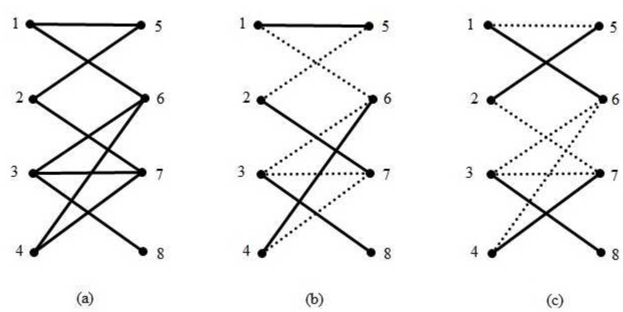

Figure 1. (b) and (c) are two different perfect matchings of graph (a)

Recurrence relation is a proof technique in mathematics. It is an equation that recursively defines a sequence or multidimensional array of values. Once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms. We will use recurrence relation to count a number of perfect matching of chain graphs.

Definition 1.1. Given a sequence $a_{g(0)}, a_{g(1)}, a_{g(2)}, ..., a_{g(n)}$, a *recurrence relation* is an equation which defines the n^{th} term in the sequence as a function f based on the previous terms:

$$a_{g(n)} = f(a_{g(0)}, a_{g(1)}, a_{g(2)}, ..., a_{g(n-1)})$$

An example of using recurrence relation is the Towers of Hanoi Problem [6]. There are three pegs: first peg having a stack of n disks, each smaller in diameter than the one below it. An allowable move consists of a disk from 1^{st} peg and put it onto another peg so that it is not over another disk of smaller size. The propose of this problem is to move the entire disks to another peg and determine the minimum number of moves. To solve the problem on moving all disks to 3^{rd} peg, we deal with the problem of moving $n-1$ disks to 2^{nd} peg, then move n^{th} disk to 3^{rd} peg, and then deal with the problem of moving the $n-1$ disks on 2^{nd} peg to 3^{rd} peg. Thus if a_n is the number of moves needed to move n disks from one to another, we have

$$a_n = 2a_{n-1} + 1$$

We have recurrence relation of *Towers of Hanoi problem* as follows:

$$a_{g(n)} = f(a_{g(0)}, a_{g(1)}, a_{g(2)}, ..., a_{g(n-1)}) = \begin{cases} 0 & \text{if } n = 0 \\ 2a_{n-1} + 1 & \text{otherwise} \end{cases}$$

2 Perfect Matching of Linear Chain Graph

A graph G is called *linear chain* if it consists of a chain of regular polygons with even number of edges and each adjacent pair of faces share exactly one edge such that all shared edges are parallel up to isomorphism. Each face is adjacent to at most two other faces.

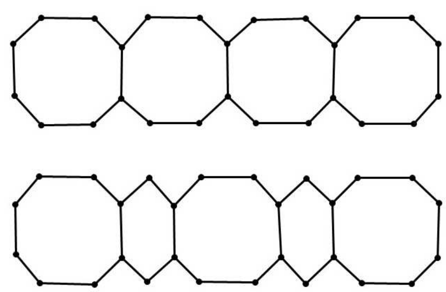

Figure 2. Example of linear chain

The face in G whose the number of edges is divisible by 4 is called a *blue face(B_i)*. The face in G whose the number of edges divided by 4 has remainder 2 is called a *red face(R_i)*. We denote the faces by its colors B_i and R_i.

Theorem 2.1. *Let G be a linear chain graph with n faces. If the number of edges in every face divided by 4 has remainder 2 then $\phi(G) = n + 1$.*

Proof. Let G be a linear chain of red faces. Let r_n be the number of all perfect matching of linear chain graph of n red faces. Then we get that $r_n = \phi(G)$. Let M be a perfect matching of G. Since every faces are red, we can define e_i as an edge shared between R_i and R_{i+1} as shown here. Let e_0 and e_n be edge opposite to e_1 in R_1 and edge opposite to e_{n-1} in R_n, respectively.

