



Common Fixed Point Theorems for Multivalued β_* - ψ -Contractive Mappings

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Abstract : The aim of this article is to present some common fixed point theorems for multivalued $\beta_* - \psi$ -contractive mappings. We also prove the existence and uniqueness of common fixed point for multivalued α_* -admissible mappings with respect to the function η_* . Our results extend several results in the literature. Suitable examples are presented to substantiate our main results.

Keywords : $\beta_* - \psi$ - contraction; common fixed point; metric space.

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1 Introduction

There has been considerable interest in the field of fixed point theory of non-expansive/contractive mappings in Functional Analysis. The classical contraction mapping principle was introduced by the great mathematician Stefan Banach in 1922. After that lots of generalization of this principle have been done in different directions by several mathematicians sometimes by changing the structure of underlying space or weakening the contraction condition (see, for example, ([1]-[16]) and references therein). In 2012, Samet et al. [1] introduced a new concept of $\alpha - \psi$ contraction for single valued mappings and proved some theorems related to such

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type of mappings. The authors of [2] extended these results from single valued function to multivalued function. The concept of $\alpha - \psi$ -contraction principle was modified by Hussain et al. [3] by introducing a new function η and they proved some fixed point results for such multivalued mappings. Very recently Berzig et al. further modified and generalized the concept of $\alpha - \psi$ -contraction principle as $\beta - \psi$ contraction mappings in [4]. Hussain et al. [5] presented some common fixed point results for single valued $\alpha - \psi$ -contractions on a complete metric space. In this paper, we discuss common fixed point results for multivalued $\beta_* - \psi$ -contraction mappings in complete metric space. Again we prove the same result for a pair of multivalued mappings which are α_* -admissible with respect to the function η_* . Moreover, some examples are presented to illustrate our main results.

2 Preliminaries

For the organization of the paper, we present some lemmas and definitions. Notations have their usual meaning in this sequel. Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ . Then the following lemma is obvious.

Lemma 2.1 ([1]). *For every function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ the following holds:*

if ψ is nondecreasing, then for each $t > 0$, $\lim_{n \rightarrow +\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.

Definition 2.1 ([6]). Let T be a self mapping on a metric space (X, d) and let $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two functions. We say that T is an α -admissible function with respect to η if for all $x, y \in X$

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

If $\eta(x, y) = 1$, then T is an α -admissible mapping. Also if $\alpha(x, y) = 1$, then T is an η -subadmissible mapping.

Definition 2.2 ([5]). Let $S, T : X \rightarrow X$ and $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two functions. We say that the pair (S, T) is α -admissible with respect to η if for all $x, y \in X$

$$\begin{aligned} \alpha(x, y) \geq \eta(x, y) &\Rightarrow \alpha(Tx, Sy) \geq \eta(Tx, Sy); \\ &\alpha(Sx, Ty) \geq \eta(Sx, Ty). \end{aligned}$$

Note that if $\eta(x, y) = 1$, then the pair (S, T) is α -admissible mapping. Also if $\alpha(x, y) = 1$, then the pair (S, T) is η -subadmissible mapping.

Definition 2.3 ([2, 3]). Let $T : X \rightarrow \mathcal{P}(X)$ be a close valued multifunction and $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two functions. We say that T is an α_* -admissible function with respect to η_* if for all $x, y \in X$

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty),$$

where

$$\begin{aligned} \alpha_*(Tx, Ty) &= \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}; \\ \eta_*(Tx, Ty) &= \sup\{\eta(a, b) : a \in Tx, b \in Ty\}. \end{aligned}$$

If we take $\eta(x, y) = 1$, then T is called α_* -admissible function.

Definition 2.4 ([2]). Let (X, d) be a complete metric space; $\beta : X \times X \rightarrow [0, +\infty)$ be a mapping and $T : X \rightarrow \mathcal{P}(X)$ be close valued multifunction and $\psi \in \Psi$. We say that T is β_* – ψ -contractive multifunction whenever

$$\beta_*(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y)) \quad \forall x, y \in X,$$

where $\beta_*(Tx, Ty) = \inf\{\beta(a, b) : a \in Tx, b \in Ty\}$.

