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Existence of Common Fixed Points for Weakly Compatible and C_q-Commuting Maps and Invariant Approximations

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Abstract : We prove the existence of common fixed points for three selfmaps A, S and T defined on a nonempty subset E of a metric space (X, d) under the assumptions that (i) A, S and T satisfy a contractive condition given by (2.1.1); (ii) $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$; and (iii) the pairs (A, S) and (A, T) are weakly compatible. We use this result to find the common fixed points of three C_q-commuting continuous selfmaps defined on q-starshaped subset E of a normed space X satisfying certain nonexpansive inequality involving rational expressions. We apply these results to prove the existence of common fixed points from the set of best approximations. Our results extend, generalize and unify the works of Al-Thagafi [1], Al-Thagafi *et. al.* [2], Brosowski [3], Habiniak [4], Hicks *et. al.* [5], Jungck [6], Jungck *et. al.* [7], Pant [8], Sahab *et. al.* [9], Sahab *et. al.* [10] and Singh [11].

Keywords : weakly compatible maps; q-affine maps; C_q-commuting maps; q-starshaped sets; nonexpansive maps; common fixed point; set of best approximation.

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1 Introduction

Let E be a nonempty subset of a normed space X. The set E is called *q-starshaped* with $q \in E$ if the segment $[q, x] = \{(1-k)q + kx : k \in [0,1]\}$ joining q to x is contained in E for all $x \in E$. Let A, S and T be selfmaps on E. We say that a selfmap A is *affine* if E is convex and $A(kx + (1-k)y) = kAx + (1-k)Ay$ for all $x, y \in E$ and for all $k \in [0,1]$. We say that a selfmap A is *q-affine* if E is q-starshaped and $A(kx + (1-k)q) = kAx + (1-k)q$ for all $x \in E$ and for all $k \in [0,1]$.

Here we note that every convex set is q-starshaped with respect to any $q \in E$ but its converse need not be true. Also every affine map is q-affine, but its converse need not be true.

Example 1.1. Let $X = R_+^2$ with the usual metric, where $R_+ = [0, \infty)$.
We write $E_1 = \{(1+\alpha, 2+3\alpha) : \alpha \in R_+\}$, $E_2 = \{(1+2\alpha, 2(1+\alpha)) : \alpha \in R_+\}$ and $E = E_1 \cup E_2$. Then E is q-starshaped subset of X with $q = (1,2)$. But E is not convex, for, by taking $x = (2,5) \in E$ and $y = (3,4) \in E$ and $k = \frac{1}{2}$, we get $kx + (1-k)y = (\frac{5}{2}, \frac{9}{2})$. We observe that $(\frac{5}{2}, \frac{9}{2})$ is not an element of E.
We define a map $A : E \to E$ by

$$A(x,y) = \begin{cases} (1,2) & \textit{if} \ \ (x,y) \in E_1 \\ 2(x,y) - (1,2) & \textit{if} \ \ (x,y) \in E_2. \end{cases}$$

Then A is q-affine with $q = (1,2)$ while A is not affine since E is not convex.

A selfmap T on E is said to be *A-contraction* if there exists $k \in [0,1)$ such that $\|Tx - Ty\| \leq k\|Ax - Ay\|$ for all $x, y \in E$. If $k = 1$, then T is called *A-nonexpansive*. The set of all fixed points of A (resp. S and T) is denoted by $F(A)$ (resp. $F(S)$ and $F(T)$). A point $x \in E$ is a *common fixed* (*coincidence*) *point of* A, S *and* T if $Ax = Sx = Tx = x$ ($Ax = Sx = Tx$). The set of coincidence points of A and S (resp. A and T) is denoted by $C(A,S)$ (resp. $C(A,T)$). The set $P_E(u) = \{x \in E : \|x - u\| = \delta(u, E)\}$ is called the *set of best approximants* to $u \in X$ out of E, where $\delta(u, E) = \inf_{y \in E} \|y - u\|$.

Though out this paper, we denote the closure of E by $\overline{E}$, the boundary of E by ∂E and the set of all nonnegative integers by Z_+.

A pair of mappings (A, T) is called

(1) *commuting* if $ATx = TAx$ for all $x, y \in E$.

(2) *R-weakly commuting* (Pant [8]), if there exists a positive real number R such that $d(ATx, TAx) \leq Rd(Ax, Tx)$ for each $x \in X$.

(3) *compatible* (Jungck [12]) if $\lim_{n\to\infty} d(ATx_n, TAx_n) = 0$ whenever $\{x_n\}$ is a sequence in E such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in E$.

(4) *weakly compatible* (Jungck *et. al.* [7]) if $ATx = TAx$ for all $x \in C(A,T)$.

(5) *C_q-commuting* (Al-Thagafi *et. al.* [2]) if E is q- starshaped and $ATx = TAx$ for all $x \in C_q(A,T)$, where $C_q(A,T) := \bigcup\{C(A, T_k) : k \in [0,1]\}$ where $T_kx = kTx + (1-k)q$.

We note that each pair of C_q-commuting mappings is weakly compatible but its converse need not be true in general as shown by the following example.

Example 1.2. Let $E = R_+$ with the usual metric. Then E is q-starshaped with $q = 0$. We define mappings $A, T : E \to E$

$$A(x) = \begin{cases} 0 & \textit{if } 0 \le x < \frac{2}{3} \\ \frac{4}{3} - x & \textit{if } \frac{2}{3} \le x < \infty \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{3} & \textit{if } 0 \le x < \frac{2}{3} \\ \frac{2}{3} & \textit{if } \frac{2}{3} \le x < \infty \end{cases}$$

Then $C(A,T) = \{\frac{2}{3}\}$ and $ATx = TAx$ for all $x \in C(A,T)$ so that the pair (A,T) is weakly compatible. We note that $C_q(A,T) = \{\frac{4}{3} - \frac{2}{3}k : 0 \le k \le 1\}$. Now let $x \in C_q(A,T)$. Then $Ax = \frac{2}{3}k$ and $Tx = \frac{2}{3}$ so that $TAx = \begin{cases} \frac{2}{3} & \textit{if } k = 1 \\ \frac{1}{3} & \textit{if } k \neq 1 \end{cases}$ and $ATx = \frac{2}{3}$ which shows that $ATx \neq TAx$ for all $x \in C_q(A,T) - \{\frac{2}{3}\}$. Hence, the pair of mappings (A,T) is not C_q-commuting maps.