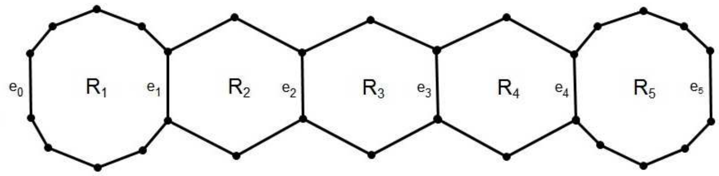

Figure 3. Definition of e_i in R_i and R_{i+1}

If M contains e_0 of R_1, it must also contains both adjacent edges of e_i in R_i for all i. Hence there is only one of perfect matching in G that contains e_0. If instead given $e_0 \notin M$, then both adjacent edges of e_0 in R_1 belong to M. Because adjacent edges of e_1 in R_1 do not belong to M, the remaining edges of M can now be in any perfect matching of G without R_1. Hence there are r_{n-1} perfect matching in G not containing e_0. Conclude that $\phi(G) = r_{n-1} + 1$. Since $r_0 = 1$ and $r_1 = 2$, $\phi(G) = n + 1$.

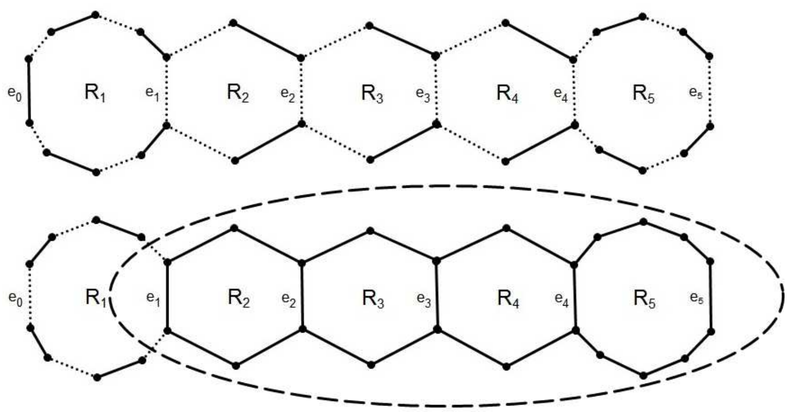

Figure 4. Counting Methods for Theorem 2.1. □

Theorem 2.2. *Let G be a linear chain graph with n faces and $b_n = \phi(G)$. If the number of edges in every face is divisible by 4 then $b_0 = 1$, $b_1 = 2$ and $b_n = b_{n-1} + b_{n-2}$ for $n \geqslant 2$.*

Proof. Let G be a linear chain of blue faces. Let b_n be the number of all perfect matching of linear chain graph of n blue faces. Then we get that $b_n = \phi(G)$. Let M be a perfect matching of G. Since every faces is blue, we can define e_i as an edge shared between B_i and B_{i+1} as shown here. Let e_0 and e_n be edge opposite to e_1 in B_1 and edge opposite to e_{n-1} in B_n, respectively.

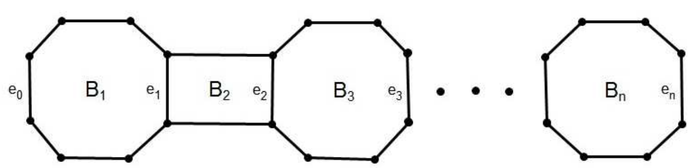

Figure 5. Definition of e_i in B_i and B_{i+1}

If M contains e_0 of B_1, then both adjacent edges of e_1 in B_1 do not belong to M and the remaining edges of M can now be in any perfect matching of G without B_1. Hence there are b_{n-1} perfect matchings in G that contains e_0. If instead given $e_0 \notin M$, then adjacent edges of e_0 in B_1 belong to M and the adjacent edges of e_1 in B_1 belong to M too. Thus adjacent edges of e_2 in B_2 do not belong to M and the remaining edges of M can now be in any perfect matching of G without B_1 and B_2. Hence there are b_{n-2} perfect matchings in G that do not contain e_0. We conclude that $b_0 = 1$, $b_1 = 2$ and $b_n = b_{n-1} + b_{n-2}$ for $n \geqslant 2$.