Definition 2.5 ([7]). Let (X, d) be a complete metric space; $\beta : X \times X \rightarrow [0, +\infty)$ be a mapping and $T : X \rightarrow \mathcal{P}(X)$ be close valued multifunction and $\psi \in \Psi$. We say that T is β – ψ -contractive multifunction whenever

$$\beta(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y)) \quad \forall x, y \in X.$$

Definition 2.6 ([7]). Let $T : X \rightarrow \mathcal{P}(X)$ be a close valued multifunction and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. We say that T is β -admissible function if for all $x, y \in X$

$$\beta(x, y) \geq 1 \Rightarrow \beta(a, b) \geq 1 \quad \forall a \in Tx, b \in Ty.$$

Definition 2.7 ([8]). Let $S, T : X \rightarrow \mathcal{P}(X)$ be two functions. A point $x \in X$ is said to be fixed point of S if $x \in Sx$ and a point $x \in X$ is said to be *common fixed point* of S and T if $x \in Sx \cap Tx$.

3 Main Results

We start this section with some definitions and lemmas and then we present our main results.

Lemma 3.1. *Let $A, B \in CL(X)$ be two compact subsets of X . Then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Proof. The proof is easy. It is obvious from the Lemma 2.1 of [8]. □

Definition 3.1. Let $S, T : X \rightarrow \mathcal{P}(X)$ and $\beta : X \times X \rightarrow [0, +\infty)$ be a function. We say that the pair (S, T) is β_* -admissible if for all $x, y \in X$

$$\begin{aligned} \beta(x, y) \geq 1 &\Rightarrow \beta_*(Tx, Sy) \geq 1; \\ &\beta_*(Sx, Ty) \geq 1. \end{aligned}$$

Again the pair (S, T) is said to be β -admissible if

$$\beta(x, y) \geq 1 \Rightarrow \beta(a, b) \geq 1 \quad \forall a \in Sx, b \in Ty.$$

Definition 3.2. Let $S, T : X \rightarrow \mathcal{P}(X)$ be two multifunctions. $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ are two functions. We say that the pair (S, T) is α_* -admissible with respect to η_* if for all $x, y \in X$

$$\begin{aligned} \alpha(x, y) \geq \eta(x, y) &\Rightarrow \alpha_*(Tx, Sy) \geq \eta_*(Tx, Sy); \\ &\alpha_*(Sx, Ty) \geq \eta_*(Sx, Ty). \end{aligned}$$

If we take $\eta(x, y) = 1$, then the pair (S, T) is called α_* -admissible and for $\alpha(x, y) = 1$, the pair (S, T) is called η_* -subadmissible mapping.

Definition 3.3. Let (X, d) be a complete metric space and $\beta : X \times X \rightarrow [0, +\infty)$ be a mapping. Let $S, T : X \rightarrow \mathcal{P}(X)$ be multifunctions and $\psi \in \Psi$. We say that the pair (S, T) is β_* - ψ -contractive multifunction whenever

$$\begin{aligned} \beta_*(Tx, Sy)H(Tx, Sy) &\leq \psi(d(x, y)) \quad \forall x, y \in X; \\ \beta_*(Sx, Ty)H(Sx, Ty) &\leq \psi(d(x, y)) \quad \forall x, y \in X, \end{aligned}$$

where $\beta_*(Tx, Sy) = \inf\{\beta(a, b) : a \in Tx, b \in Sy\}$.

Now we are in a position to state our main result.

Theorem 3.2. Let (X, d) be a complete metric space. $S, T : X \rightarrow \mathcal{P}(X)$ are two compact valued multifunctions. Assume that the pair (S, T) is β_* - ψ -contractive mapping. If the following assertions hold:

1. (x_n) is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$ then $\beta(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$;
2. there exists some $x_0 \in X$ such that $\beta(x_0, y) \geq 1$ whenever $y \in Tx_0$ or $y \in Sx_0$

then S and T have a common fixed point i.e. there exists $x \in X$ such that $x \in Sx \cap Tx$.

Proof. By the hypothesis of the theorem suppose $x_0 \in X$ and $x_1 \in Tx_0$. So $\beta(x_0, x_1) \geq 1$ which implies $\beta_*(Tx_0, Sx_1) \geq 1$. Again Tx_0 and Sx_1 are compact subsets and for $x_1 \in Tx_0$, we can find $x_2 \in Sx_1$, such that,

$$d(x_1, x_2) \leq H(Tx_0, Sx_1).$$

Since the pair (S, T) is β_* - ψ -admissible mapping so we get

$$d(x_1, x_2) \leq \beta_*(Tx_0, Sx_1)H(Tx_0, Sx_1) \leq \psi(d(x_0, x_1)).$$

So for $x_1 \in Tx_0$, $x_2 \in Sx_1$, we have,

$$\beta(x_1, x_2) \geq 1 \Rightarrow \beta_*(Sx_1, Tx_2) \geq 1,$$

and we get

$$\beta_*(Sx_1, Tx_2)H(Sx_1, Tx_2) \leq \psi(d(x_1, x_2)).$$

Again by the Lemma 2.1, for $x_2 \in Sx_1$, we can find $x_3 \in Tx_2$, such that

$$d(x_2, x_3) \leq H(Sx_1, Tx_2).$$

Therefore we obtain

$$d(x_2, x_3) \leq \beta_*(Sx_1, Tx_2)H(Sx_1, Tx_2) \leq \psi(d(x_1, x_2)) \leq \psi^2(d(x_0, x_1)).$$