A Banach space X is siad to satisfy *Opial's condition* (Opial [13]) if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $x \neq y$.

A selfmap $T : E \to E$ is said to be *demiclosed* at 0 if every sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges to $0 \in E$, then $0 = Tx$. A selfmap $T : E \to E$ is said to be *weakly continuous* if $Tx_n \to Tx$ weakly whenever $x_n \to x$ weakly. If $\{x_n\}$ converges weakly to x as $n \to \infty$, we write it by $x = w - \lim_{n\to\infty} x_n$.

In 1998, Jungck *et. al.* [7], introduced the notion of weakly compatible maps which is found to be very helpful in obtaining coincidence points and common fixed points of various classes of mappings on a metric space. For example we refer Abbas [14], Ahmed [15], Al-Thagafi *et. al.* [2], Aydi *et. al.*[16], Bari *et. al.* [17], Beg *et. al.* [18], Ding *et. al.*[19], Jha [20], Khan *et. al.* [21], Karapinar *et. al.*[22], Song [23]and the references therein.

In 1994, Pant [8] proved the following common fixed theorem for a pair of selfmaps.

Theorem 1.3. (*Pant* [8]). *Let (X,d) be a complete metric space and let A and T be R-weakly commuting selfmaps of X satisfying the condition:*

$d(Tx,Ty) \le \varphi(d(Ax,Ay))$

for all $x, y \in X$, where $\varphi : R_+ \to R_+$ is a continuous function such that $\varphi(t) < t$ for each $t > 0$. If $T(X) \subset A(X)$, and if either A or T is continuous on X, then A and T have a unique common fixed point in X.

In 2006, Al-Thagafi *et. al.* [2] proved the following theorem.

Theorem 1.4. (*Al-Thagafi et. al.* [2]) *Let E be a subset of a metric space (X,d), A and T be selfmaps of E such that $T(E) \subseteq A(E)$. Suppose that A and T are weakly compatible on E, T is A-contraction and $T(E)$ is complete. Then $F(T) \cap F(A)$ is a singleton.*

In 1963, Meinardus [24] combined the concepts of existence of fixed points and best approximation, and in 1969, Brosowski [3] extended and simplified the result established by Meinardus as follows.

Theorem 1.5. (*Brosowski* [3]) *Let T be a linear and nonexpansive operator on a normed space X and let $E \subseteq X$ with $T(E) \subset E$. Let $u \in F(T)$. If $P_E(u)$ is nonempty, compact and convex, then $P_E(u) \cap F(T) \neq \emptyset$.*

In 1979, Singh [11] relaxed the linearity of the operator T and the convexity of $P_E(u)$ in Brosowski's theorem and proved the following theorem.

Theorem 1.6. (*Singh* [11]) *Let X be a normed space, $T : X \to X$ a nonexpansive operator. Suppose that $E \subset X$ with $T(E) \subset E$ and $u \in F(T)$. If $P_E(u)$ is a nonempty compact and starshaped, then $P_E(u) \cap F(T) \neq \emptyset$.*

Hicks *et. al.* [5] relaxed the condition $T(E) \subset E$ by $T(\partial E) \subset E$ in Singh's result. In 1988, Sahab *et. al.* [10] generalized the results due to Brosowski [3] and Singh [11], and proved the following theorem.

Theorem 1.7. (*Sahab et. al.* [10]) *Let T and A be selfmaps on a normed linear space X with $E \subseteq X$ such that $T(\partial E) \subset E$ and $u \in F(T) \cap F(A)$. Suppose T is A-nonexpansive on $P_E(u) \cup \{u\}$, T and A are commuting on $P_E(u)$ and A is linear and continuous on $P_E(u)$. If $P_E(u)$ is nonempty compact and q-starshaped with $q \in F(A)$ and if $A(P_E(u)) = P_E(u)$, then $P_E(u) \cap F(T) \cap F(A) \neq \emptyset$.*

In 1995, Jungck *et. al.* [25] proved the following result which extends the aforementioned theorems.

Theorem 1.8. (*Jungck* et. al. [25]) *Let T and A be selfmaps of a Banach space X with $E \subseteq X$ such that $T(\partial E) \subset E$ and $u \in F(T) \cap F(A)$. Suppose that $P_E(u)$ is q-starshaped with $q \in F(A)$ and $A(P_E(u)) = P_E(u)$, A is affine and continuous in the weak topology and strong topology on $P_E(u)$. If A and T are commuting on $P_E(u)$ and T is A - nonexpansive on $P_E(u) \cup \{u\}$, then $P_E(u) \cap F(T) \cap F(A) \neq \emptyset$ provided either*

(i) $P_E(u)$ is weakly compact and $I - T$ is demiclosed; or

(ii) $P_E(u)$ is weakly compact and E satisfies Opial's condition.

