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The Quasi-Hadamard Product of Certain Analytic and p-Valent Functions

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Abstract : The author establishes certain results concerning the quasi-Hadamard product of certain analytic and p-valent functions with negative coefficients in the open unit disc.

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1 Introduction

Throughout the paper, let the functions of the form

$$f(z) = a_p z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_p > 0;\ a_k \geq 0)\,, \tag{1.1}$$

$$f_i(z) = a_{p,i} z^p - \sum_{k=p+1}^{\infty} a_{k,i} z^k \qquad (a_{p,i} > 0;\ a_{k,i} \geq 0)\,, \tag{1.2}$$

$$g(z) = b_p z^p - \sum_{k=p+1}^{\infty} b_k z^k \qquad (b_p > 0;\ b_k \geq 0)\,, \tag{1.3}$$

and

$$g_j(z) = b_{p,j}z^p - \sum_{k=p+1}^{\infty} b_{k,j}z^k \qquad (b_{p,j} > 0;\ b_{k,j} \geq 0)\,, \tag{1.4}$$

where $p, i, j \in \mathbb{N} = \{1, 2, \ldots\}$, be analytic and p-valent in

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}\,.$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence [1, p. 4]:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{T}_p\,(A, B)$ be the class of functions f of the form (1.1) and satisfying

$$\frac{1}{p}\frac{zf'\,(z)}{f\,(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq A < B \leq 1;\, z \in \mathbb{U})\,. \tag{1.5}$$

Also let $\mathcal{C}_p\,(A, B)$ denote the class of functions of the form (1.1) such that $\frac{1}{p}zf' \in \mathcal{T}_p\,(A, B)\,.$

Using similar arguments as given by Reddy and Padmanabhan [2] we can easily prove the following analogous results for functions in the classes $\mathcal{T}_p\,(A, B)$ and $\mathcal{C}_p\,(A, B)$.

A function f defined by (1.1) belongs to the class $\mathcal{T}_p\,(A, B)$ if and only if

$$\sum_{k=p+1}^{\infty} \left[\{k\,(B + 1) - p\,(A + 1)\}\,a_k\right] \leq p\,(B - A)\,a_p \tag{1.6}$$

and f defined by (1.1) belongs to the class $\mathcal{C}_p\,(A, B)$ if and only if

$$\sum_{k=p+1}^{\infty} \left[\frac{k}{p}\,\{k\,(B + 1) - p\,(A + 1)\}\,a_k\right] \leq p\,(B - A)\,a_p. \tag{1.7}$$

We now introduce a new class of analytic and p-valent functions which plays an important role in the discussion that follows.

Definition 1.1. A function f, defined by (1.1), belongs to the class $f \in \mathcal{T}_p^c(A,B)$ if and only if

$$\sum_{k=p+1}^{\infty}\left[\left(\frac{k}{p}\right)^c \{k(B+1)-p(A+1)\}\, a_k\right] \leq p(B-A)\, a_p, \tag{1.8}$$

where $-1 \leq A < B \leq 1$ and c is any fixed nonnegative real number.

We note that, for every nonnegative real number c, the class $\mathcal{T}_p^c(A,B)$ is nonempty as the functions of the form

$$f(z) = a_p z^p - \sum_{k=p+1}^{\infty} \frac{p(B-A)\, a_p}{\left(\frac{k}{p}\right)^c \{k(B+1)-p(A+1)\}} \lambda_k z^k, \qquad (z \in \mathbb{U}), \tag{1.9}$$

where $a_p > 0$, $\lambda_k \geq 0$ and $\sum_{k=p+1}^{\infty} \lambda_k \leq 1$, satisfy the inequality (1.8).

It is evident that $\mathcal{T}_p^1(A,B) \equiv \mathcal{C}_p(A,B)$ and, for $c=0$, $\mathcal{T}_p^c(A,B)$ is identical to $\mathcal{T}_p(A,B)$. Further, $\mathcal{T}_p^c(A,B) \subset \mathcal{T}_p^h(A,B)$ if $c > h \geq 0$, the containment being proper. Hence, for any positive integer c, we have the inclusion relation

$$\mathcal{T}_p^c(A,B) \subset \mathcal{T}_p^{c-1}(A,B) \subset \cdots \subset \mathcal{T}_p^2(A,B) \subset \mathcal{C}_p(A,B) \subset \mathcal{T}_p(A,B).$$

Let us define the quasi-Hadamard product of the functions f and g by

$$f * g(z) = a_p b_p z^p - \sum_{k=p+1}^{\infty} a_k b_k z^k. \tag{1.10}$$

Similarly, we can define the quasi-Hadamard product of more than two functions.

In this work we establish certain results concerning the quasi-Hadamard product of functions belonging to the classes $\mathcal{T}_p^c(A,B)$, $\mathcal{T}_p(A,B)$ and $\mathcal{C}_p(A,B)$.

2 The Main Theorem

Theorem 2.1. *Let the functions f_i defined by (1.2) be in the class $\mathcal{C}_p(A,B)$ for every $i = 1,2,\ldots,r$; and let the functions g_j defined by (1.4) be in the class $\mathcal{T}_p(A,B)$ for every $j = 1,2,\ldots,s$. Then the quasi-Hadamard product $f_1 * f_2 * \cdots * f_r * g_1 * g_2 * \cdots * g_s(z)$ belongs to the class $\mathcal{T}_p^{2r+s-1}(A,B)$.*

Proof. We denote the quasi-Hadamard product $f_1 * f_2 * \cdots * f_r * g_1 * g_2 * \cdots * g_s(z)$ by the function $h(z)$, for the sake of convenience.