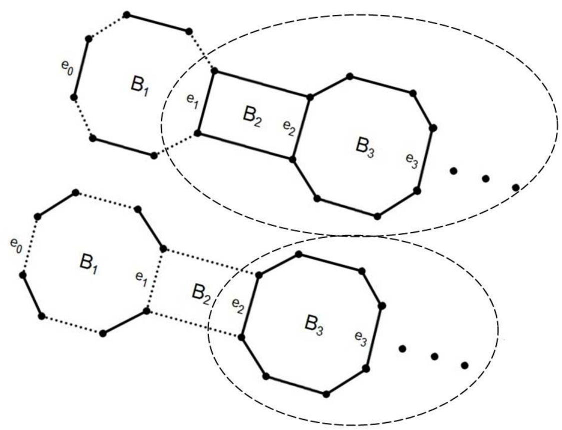

Figure 6. Counting Methods for Theorem 2.2. □

Since the recurrence relation $b_n = b_{n-1} + b_{n-2}$ is linear homogeneous of degree 2 with the initial conditions $b_0 = 1$, $b_1 = 2$, we get the general solution

$$b_n = (\tfrac{\sqrt{5}+3}{2\sqrt{5}})(\tfrac{1+\sqrt{5}}{2})^n + (\tfrac{\sqrt{5}-3}{2\sqrt{5}})(\tfrac{1-\sqrt{5}}{2})^n$$

For example, graph G is a linear chain of 3 blue faces. By theorem 2.2, we have $b_3 = b_2 + b_1 = 2b_1 + b_0 = 5$. If we use the general solution, then

$$b_3 = (\tfrac{\sqrt{5}+3}{2\sqrt{5}})(\tfrac{1+\sqrt{5}}{2})^3 + (\tfrac{\sqrt{5}-3}{2\sqrt{5}})(\tfrac{1-\sqrt{5}}{2})^3 = 5$$

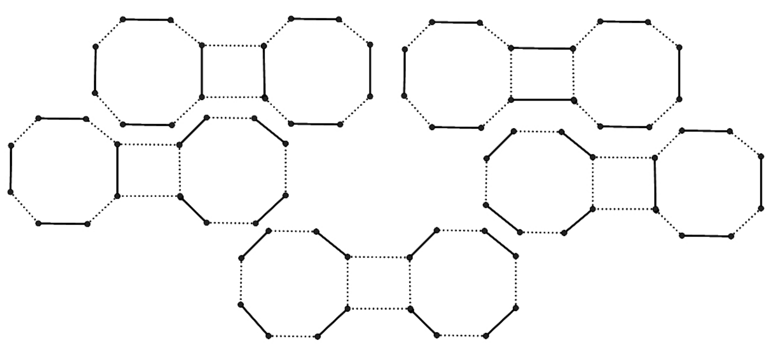

Figure 7. All perfect matching of linear chain with 3 blue faces.

In the case of G being a linear chain graph which alternates between blue faces and red faces, if the number of faces is even, we call *even linear chain*. Given the number of faces is $2k$ when $k \in \mathbb{N}$, then by symmetry $b_{2k} = r_{2k}$, where b_{2k} is the number of all perfect matching of G which the first face is blue, r_{2k} is the number of all perfect matching of G which the first face is red. If the number of faces is odd, we call *odd linear chain*. They are many cases such that we will use recurrence relation again.

Theorem 2.3. *Let G be an odd linear chain graph with n faces. Let the first face of G be a blue face and $b_n = \phi(G)$. Then $b_0 = 1$, $b_1 = 2$, $b_2 = 3$ and $b_n = b_{n-1} + b_{n-3}$ for odd $n \geqslant 3$.*

Proof. Let G be an odd linear chain graph. Given $b_n = \phi(G)$ and n be odd. Let M be a perfect matching of G. Since the number of edges in every face is even, we can define e_i as an edge shared between blue faces and red faces as shown in Figure 8. Let e_0 and e_n be edge opposite to e_1 in B_1 and edge opposite to e_{n-1} in B_n, respectively.