Continuing this process, we get a sequence (x_n) with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ such that

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq \beta_*(Sx_{2n-1}, Tx_{2n})H(Sx_{2n-1}, Tx_{2n}) \\ &\leq \psi(d(x_{2n-1}, x_{2n})) \\ &\vdots \\ &\leq \psi^{2n}(d(x_0, x_1)) \end{aligned}$$

and

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq \beta_*(Tx_{2n}, Sx_{2n+1})H(Tx_{2n}, Sx_{2n+1}) \\ &\leq \psi(d(x_{2n}, x_{2n+1})) \\ &\vdots \\ &\leq \psi^{2n+1}(d(x_0, x_1)). \end{aligned}$$

So for any $n \in \mathbb{N}$ we get

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$$

and

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} \psi^n(d(x_0, x_1)) < \infty.$$

We now prove that the sequence (x_n) is a Cauchy sequence. For this let $\epsilon > 0$ be fixed and $n(\epsilon) \in \mathbb{N}$ such that $\sum_n^{\infty} \psi^n(d(x_0, x_1)) < \epsilon$ for all $n > n(\epsilon)$. Let $n, m \in \mathbb{N}$ such that $m > n > n(\epsilon)$. Using triangle inequality, we get,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq \sum_n^{\infty} \psi^n(d(x_0, x_1)) \\ &< \epsilon. \end{aligned}$$

$\Rightarrow (x_n)$ is a Cauchy sequence. Since X is complete and (x_n) is a Cauchy sequence in X , it converges in X .

Suppose $x_n \rightarrow z$ as $n \rightarrow \infty$. Now we prove that z is a common fixed point of T and S . For all $n \geq 0$, we get $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$. Since (x_n) is a

convergent sequence converges to z and (X, d) is a T_2 -space so every subsequence of (x_n) also converges to z . Therefore

$$x_{2n+1} \rightarrow z \text{ as } n \rightarrow \infty;$$

$$x_{2n+2} \rightarrow z \text{ as } n \rightarrow \infty.$$

Now for all $n \geq 0$, $x_{2n+2} \in Sx_{2n+1}$. Hence

$$d(x_{2n+2}, Tz) \leq H(Sx_{2n+1}, Tz).$$

Since $x_n \rightarrow z$ as $n \rightarrow \infty$ so $\beta(x_n, z) \geq 1$ for all n which implies

$$\beta_*(Sx_{2n+1}, Tz)H(Sx_{2n+1}, Tz) \leq \psi(d(x_{2n+1}, z)).$$

Therefore

$$d(x_{2n+2}, Tz) \leq \beta_*(Sx_{2n+1}, Tz)H(Sx_{2n+1}, Tz) \leq \psi(d(x_{2n+1}, z)).$$

Taking $n \rightarrow \infty$, we get, $d(z, Tz) \leq \psi(d(z, z))$. Since $\psi(t) < t$, $\forall t > 0$ and continuous so $d(z, Tz) = 0 \Rightarrow z \in Tz$. Hence z is a fixed point of T . In a similar fashion, we can prove that z is also a fixed point of S . Therefore we can conclude that z is a common fixed point of T and S . \square

Corollary 3.3. *Let (X, d) be a complete metric space and $S, T : X \rightarrow \mathcal{P}(X)$ are two compact valued multifunctions. Assume that the pair (S, T) is β - ψ -contractive mapping. If the following assertions hold:*

1. (x_n) is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$ then $\beta(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$;
2. there exists some $x_0 \in X$ such that $\beta(x_0, y) \geq 1$ whenever $y \in Tx_0$ or $y \in Sx_0$

then S and T have common fixed point.