In 1996, Al-Thagafi [1] extended, generalized and unified the works of Brosowski [3], Hicks *et. al.* [5], Sahab *et. al.* [10] and Singh [11] as follows:

Theorem 1.9. (*Al-Thagafi* [1]) *Let A and T be selfmaps of a normed space X with $u \in F(A) \cap F(T)$ and $E \subset X$ with $T(\partial E \cap E) \subset E$. Suppose that $D = P_E(u) \cap C_E^A(u)$, where $C_E^A(u) = \{x \in E : Ax \in P_E(u)\}$, is closed and q-starshaped with $q \in F(A)$. Let T be A-nonexpansive on $D \cup \{u\}$ with $\overline{T(D)}$ is compact, A be continuous, linear, $A(D) = D$ and A commutes with T on D. Then $P_E(u) \cap F(T) \cap F(A) \neq \emptyset$.*

Throughout this paper, let Φ be the set of all continuous selfmaps $\varphi : R_+ \to R_+$ satisfying

(i) φ is monotone increasing; and

(ii) $0 \le \varphi(t) < t$ for all $t > 0$.

Lemma 1.10. (*Singh* et. al. [26]) *If $\varphi \in \Phi$, then $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t \in (0,\infty)$, where φ^n denotes the n-times repeated composition of φ with itself.*

In section 2 of this paper, we prove the existence of common fixed points for three selfmaps A, S and T defined on a nonempty subset E of a metric space (X, d) under the assumption that (i) A, S and T satisfy a contractive condition given by (2.1.1); (ii) $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$; and (iii) the pairs (A, S) and (A, T) are weakly compatible. We use this result to find common fixed points of three C_q-commuting continuous selfmaps defined on q-starshaped subset E of a normed space X satisfying certain nonexpansive inequality involving rational expressions. In section 3, we apply the results of section 2 to prove the existence of common fixed points from the set of best approximations. These results extend, generalize and unify the works of Al-Thagafi [1], Al-Thagafi *et. al.* [2], Brosowski [3], Habiniak[4], Hicks *et. al.* [5], Jungck [6], Jungck *et. al.* [25], Pant [8], Sahab *et. al.* [9], Sahab *et. al.* [10] and Singh [11].

2 Main Results

Theorem 2.1. *Let E be a nonempty subset of a metric space (X, d) and $A, S, T : E \to E$ be three selfmaps such that $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$. Assume that there exists a $\varphi \in \Phi$ such that*

$$d(Sx, Ty) \le \varphi(max\{d(Ax, Ay), \frac{d(Ax,\ Sx)\ d(Ay,\ Ty)}{1 + d(Ax,\ Ay)}, \frac{d(Ax,\ Ty)\ d(Ay,\ Sx)}{1 + d(Ax,\ Ay)}\}) \tag{2.1.1}$$

for all $x, y \in E$. Assume also that the pairs of mappings (A, S) and (A, T) are weakly compatible. If $A(E)$ is complete, then A, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in E$. Since $S(E) \subseteq A(E)$, we can find $x_1 \in E$ such that $Sx_0 = Ax_1$ and since $T(E) \subseteq A(E)$, we can find $x_2 \in E$ such that $Tx_1 = Ax_2$. Continuing this way inductively, we get a sequence $\{x_n\}$ in E such that $Ax_n = Sx_{n-1}$ if n is odd and $Ax_n = Tx_{n-1}$ if n is even. Now we claim that the sequence $\{Ax_n\}_{n=1}^{\infty}$ is Cauchy in E.

<u>*Case* (i)</u>: $Ax_n = Ax_{n+1}$ for some n.

Without loss of generality, we assume that n is even. Then $n = 2m$ for some $m \in Z_+$. So $Ax_{2m} = Ax_{2m+1}$.

Thus, we have

$$
\begin{aligned}
d(Ax_{2m+1}, Ax_{2m+2}) &= d(Sx_{2m}, Tx_{2m+1}) \\
&\le \varphi(max\{d(Ax_{2m}, Ax_{2m+1}), \frac{d(Ax_{2m},\ Sx_{2m})\ d(Ax_{2m+1},\ Tx_{2m+1})}{1 + d(Ax_{2m},\ Ax_{2m+1})}, \\
&\qquad \frac{d(Ax_{2m},\ Tx_{2m+1})\ d(Ax_{2m+1},\ Sx_{2m})}{1 + d(Ax_{2m},\ Ax_{2m+1})}\}) \\
&\le \varphi(max\{d(Ax_{2m}, Ax_{2m+1}), \frac{d(Ax_{2m},\ Ax_{2m+1})\ d(Ax_{2m+1},\ Ax_{2m+2})}{1+d(Ax_{2m},\ Ax_{2m+1})}, \\
&\qquad \frac{d(Ax_{2m},\ Ax_{2m+2})\ d(Ax_{2m+1},\ Ax_{2m+1})}{1 + d(Ax_{2m},\ Ax_{2m+1})}\}) \\
&= \varphi(0)
\end{aligned}
$$

This implies that $d(Ax_{2m+1}, Ax_{2m+2}) = 0$ and hence $Ax_{2m+1} = Ax_{2m+2}$, i.e., $Ax_{n+1} = Ax_{n+2}$ which shows that $Ax_n = Ax_{n+2}$. Repeating this procedure inductively, we get $Ax_n = Ax_{n+k}$ for $k \ge 1$.

Hence, $\{Ax_m\}_{m \ge n}$ is a constant sequence and hence is Cauchy in E.

Case (ii): $Ax_n \ne Ax_{n+1}$ for all $n = 1, 2, \cdots$.

First we assume that n is even. Then $n = 2m$ for some $m \in Z_+$.