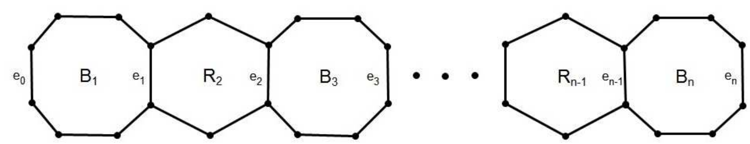

Figure 8. Definition of e_i in G

If M contains e_0 of B_1, then both adjacent edges of e_1 in B_1 do not belong to M, because B_1 is a blue face, and the remaining edges of M can now be in any perfect matching of G without B_1. Hence there are $r_{n-1} = b_{n-1}$ perfect matching in G that contain e_0. If instead given $e_0 \notin M$, then both adjacent edges of e_0 in B_1 belong to M. Since adjacent edges of e_1 in B_1 and e_2 in R_2 belong to M too, then both adjacent edges of e_3 in B_3 do not belong to M and the remaining edges of M can now be in any perfect matching of G without B_1, R_2 and B_3. Hence there are $r_{n-3} = b_{n-3}$ perfect matching in G not containing e_0. We conclude that $b_n = b_{n-1} + b_{n-3}$ for odd $n \geqslant 3$ when $b_0 = 1$, $b_1 = 2$ and $b_2 = 3$.

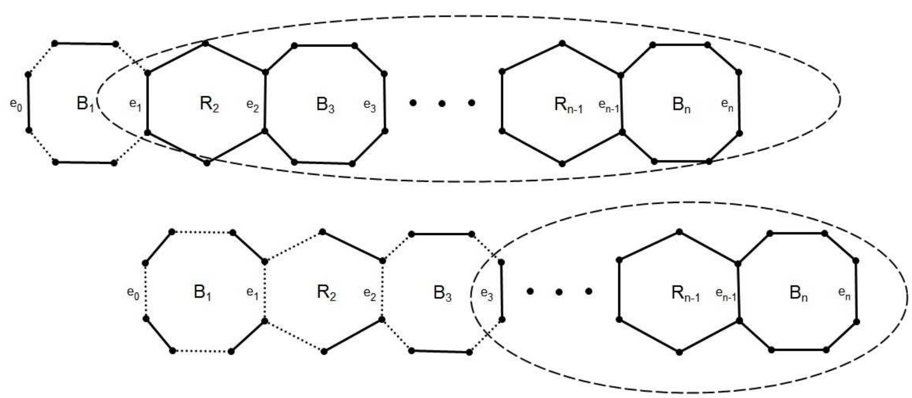

Figure 9. Counting Methods for Theorem 2.3. □

Theorem 2.4. *Let G be an odd linear chain graph with n faces and $r_n = \phi(G)$. Let the first face of G be a red face. Then $r_0 = 1$, $r_1 = 2$ and $r_n = r_{n-1} + r_{n-2}$ for odd $n \geqslant 3$.*

Proof. Let G be an odd linear chain graph. Given $r_n = \phi(G)$ and n be odd. Let M be a perfect matching of G. Since the number of edges in every faces is even, we can define e_i as an edge shared between red faces and blue faces as shown here. Let e_0 and e_n be edge opposite to e_1 in R_1 and edge opposite to e_{n-1} in R_n, respectively.

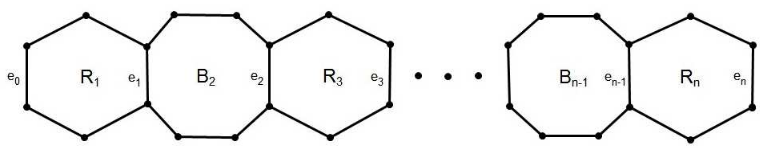

Figure 10. Definition of e_i in G

If M contains e_0 of R_1, then both adjacent edges of e_1 in R_1 belong to M. Because adjacent edges of e_2 of B_2 do not belong to M, the remaining edges of M can now be in any perfect matching of G without R_1 and B_2. Hence there are r_{n-2} perfect matchings in G that contain e_0. If instead given $e_0 \notin M$ then both adjacent edges of e_0 of R_1 belong to M. Because of adjacent edges of e_1 in R_1 not belong to M, and the remaining edges of M can now be in any perfect matching of G without R_1. Hence there are $b_{n-1} = r_{n-1}$ perfect matchings in G not containing e_0. Conclude that $r_0 = 1$, $r_1 = 2$ and $r_n = r_{n-1} + r_{n-2}$ for odd $n \geqslant 3$.