Clearly,

$$h(z) = \left\{\prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s} b_{p,j}\right\} z^p - \sum_{k=p+1}^{\infty}\left\{\prod_{i=1}^{r} a_{k,i} \prod_{j=1}^{s} b_{k,j}\right\} z^k. \tag{2.1}$$

To prove the theorem, we need to show that

$$\sum_{k=p+1}^{\infty}\left[\left(\frac{k}{p}\right)^{2r+s-1}\{k(B+1)-p(A+1)\}\left\{\prod_{i=1}^{r}a_{k,i}\prod_{j=1}^{s}b_{k,j}\right\}\right] \leq p(B-A)\left(\prod_{i=1}^{r}a_{p,i}\prod_{j=1}^{s}b_{p,j}\right). \tag{2.2}$$

Since $f_i \in \mathcal{C}_p(A,B)$, we have

$$\sum_{k=p+1}^{\infty}\left[\frac{k}{p}\{k(B+1)-p(A+1)\}a_{k,i}\right] \leq p(B-A)a_{p,i}, \tag{2.3}$$

for every $i=1,2,\ldots,r$. Therefore

$$\frac{k}{p}\{k(B+1)-p(A+1)\}a_{k,i} \leq p(B-A)a_{p,i}$$

or

$$a_{k,i} \leq \frac{p(B-A)a_{p,i}}{\frac{k}{p}\{k(B+1)-p(A+1)\}}$$

for every $i=1,2,\ldots,r$. The right side of the above inequality is not greater than $\left(\frac{k}{p}\right)^{-2}a_{p,i}$. Hence

$$a_{k,i} \leq \left(\frac{k}{p}\right)^{-2}a_{p,i}, \tag{2.4}$$

for every $i=1,2,\ldots,r$. Similarly, for $g_j \in \mathcal{T}_p(A,B)$, we have

$$\sum_{k=p+1}^{\infty}\left[\{k(B+1)-p(A+1)\}b_{k,j}\right] \leq p(B-A)b_{p,j}, \tag{2.5}$$

for every $j=1,2,\ldots,s$. Whence we obtain

$$b_{k,j} \leq \left(\frac{k}{p}\right)^{-1}b_{p,j}, \tag{2.6}$$

for every $j=1,2,\ldots,s$.

Using (2.4) for $i=1,2,\ldots,r$, (2.6) for $j=1,2,\ldots,s-1$ and (2.5) for $j=s$, we get

$$\sum_{k=p+1}^{\infty}\left[\left(\frac{k}{p}\right)^{2r+s-1}\{k(B+1)-p(A+1)\}\left\{\prod_{i=1}^{r}a_{k,i}\prod_{j=1}^{s}b_{k,j}\right\}\right]$$
$$\leq \sum_{k=p+1}^{\infty}\left[\left(\frac{k}{p}\right)^{2r+s-1}\{k(B+1)-p(A+1)\}b_{k,s}\left\{\left(\frac{k}{p}\right)^{-2r}\left(\frac{k}{p}\right)^{-(s-1)}\right\}\right.$$

$$\times\left(\prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s-1} b_{p,j}\right)\Bigg]$$

$$= \left(\sum_{k=p+1}^{\infty} \left[\{k(B+1) - p(A+1)\} b_{k,s}\right]\right)\left(\prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s-1} b_{p,j}\right)$$

$$\leq p(B-A)\left(\prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s} b_{p,j}\right),$$

and therefore $h \in \mathcal{T}_p^{2r+s-1}(A,B)$, completing the proof of the theorem. □

We note that the required estimate can be also obtained by using (2.4) for $i = 1, 2, \ldots, r-1$, (2.6) for $j = 1, 2, \ldots, s$, and (2.3) for $i = r$.

Now we discuss some applications of Theorem 2.1. Taking into account the quasi-Hadamard product of functions $f_1, f_2, \ldots, f_r$ only, in the proof of Theorem 2.1, and using (2.4) for $i = 1, 2, \ldots, r-1$ and (2.3) for $i = r$ we are led to

Corollary 2.2. *Let the functions f_i defined by* (1.2) *belong to the class $\mathcal{C}_p(A,B)$ for every $i = 1, 2, \ldots, r$. Then the quasi-Hadamard product $f_1 * f_2 * \cdots * f_r(z)$ belongs to the class $\mathcal{T}_p^{2r-1}(A,B)$.*

Next, taking into account the quasi-Hadamard product of the functions g_1, $g_2, \ldots, g_s$ only, in the proof of Theorem 2.1, and using (2.6) for $j = 1, 2, \ldots, s-1$, and (2.3) for $j = s$, we have the following corollary.

Corollary 2.3. *Let the functions g_j defined by* (1.4) *belong to the class $\mathcal{T}_p(A,B)$ for every $j = 1, 2, \ldots, s$. Then the quasi-Hadamard product $g_1 * g_2 * \cdots * g_s(z)$ belongs to the class $\mathcal{T}_p^{s-1}(A,B)$.*

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