Now consider

$$
\begin{aligned}
d(Ax_{2m+1}, Ax_{2m}) &= d(Sx_{2m}, Tx_{2m-1}) \\
&\le \varphi(max\{d(Ax_{2m}, Ax_{2m-1}), \frac{d(Ax_{2m},\ Sx_{2m})\ d(Ax_{2m-1},\ Tx_{2m-1})}{1+d(Ax_{2m},\ Ax_{2m-1})}, \\
&\qquad \frac{d(Ax_{2m},\ Tx_{2m-1})\ d(Ax_{2m-1},\ Sx_{2m})}{1+d(Ax_{2m},\ Ax_{2m-1})}\}) \\
&= \varphi(max\{d(Ax_{2m}, Ax_{2m-1}), \frac{d(Ax_{2m},\ Ax_{2m+1})\ d(Ax_{2m-1},\ Ax_{2m})}{1+d(Ax_{2m},\ Ax_{2m-1})}, \\
&\qquad \frac{d(Ax_{2m},\ Ax_{2m})\ d(Ax_{2m-1},\ Ax_{2m+1})}{1+d(Ax_{2m},\ Ax_{2m-1})}\}) \\
&\le \varphi(max\{d(Ax_{2m}, Ax_{2m-1}), d(Ax_{2m+1}, Ax_{2m})\}).
\end{aligned}
$$

Hence,

$$
d(Ax_{2m+1}, Ax_{2m}) \le \varphi(d(Ax_{2m}, Ax_{2m-1})),\ m = 1, 2, \cdots, \tag{2.1.2}
$$

since the other possibility leads us to a contradiction.

Similarly, if $n = 2m + 1$ for some $m \in Z_+$, we get

$$
\begin{aligned}
d(Ax_{2m+2}, Ax_{2m+1}) &= d(Ax_{2m+1}, Ax_{2m+2}) \\
&\le \varphi(d(Ax_{2m}, Ax_{2m+1})) \\
&= \varphi(d(Ax_{2m+1}, Ax_{2m})),\ m = 1, 2, \cdots. \qquad (2.1.3)
\end{aligned}
$$

Hence, from (2.1.2) and (2.1.3), we get

$$
d(Ax_{n+1}, Ax_n) \le \varphi(d(Ax_n, Ax_{n-1})), \text{ for all } n = 2, 3, \cdots. \tag{2.1.4}
$$

This implies that

$$d(Ax_{n+1}, Ax_n) < d(Ax_n, Ax_{n-1}), \text{ for all } n = 2, 3, \cdots. \qquad (2.1.5)$$

Hence, $\{d(Ax_{n+1}, Ax_n)\}_{n=2}^{\infty}$ is a decreasing sequence of nonnegative reals.

Now by repeated applications of (2.1.4), we get

$$d(Ax_{n+1}, Ax_n) \leq \varphi^{n-1}(d(Ax_2, Ax_1)), \text{ for all } n = 2, 3, \cdots. \qquad (2.1.6)$$

Letting $n \to \infty$, by Lemma 1.10, the right hand side of (2.1.6) tends to zero, *i.e.*,

$$\lim_{n\to\infty} d(Ax_{n+1}, Ax_n) = 0. \qquad (2.1.7)$$

By (2.1.5) and (2.1.7), it is sufficient to show that $\{Ax_{2n}\}$ is Cauchy. Otherwise, there exists an $\varepsilon > 0$ and there exist sequences $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$d(Ax_{2m_k}, Ax_{2n_k}) \geq \varepsilon \quad \text{and} \quad d(Ax_{2m_k-2}, Ax_{2n_k}) < \varepsilon. \qquad (2.1.8)$$

Hence,

$$\varepsilon \leq \liminf_{k\to\infty} d(Ax_{2m_k}, Ax_{2n_k}). \qquad (2.1.9)$$

Now for each positive integer k, by triangle inequality, we get

$$\begin{aligned} d(Ax_{2m_k}, Ax_{2n_k}) \leq\ & d(Ax_{2m_k}, Ax_{2m_k-1}) + d(Ax_{2m_k-1}, Ax_{2m_k-2}) \\ & + d(Ax_{2m_k-2}, Ax_{2n_k}). \end{aligned}$$

On taking limit supremum of both sides, as $k \to \infty$, we get

$$\limsup_{n\to\infty} d(Ax_{2m_k}, Ax_{2n_k}) \leq \varepsilon. \qquad (2.1.10)$$

Hence, from (2.1.9) and (2.1.10), we have

$$\lim_{k\to\infty} d(Ax_{2m_k}, Ax_{2n_k}) = \varepsilon. \qquad (2.1.11)$$

Now from the triangle inequality, we have

$$d(Ax_{2m_k}, Ax_{2n_k-1}) \leq d(Ax_{2m_k}, Ax_{2n_k}) + d(Ax_{2n_k}, Ax_{2n_k-1}).$$

On taking limit supremum, as $k \to \infty$, we get

$$\limsup_{n\to\infty} d(Ax_{2m_k}, Ax_{2n_k-1}) \leq \varepsilon. \qquad (2.1.12)$$

Again from the triangle inequality, we get

$$d(Ax_{2m_k}, Ax_{2n_k}) \leq d(Ax_{2m_k}, Ax_{2n_k-1}) + d(Ax_{2n_k-1}, Ax_{2n_k});$$

On taking limit infimum, as $k \to \infty$, we get

$$\varepsilon \leq \liminf_{n\to\infty} d(Ax_{2m_k}, Ax_{2n_k-1}). \qquad (2.1.13)$$