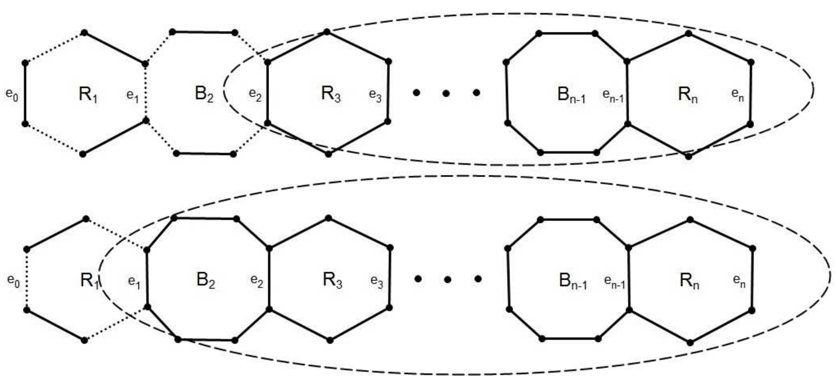

Figure 11. Counting Methods for Theorem 2.4. □

In particular, $r_n = r_{n-1} + r_{n-2}$ is linear homogeneous of degree 2 with the initial conditions $r_0 = 1$, $r_1 = 2$, we get the general solution

$$r_n = (\tfrac{\sqrt{5}+3}{2\sqrt{5}})(\tfrac{1+\sqrt{5}}{2})^n + (\tfrac{\sqrt{5}-3}{2\sqrt{5}})(\tfrac{1-\sqrt{5}}{2})^n$$

As mentioned previously, we have $b_{2k} = r_{2k}$, where b_{2k} is the number of perfect matching in G which the first face is blue, r_{2k} is the number of perfect matching of G which the first face is red and $k \in \mathbb{N}$. Now we can apply counting concepts in Theorem 2.3 and 2.4 to the case of the number of faces is an even number.

Theorem 2.5. *Let G be an even linear chain graph with n faces.*

(1) If the first face of G be a blue face and G alternates between blue faces and red faces. Then $r_0 = 1$, $r_1 = 2$, $r_2 = 3$ and $b_n = r_{n-1} + r_{n-3}$ for even $n \geqslant 4$.

(2) If the first face of G be a red face and G alternates between red faces and blue faces. Then $b_0 = 1$, $b_1 = 2$ and $r_n = b_{n-1} + b_{n-2}$ for even $n \geqslant 2$.

For example, graph G is an even linear chain graph with 4 faces as shown in Figure 12. By theorem 2.4 and 2.5, we have $b_4 = r_3 + r_1 = (r_2 + r_1) + r_1 = (3 + 2) + 2 = 7$.

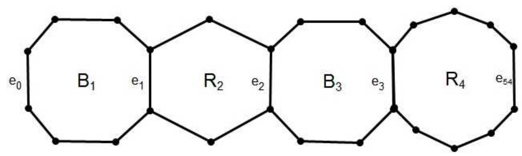

Figure 12. An example of linear chain with 4 faces(B & R).

The followings are all perfect matchings of G,

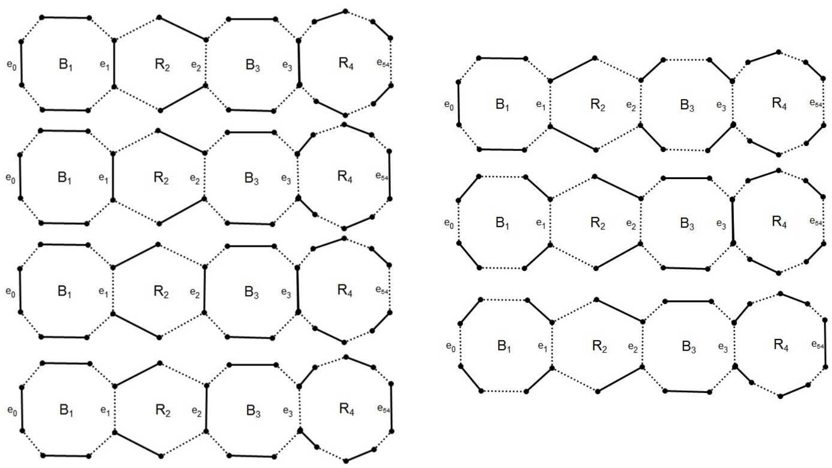

Figure 13. All perfect matchings of even linear chain with 4 faces(B & R).

3 Perfect Matching of Snake Chain Graphs

A graph of chain, which is not linear, is called *snake chain* if the shared edges are not all parallel. Let G consist of a chain of several faces $G_1, G_2, G_3, ..., G_p$. For $i = 1, 2, 3, ..., p$, edges shared by G_{i-1} and G_i is called $e_{i,1}$. Define the first shared edge $e_{2,1}$ in G_2 to be the same edge as e_{1,k_1} in G_1 and the opposite of this edge in G_1 is called $e_{1,1}$. Then define the rest edges in each G_i as $e_{i,j}$ clockwise as shown in Figure 14(c). All shared edges are called $e_{1,k_1} = e_{2,1}$, $e_{2,k_2} = e_{3,1}$, ..., $e_{p-1,k_{p-1}} = e_{p,1}$. It is called an *odd snake chain* if k_i is odd for all $i = 2, 3, ..., p-1$. It is called an *even snake chain* if k_i is even for all $i = 2, 3, ..., p-1$.

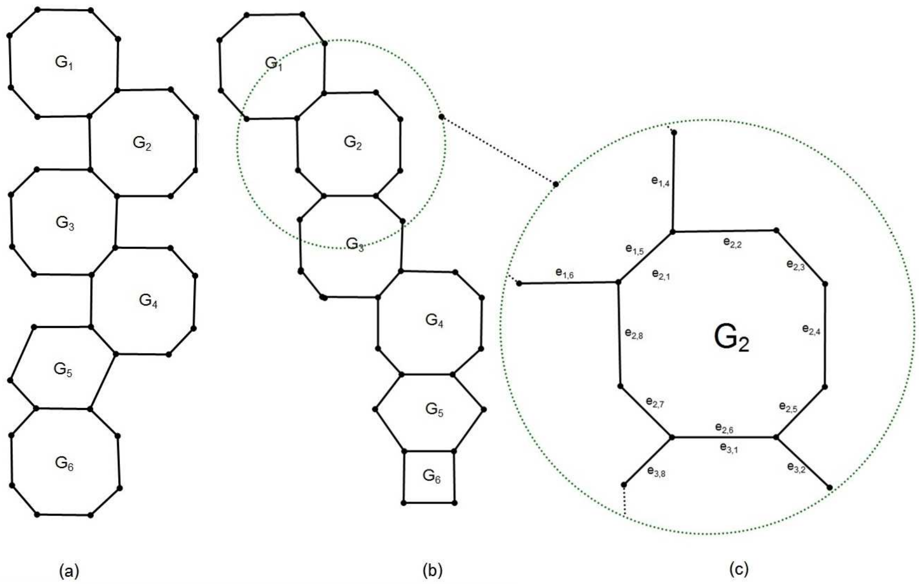

Figure 14. Example of (a)odd snake chain, (b)even snake chain graphs and (c) $e_{i,j}$ in G_2

Theorem 3.1. *Let G be an odd snake chain graph with p faces and $a_p = \phi(G)$. Then $a_0 = 1$, $a_1 = 2$ and $a_p = a_{p-1} + a_{p-2}$ for $p \geqslant 2$.*

Proof. Let G be an odd snake chain graph which consists of a chain of several faces $G_1, G_2, G_3, ..., G_p$. We define $e_{i,j}$ by definition of an odd snake chain.

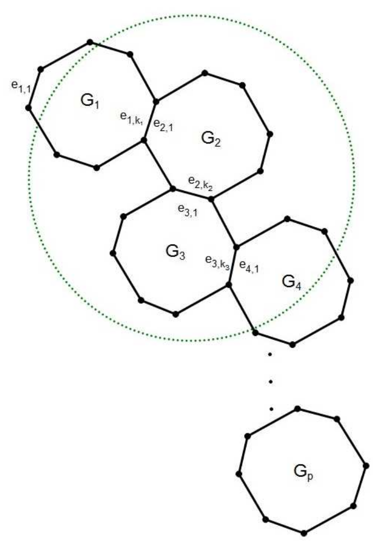

Figure 15. Definition of $e_{i,j}$ in G

Case I G_1 is a blue face. If M contains $e_{1,1}$ of G_1, then both adjacent edges of $e_{2,1}$ in G_1 do not belong to M. The remaining edges of M can now be in any perfect matching of G without G_1. Hence there are a_{p-1} perfect matchings in G that contain $e_{1,1}$. If instead given $e_{1,1} \notin M$, then M must contain $e_{1,2}$ and $e_{1,p}$ of G_1. It must also contain both adjacent edges of $e_{2,1}$ in G_1. Since k_i is odd for all $i = 2, 3, ..., p-1$, then e_{2,k_2-1} and e_{2,k_2+1} do not belong to M. The remaining edges of M can now be in any perfect matching of G without G_1 and G_2. Hence there are a_{p-2} perfect matching in G not containing $e_{1,1}$. Conclude that $a_0 = 1$, $a_1 = 2$ and $a_p = a_{p-1} + a_{p-2}$ for $p \geqslant 2$.