Hence, from (2.1.12) and (2.1.13), we have

$$\lim_{k\to\infty} d(Ax_{2m_k}, Ax_{2n_k-1}) = \varepsilon. \qquad (2.1.14)$$

Similarly, using the triangle inequality and (2.1.11), we obtain

$$\lim_{k\to\infty} d(Ax_{2m_k+1}, Ax_{2n_k-1}) = \varepsilon. \tag{2.1.15}$$

Now consider

$$\begin{aligned} d(Ax_{2m_k}, Ax_{2n_k}) &\le d(Ax_{2m_k}, Ax_{2m_k+1}) + d(Ax_{2m_k+1}, Ax_{2n_k}) \\ &= d(Ax_{2m_k}, Ax_{2m_k+1}) + d(Sx_{2m_k}, Tx_{2n_k-1}) \\ &\le d(Ax_{2m_k}, Ax_{2m_k+1}) + \varphi(max\{d(Ax_{2m_k}, Ax_{2n_k-1}), \\ &\quad \frac{d(Ax_{2m_k},\ Sx_{2m_k})\ d(Ax_{2n_k-1},\ Tx_{2n_k-1})}{1+d(Ax_{2m_k},\ Ax_{2n_k-1})}, \\ &\quad \frac{d(Ax_{2m_k},\ Tx_{2n_k-1})\ d(Ax_{2n_k-1},\ Sx_{2m_k})}{1+d(Ax_{2m_k},\ Ax_{2n_k-1})}\}) \\ &= d(Ax_{2m_k}, Ax_{2m_k+1}) + \varphi(max\{d(Ax_{2m_k}, Ax_{2n_k-1}), \\ &\quad \frac{d(Ax_{2m_k},\ Ax_{2m_k+1})\ d(Ax_{2n_k-1},\ Ax_{2n_k})}{1+d(Ax_{2m_k},\ Ax_{2n_k-1})}, \\ &\quad \frac{d(Ax_{2m_k},\ Ax_{2n_k})\ d(Ax_{2n_k-1},\ Ax_{2m_k+1})}{1+d(Ax_{2m_k},\ Ax_{2n_k-1})}\}). \end{aligned}$$

Letting $k \to \infty$, using the continuity of φ, and using (2.1.7),(2.1.11), (2.1.14) and (2.1.15), we get

$\varepsilon \le 0 + \varphi(max\{\varepsilon, \frac{0}{1+\varepsilon}, \frac{\varepsilon^2}{1+\varepsilon}\}) = \varphi(\varepsilon)$,

a contradiction. Thus $\{Ax_{2n}\}_{n=1}^{\infty}$ is Cauchy and hence $\{Ax_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $A(E)$.

As $A(E)$ is complete, there exists $z \in A(E)$ such that $\lim_{n\to\infty} Ax_n = z$.

Hence,

$$\lim_{n\to\infty} Ax_{2n+1} = \lim_{n\to\infty} Sx_{2n} = z. \tag{2.1.16}$$

and

$$\lim_{n\to\infty} Ax_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = z. \tag{2.1.17}$$

Since $z \in A(E)$, there exists $u \in E$ such that $z = Au$. (2.1.18)

Now consider

$$d(Sx_{2n}, Tu) \le \varphi(max\{d(Ax_{2n}, Au), \frac{d(Ax_{2n},\ Sx_{2n})\ d(Au,\ Tu)}{1+d(Ax_{2n},\ Au)}, \frac{d(Ax_{2n},\ Tu)\ d(Sx_{2n},\ Au)}{1+d(Ax_{2n},\ Au)}\}).$$

Letting $n \to \infty$, by using (2.1.16), (2.1.17) and (2.1.18) and the continuity of φ, we get

$$d(z,Tu) \leq \varphi(max\{d(z,Au), \frac{d(z,\ z)\ d(Au,\ Tu)}{1+d(z,\ Au)}, \frac{d(z,\ Tu)\ d(z,\ Au)}{1+d(z,\ Au)}\})$$
$$= \varphi(0).$$

Hence, $z = Tu$. (2.1.19)

Again consider

$$d(Su,Tx_{2n+1}) \leq \varphi(max\{d(Au,Ax_{2n+1}), \frac{d(Au,\ Su)\ d(Ax_{2n+1},\ Tx_{2n+1})}{1+d(Au,\ Ax_{2n+1})}, \frac{d(Au,\ Tx_{2n+1})\ d(Su,\ Ax_{2n+1})}{1+d(Au,\ Ax_{2n+1})}\}).$$

Letting $n \to \infty$, by using (2.1.16), (2.1.17) and (2.1.18) and the continuity of φ, we get

$$d(Su,z) \leq \varphi(max\{d(Au,z), \frac{d(Au,\ Su)\ d(z,\ z)}{1+d(Au,\ z)}, \frac{d(Su,\ z)\ d(Au,\ z)}{1+d(Au,\ z)}\})$$
$$= \varphi(0).$$

Hence, $z = Su$. (2.1.20)

Hence, from (2.1.18), (2.1.19) and (2.1.20), it follows that

$$z = Au = Su = Tu. \quad (2.1.21)$$

Since (A,S) and (A,T) are weakly compatible pairs of mappings, it follows from (2.1.21) that $ASu = SAu$ and $ATu = TAu$ and hence, we have

$$Sz = SAu = ASu = Az = ATu = TAu = Tz.$$

Hence, $Sz = Az = Tz$. (2.1.22)

Now we claim that z is a common fixed point of A, S and T.

Consider

$$d(Sz,z) = d(Sz,Tu) \leq \varphi(max\{d(Az,Au), \frac{d(Az,\ Sz)\ d(Au,\ Tu)}{1+d(Az,\ Au)}, \frac{d(Az,\ Tu)\ d(Au,\ Sz)}{1+d(Az,\ Au)}\})$$
$$= \varphi(d(Sz,z)).$$

Hence, $d(Sz,z) = 0$, i.e., $Sz = z$.

Hence, $Az = Sz = Tz = z$.

Uniqueness of z follows from the inequality (2.1.1). Hence this completes the proof of the theorem.