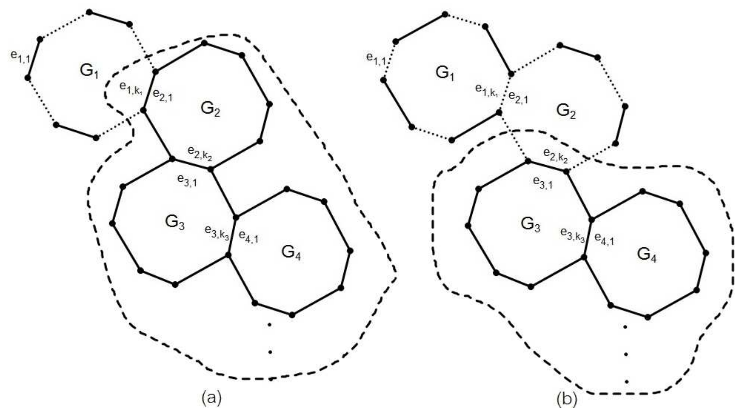

Figure 16. G_1 is a blue face which (a) M contains $e_{1,1}$ of G_1 and (b) $e_{1,1} \notin M$

Case II G_1 is a red face.

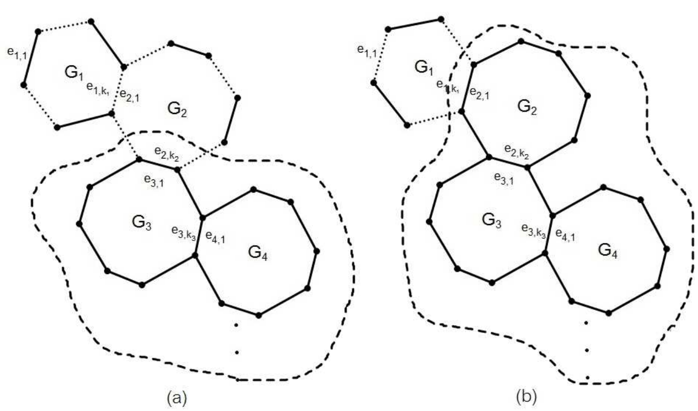

Figure 17. G_1 is a red face. (a) M contains $e_{1,1}$ of G_1 (b) $e_{1,1} \notin M$

If M contains $e_{1,1}$ of G_1, then it must also contain both adjacent edges of $e_{2,1}$ in G_1. Since k_i is odd for all $i = 2, 3, ..., p-1$, then e_{2,k_2-1} and e_{2,k_2+1} do not belong to M. The remaining edges of M can now be in any perfect matching of G without G_1 and G_2. Hence there are a_{p-2} perfect matchings in G that contain $e_{1,1}$.

If instead given $e_{1,1} \notin M$, then M must contain $e_{1,2}$ and $e_{1,p}$ of G_1 . The adjacent edges of $e_{2,1}$ in G_1 do not belong to M. The remaining edges of M can now be in any perfect matching of G without G_1. Hence there are a_{p-1} perfect matchings in G not containing $e_{1,1}$. Conclude that $a_0 = 1$, $a_1 = 2$ and $a_p = a_{p-1} + a_{p-2}$ for $p \geqslant 2$. □

In particular, $a_p = a_{p-1} + a_{p-2}$ is linear homogeneous of degree 2 with the initial conditions $a_0 = 1$, $a_1 = 2$, we get the general solution

$$a_p = (\frac{\sqrt{5}+3}{2\sqrt{5}})(\frac{1+\sqrt{5}}{2})^p + (\frac{\sqrt{5}-3}{2\sqrt{5}})(\frac{1-\sqrt{5}}{2})^p$$

Theorem 3.2. *Let G be an even snake chain graph with p faces and $a_p = \phi(G)$. Then $a_p = p + 1$.*

Proof. Let G be an even snake chain graph which consists of a chain of several faces $G_1, G_2, G_3, ..., G_p$. We define $e_{i,j}$ by definition of an even snake chain.