□

Corollary 2.2. *Let E be a nonempty subset of a metric space (X,d) and $A,S,T : E \to E$ be three selfmaps such that $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$. Assume that* *there exists a $k \in [0, 1)$ such that*

$$d(Sx, Ty) \leq k\ max\{d(Ax, Ay), \frac{d(Ax,\ Sx)\ d(Ay,\ Ty)}{1 + d(Ax,\ Ay)}, \frac{d(Ax,\ Ty)\ d(Ay,\ Sx)}{1 + d(Ax,\ Ay)}\} \tag{2.2.1}$$

for all $x, y \in E$. Assume also that the pairs of mappings (A, S) and (A, T) are weakly compatible. If $A(E)$ is complete, then A, S and T have a unique common fixed point in X.

Proof. Follows from Theorem 2.1, by taking $\varphi(t) = kt$ for some $k \in [0, 1)$. □

Remark 2.3. *By choosing $S = T$ in Corollary 2.2, clearly Theorem 1.4 follows as a corollary to Corollary 2.2. In Theorem 1.4, the assumption is $T(E)$ is complete, where as here $A(E)$ is complete.*

If we relax the continuity condition of 'either A or T ' and impose the assumption 'φ is monotonically increasing on R_+' on φ in Theorem 1.3, the following corollary suggests that the conclusion of Theorem 1.3 still holds.

Corollary 2.4. *Let (X, d) be a complete metric space and let A and T be R-weakly commuting selfmaps of X satisfying the condition:*

$$d(Tx, Ty) \leq \varphi(d(Ax, Ay)) \tag{2.4.1}$$

for all $x, y \in X$, where $\varphi : R_+ \to R_+$ is a continuous, monotone increasing function such that $\varphi(t) < t$ for each $t > 0$. If $T(X) \subset A(X)$, then A and T have a unique common fixed point in X.

Proof. Since A and T are R-weakly commuting, they are weakly compatible and since the inequality (2.4.1) implies the inequality (2.1.1) with $S = T$, the conclusion of Corollary 2.4 follows from Theorem 2.1. □

Remark 2.5. *Since Theorem 1 of Jungck* [6] *is a corollary to Corollary 2.4, it also follows as a corollary to Theorem 2.1.*

Example 2.6. Let $X = R$, the real line with the usual metric and $E = [0, 1]$. We define mappings A, S and T on E by

$$A(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{2}{3} \\ \frac{4}{3} - x & \text{if } \frac{2}{3} \leq x \leq 1, \end{cases} \qquad S(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x < \frac{2}{3} \\ 1 - \frac{1}{2}x & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}$$

and

$$T(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \leq x < \frac{2}{3} \\ 1 - \frac{1}{2}x & \text{if } \frac{2}{3} \leq x \leq 1. \end{cases}$$

Here $A(E) = \{0\} \cup [\frac{1}{3}, \frac{2}{3}]$, $S(E) = T(E) = [\frac{1}{2}, \frac{2}{3}]$ so that $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$. Besides $A(E)$ is compact and the pairs of mappings (A, S) and (A, T) are weakly compatible. We also observe that the mappings A, S and T satisfy the inequality (2.1.1) with $\varphi : R_+ \to R_+$ defined by $\varphi(t) = \frac{1}{2}t$, $t \in R_+$.

Hence the mappings A, S and T satisfy all the conditions of the hypotheses of Theorem 2.1 and $\frac{2}{3}$ is the unique common fixed point of the mappings A, S and T.

Further, we observe that the pairs of mappings (A, S) and (A, T) are not compatible, for, if $x_n = \frac{2}{3} + \frac{1}{n}, n \geq 3$; $Ax_n = \frac{2}{3} - \frac{1}{n}, n \geq 3$, $Sx_n = Tx_n = \frac{2}{3} - \frac{1}{2n}, n \geq 3$ so that $Ax_n \to \frac{2}{3}$, $Sx_n \to \frac{2}{3}$ and $Tx_n \to \frac{2}{3}$ as $n \to \infty$. Now $SAx_n = \frac{1}{2}$, $ASx_n = 0$, $TAx_n = \frac{2}{3}$, and $ATx_n = 0$ so that $\lim_{n\to\infty} d(SAx_n, ASx_n) = \frac{1}{2} \neq 0$, $\lim_{n\to\infty} d(TAx_n, ATx_n) = \frac{2}{3} \neq 0$.

Here if we choose A as above and $S(x) = T(x) = \begin{cases} \frac{1}{2} & if\ 0 \leq x < \frac{2}{3} \\ 1 - \frac{1}{2}x & if\ \frac{2}{3} \leq x \leq 1, \end{cases}$

the pair of mappins (A, T) is not compatible which implies that (A, T) is not R-weakly commuting. Hence Corollary 2.4 is not applicable. Also, since the mappings A and T are not commuting, Theorem 1 of Jungck [6] is not applicable.

Example 2.7. Let $X = R_+$ with the usual metric and $E = X$. We define mappings A and T on E by $Ax = x$ and $Tx = \frac{x}{1+x}$.

Here if we choose $S = T$, though A and T are continuous and commuting maps; A and T satisfy all the conditions of the hypotheses of Theorem 2.1 with $\varphi : R_+ \to R_+$ defined by $\varphi(t) = \frac{t}{1+t}$.

But T is not A-contraction, for, fixing $y = 0$ and for all $x \in X$, $|Tx - T0| = \frac{x}{1+x}$ and $|Ax - A0| = x$. Hence, for all $x \in X$, there does not exist a $k \in [0, 1)$ satisfying $\frac{x}{1+x} \leq kx$. In fact, for all $k \in [0, 1)$, T is not a A-contraction when $y = 0$ and $0 < x < \frac{1-k}{k}$.

Hence, Theorem 1.4 fails to hold.

THerefore, Example 2.6 and Example 2.7 show that Theorem 2.1 is a generalization of Theorem 1.4, Corollary 2.4 and Jungck ([6], Theorem 1).

We now prove the following results as an application of Corollary 2.2.

Theorem 2.8. *Let E be a nonempty q-starshaped subset of a normed space X and let $S, T, A : E \to E$ be three continuous selfmaps such that $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$ and satisfying*

$$\|Sx - Ty\| \leq max\{\|Ax - Ay\|, \frac{\delta(Ax,\ [Sx,\ q])\ \delta(Ay,\ [Ty,\ q])}{1 + \|Ax - Ay\|}, \frac{\delta(Ax,\ [Ty,\ q])\ \delta(Ay,\ [Sx,\ q])}{1 + \|Ax - Ay\|}\} \tag{2.8.1}$$

for all $x, y \in E$. Suppose that both (T, A) and (S, A) are C_q-commuting and A is q-affine. If $A(E)$ is a compact subset of E, then $F(T) \cap F(S) \cap F(A) \neq \emptyset$.

Proof. We choose a sequence $\{k_n\} \subseteq (0,1)$ with $\lim_{n\to\infty} k_n = 1$. For each $n = 1, 2, \cdots$ and for all $x \in E$, define mappings S_n and T_n by

$$S_n x = k_n Sx + (1-k_n)q \text{ and } T_n x = k_n Tx + (1-k_n)q.$$

Since E is q-starshaped, A is q-affine, $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$, we get $S_n x \in A(E)$ and $T_n x \in A(E)$.

Hence, $S_n(E) \subseteq A(E)$ and $T_n(E) \subseteq A(E)$.

Also, for all $x, y \in E$,

$$\begin{aligned}\|S_n x - T_n y\| &= k_n \|Sx - Ty\| \\ &\leq k_n\ max\{\|Ax - Ay\|, \frac{\delta(Ax,\ [Sx,\ q])\ \delta(Ay,\ [Ty,\ q])}{1 + \|Ax - Ay\|}, \\ &\qquad \frac{\delta(Ax,\ [Ty,\ q])\ \delta(Ay,\ [Sx,\ q])}{1 + \|Ax - Ay\|}\} \\ &\leq k_n\ max\{\|Ax - Ay\|, \frac{\|Ax - S_n x\|\ \|Ay - T_n y\|}{1 + \|Ax - Ay\|}, \frac{\|Ax - T_n y\|\ \|Ay - S_n x\|}{1 + \|Ax - Ay\|}\}\end{aligned}$$

where $\delta(Ax, [Sx, q]) = \inf\{\|Ax - y\| : y \in [Sx, q]\} \leq \|Ax - S_n x\|$ for each $n = 1, 2, \cdots$.

Hence,for each $n = 1, 2, \cdots$, mappings S_n, T_n and A satisfy the inequality (2.2.1).

Since the pairs (T, A) and (S, A) are C_q-commuting and A is q-affine, if $S_n x = Ax = T_n x$, we get $S_n Ax = AS_n x$ and $T_n Ax = AT_n x$.

This implies that the pairs (S_n, A) and (T_n, A) are weakly compatible maps. As $A(E)$ is compact, then $A(E)$ is complete. Therefore, maps S_n, T_n and A satisfy all the conditions of Corollary 2.2 and hence for each $n = 1, 2, \cdots$, there exists a unique $x_n \in E$ such that $S_n x_n = Ax_n = T_n x_n = x_n$.

Since $A(E)$ is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} Ax_{n_j} = z$ (say) in $A(E)$, and hence $x_{n_j} \to z$ as $j \to \infty$.

Now as $j \to \infty$, the continuity of A, S and T, we obtain

$$Tx_{n_j} \to Tz,\ Ax_{n_j} \to Az, \text{ and } Sx_{n_j} \to Sz\ .$$

Hence,we have

$$z = \lim_{j\to\infty} x_{n_j} = \lim_{j\to\infty} Ax_{n_j} = Az;$$

$$z = \lim_{j\to\infty} x_{n_j} = \lim_{j\to\infty} S_{n_j} x_{n_j} = Sz; \text{ and}$$

$$z = \lim_{j\to\infty} x_{n_j} = \lim_{j\to\infty} T_{n_j} x_{n_j} = Tz.$$

Hence, $z = Az = Tz = Sz$. Hence the theorem follows.

□

Now we give an example in support of Theorem 2.8.

Example 2.9. Let $X = [0, \infty)$ with the usual metric and $E = [0, \frac{3}{2}]$. We define mappings $A, S, T : E \to E$ by

$$A(x) = \begin{cases} x & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{4} & \text{if } \frac{1}{2} \le x \le \frac{3}{2}, \end{cases} \qquad S(x) = \begin{cases} x & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \le x \le \frac{3}{4} \\ \frac{1}{2}x + \frac{1}{8} & \text{if } \frac{3}{4} \le x \le \frac{3}{2} \end{cases}$$

and

$$T(x) = \begin{cases} x & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \le x \le \frac{3}{4} \\ \frac{1}{2}x + \frac{1}{8} & \text{if } \frac{3}{4} \le x \le 1 \\ \frac{5}{8} & \text{if } 1 \le x \le \frac{3}{2}. \end{cases}$$

Here we observe that $A(E) = [0, 1]$, $S(E) = [0, \frac{7}{8}]$ and $T(E) = [0, \frac{5}{8}]$ so that $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$. Also, E is convex and hence q-starshaped at any $q \in E$ and A is q-affine with $q = \frac{1}{2}$. We also observe that the pairs of mappings (A, S) and (A, T) are C_q-commuting with $C_q(A, S) = C_q(A, T) = \{\frac{1}{2}\}$ and $A(E)$ is compact. Further, the mappings A, S and T satisfy the inequality (2.8.1). Hence, the mappings A, S and T satisfy all the conditions of Theorem 2.8 and $\frac{1}{2}$ is the unique common fixed point of A, S and T.