Case I G_1 is a blue face. If M contains $e_{1,1}$ of G_1, then both adjacent edges of $e_{2,1}$ in G_1 do not belong to M. The remaining edges of M can now be in any perfect matching of G without G_1. Hence there are a_{p-1} perfect matchings in G that contain $e_{1,1}$. If instead given $e_{1,1} \notin M$, then M must contain $e_{1,2}$ and $e_{1,|E(G_1)|}$ of G_1. It must also contain both adjacent edges of $e_{2,1}$ in G_1. Since k_i is even for all $i = 2, 3, ..., p-1$, then both adjacent edges of e_{i,k_i} in G_i must belong to M. Thus there is only one perfect matching in G that contains $e_{1,1}$. Conclude that $a_p = a_{p-1} + 1$. Since $a_0 = 1, a_1 = 2$, then $a_p = p + 1$.

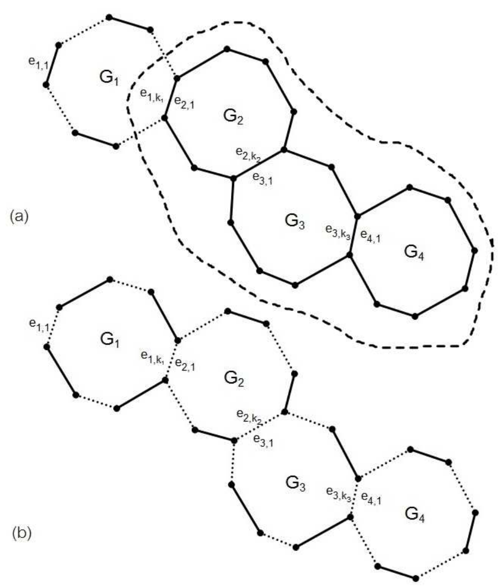

Figure 18. Example of $p = 4$ and G_1 is a blue face. (a) $e_{1,1} \in M$ (b) $e_{1,1} \notin M$

Case II G_1 is a red face. If M contains $e_{1,1}$ of G_1, then it must also contain adjacent edges of $e_{2,1}$ in G_1. Since k_i is even for all $i = 2, 3, ..., p-1$, then both adjacent edges of e_{i,k_i} in G_i must belong to M. Thus there is only one perfect matching in G that contains $e_{1,1}$. If instead given $e_{1,1} \notin M$, then M must contain $e_{1,2}$ and $e_{1,|E(G_1)|}$ of G_1 . Both adjacent edges of $e_{2,1}$ in G_1 do not belong to M. The remaining edges of M can now be in any perfect matching of G without G_1. Hence there are a_{p-1} perfect matching in G not containing $e_{1,1}$. Conclude that $a_p = a_{p-1} + 1$. Since $a_0 = 1, a_1 = 2$, $a_p = p + 1$.

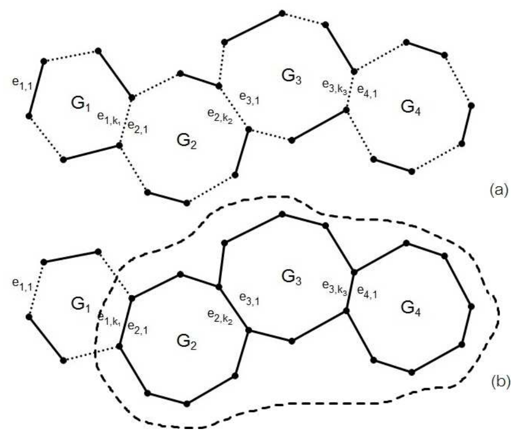

Figure 19. Example of $p = 4$ and G_1 is a red face. (a) $e_{1,1} \in M$ (b) $e_{1,1} \notin M$ □

The recurrence relation is a useful tool for counting a number of perfect matching of other graphs. Moreover, we can apply it to other part of graph theory, such as counting number of vertex independent set, face independent set, chromatic polynomial of graph and counting acyclic orientation, etc.

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