Theorem 2.10. *Let E be a nonempty q-starshaped subset of a Banach space X and let $S, T, A : E \to E$ be three weakly continuous selfmaps satisfying the inequality* (2.8.1)*. Assume that $S(E) \subseteq A(E)$ and $T(E) \subseteq A(E)$. Further suppose that the pairs (T, A) and (S, A) are C_q-commuting and A is q-affine. If $A(E)$ is a weakly compact subset of E, then $F(T) \cap F(S) \cap F(A) \ne \emptyset$.*

Proof. We choose a sequence $\{k_n\} \subseteq (0, 1)$ with $\lim_{n\to\infty} k_n = 1$. For $n = 1, 2, \cdots$ and for all $x \in E$, define mappings S_n and T_n by

$$S_n x = k_n Sx + (1 - k_n)q \text{ and } T_n x = k_n Tx + (1 - k_n)q.$$

Here we note that $A(E)$ is complete, since the weak topology is Hausdorff and $A(E)$ is weakly compact. Hence by the proof of Theorem 2.8, for each n, there exists a unique $x_n \in E$ such that $S_n x_n = Ax_n = T_n x_n = x_n$.

Since $A(E)$ is weakly compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $Ax_{n_j} \to z$ (say) weakly in $A(E)$ as $j \to \infty$and hence $x_{n_j} \to z$ weakly as $j \to \infty$.

By the weak continuity of A, S and T, as $j \to \infty$, we obtain

$$Tx_{n_j} \to Tz \text{ weakly, } Ax_{n_j} \to Az \text{ weakly,and } Sx_{n_j} \to Sz \text{ weakly .}$$

Hence,we have

$$z = w - \lim_{j\to\infty} x_{n_j} = w - \lim_{j\to\infty} Ax_{n_j} = Az;$$

$$z = w - \lim_{j\to\infty} x_{n_j} = w - \lim_{j\to\infty} S_{n_j}x_{n_j} = Sz; \text{ and}$$

$$z = w - \lim_{j\to\infty} x_{n_j} = w - \lim_{j\to\infty} T_{n_j}x_{n_j} = Tz.$$

Hence, $z = Az = Tz = Sz$. Hence the theorem follows. □

3 Invariant Approximation Results

In this section, we prove the existence of common fixed points in the set of best approximations by using Theorem 2.8 and Theorem 2.10.

Theorem 3.1. *Let E be a subset of a normed space X and $A, S, T : E \to E$ be three continuous selfmaps such that $u \in F(T) \cap F(S) \cap F(A)$ for some $u \in X$ and $S(\partial E \cap E) \subset E$ and $T(\partial E \cap E) \subset E$. Assume that $P_E(u)$ is q-starshaped, A is q-affine and $A(P_E(u)) = P_E(u)$ is compact. Suppose the pairs (S, A) and (T, A) are C_q-commuting and for all $x, y \in P_E(u) \cup \{u\}$, satisfy the inequality*

$$\|Sx - Ty\| \le \begin{cases} \|Ax - Au\| & \text{if } y = u \\ \|Au - Ay\| & \text{if } x = u \\ max\{\|Ax - Ay\|, & \\ \qquad \frac{\delta(Ax,\ [Sx,\ q])\ \delta(Ay,\ [Ty,\ q])}{1+\|Ax-Ay\|}, & \\ \qquad \frac{\delta(Ax,\ [Ty,\ q])\ \delta(Ay,\ [Sx,\ q])}{1+\|Ax-Ay\|}\} & \text{if } x, y \in P_E(u). \end{cases} \tag{3.1.1}$$

Then $P_E(u) \cap F(T) \cap F(S) \cap F(A) \neq \emptyset$.

Proof. Let $x \in P_E(u)$. Then $\|x - u\| = \delta(u, E)$. Since for any $k \in (0, 1)$, $\|ku + (1 - k)x - u\| = (1 - k)\|x - u\| < \delta(u, E)$, the line segment $\{ku + (1 - k)x : 0 < k < 1\}$ and the set E are disjoint. Thus x is not an interior point of E and so $x \in \partial E \cap E$.

Since $S(\partial E \cap E) \subset E$ and $T(\partial E \cap E) \subset E$, we have $Sx, Tx \in E$. Also, since $Ax \in P_E(u)$, $u \in F(T) \cap F(S) \cap F(A)$, and S, T and A satisfy the inequality (3.1.1), we have $\|Sx - u\| \le \delta(u, E)$ and $\|u - Tx\| \le \delta(u, E)$.

Hence, $Sx, Tx \in P_E(u)$ and hence $S(P_E(u)) \subseteq A(P_E(u))$ and $T(P_E(u)) \subseteq A(P_E(u))$.

Therefore, by Theorem 2.8, there exists $z \in P_E(u)$ such that $Sz = Tz = Az = z$.

Hence, $P_E(u) \cap F(T) \cap F(S) \cap F(A) \neq \emptyset$. □

Theorem 3.2. *Let E be a subset of a Banach space X and $A, S, T : E \to E$ be three weakly continuous selfmappings such that $u \in F(T) \cap F(S) \cap F(A)$ for some $u \in X$ and $S(\partial E \cap E) \subset E$ and $T(\partial E \cap E) \subset E$. Assume that $P_E(u)$ is q-starshaped, A is q-affine and $A(P_E(u)) = P_E(u)$ is weakly compact. Suppose the* *pairs* (S, A) *and* (T, A) *are* C_q*-commuting and satisfy the inequality* (3.1.1)*. Then* $P_E(u) \cap F(T) \cap F(S) \cap F(A) \neq \emptyset$.

Proof. Runs on the same lines as that of the proof of Theorem 3.1, where we use Theorem 2.10 instead of Theorem 2.8. □

Remark 3.3. *Theorem 3.1 extends Theorem 1.5, Theorem 1.6, Theorem 1.7, Theorem 1.9 and Habiniak* ([4]*, Theorem 8*) *to three selfmaps.*

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