Theorem 3.4. *If we add the condition ' x_0 is any other common fixed point of S and T with $\beta(z, x_0) \geq 1$ and $d(z, x_0) \leq H(Tz, Sx_0)$ to the hypotheses of Theorem 3.2 then the functions S and T have a unique common fixed point.*

Proof. Given that x_0 is another common fixed point with $\beta(z, x_0) \geq 1$. For $z \in Tz$ and $x_0 \in Sx_0$ we have $d(z, x_0) \leq H(Tz, Sx_0)$. Again $\beta(z, x_0) \geq 1 \Rightarrow \beta_*(Tz, Sx_0) \geq 1$ and hence $\beta_*(Tz, Sx_0)H(Tz, Sx_0) \leq \psi(d(z, x_0))$. Therefore, we have,

$$d(z, x_0) \leq \psi(d(z, x_0)) < d(z, x_0),$$

which is a contradiction. Hence the proof follows. \square

Theorem 3.5. *Let (X, d) be a complete metric space and $S, T : X \rightarrow \mathcal{P}(X)$ be two compact valued multifunctions. Suppose there exist two functions $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ such that the pair (S, T) is α_* -admissible with respect to η_* . Assume that*

$$\begin{aligned} \alpha(x, y) \geq \eta(x, y) &\Rightarrow H(Tx, Sy) \leq \psi(d(x, y)); \\ H(Sx, Ty) &\leq \psi(d(x, y)). \end{aligned}$$

Suppose the following assertions hold:

1. *there exists some x_0 such that $\alpha(x_0, y) \geq \eta(x_0, y)$ whenever $y \in Tx_0$ or $y \in Sx_0$;*
2. *for any sequence (x_n) such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow z$ as $n \rightarrow +\infty$ then $\alpha(x_n, z) \geq \eta(x_n, z)$ for all $n \in \mathbb{N} \cup \{0\}$*

then S and T have a common fixed point.

Remark 3.6. *Theorem 3.5 follows from the Theorem 3.2 if we take the particular form of the function $\beta : X \times X \rightarrow [0, +\infty)$ as*

$$\beta(x, y) = \begin{cases} 1 & \text{whenever } \alpha(x, y) \geq \eta(x, y); \\ 0 & \text{otherwise.} \end{cases}$$

Suppose all the conditions of Theorem 3.5 are satisfied. We show that all the conditions of Theorem 3.2 are also satisfied.

1. Let the pair (S, T) is α_* -admissible w.r.t the function η_* . So

$$\begin{aligned} \alpha(x, y) \geq \eta(x, y) &\Rightarrow \alpha_*(Tx, Sy) \geq \eta_*(Tx, Sy); \\ \alpha_*(Sx, Ty) &\geq \eta_*(Sx, Ty). \end{aligned}$$

Now

$$\begin{aligned} \alpha_*(Tx, Sy) &\geq \eta_*(Tx, Sy) \\ \Rightarrow \alpha(a, b) &\geq \eta(a, b) \quad \forall a \in Tx, b \in Sy \\ \Rightarrow \beta(a, b) &= 1 \quad \forall a \in Tx, b \in Sy. \end{aligned}$$

Thus $\beta(x, y) = 1 \Rightarrow \beta_*(Tx, Sy) = 1$ which shows that the pair (S, T) is β_* -admissible.

2. Suppose there exists $x_0 \in X$ such that $\alpha(x_0, y) \geq \eta(x_0, y)$ for $y \in Tx_0$ or $y \in Sx_0$. Then clearly $\beta(x_0, y) = 1$ whenever $y \in Tx_0$ or $y \in Sx_0$.
3. Suppose for any sequence (x_n) such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \rightarrow z$ as $n \rightarrow +\infty$ then $\alpha(x_n, z) \geq \eta(x_n, z)$ for all $n \in \mathbb{N}$. So $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \Rightarrow \beta(x_n, x_{n+1}) = 1$ and $\alpha(x_n, z) \geq \eta(x_n, z) \Rightarrow \beta(x_n, z) = 1$ for all $n \in \mathbb{N}$.

All the conditions are satisfied and hence the Theorem 3.5 follows from Theorem 3.2.

Note: If we consider $T = S$ in the above theorem then we get the Result 4.1 of [3].

Corollary 3.7. *Let (X, d) be a complete metric space and $S, T : X \rightarrow \mathcal{P}(X)$ be two compact valued multifunctions. Suppose there exists a function $\alpha : X \times X \rightarrow [0, +\infty)$ such that the pair (S, T) is α_* -admissible. Assume that*

$$\begin{aligned}\alpha(x, y) \geq 1 &\Rightarrow H(Tx, Sy) \leq \psi(d(x, y)); \\ &H(Sx, Ty) \leq \psi(d(x, y)).\end{aligned}$$

Suppose the following assertions hold:

1. there exists some x_0 such that $\alpha(x_0, y) \geq 1$ whenever $y \in Tx_0$ or $y \in Sx_0$;
2. for any sequence (x_n) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow z$ as $n \rightarrow +\infty$ then $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$

then S and T have a common fixed point.

Remark 3.8. Here if we consider the function $\beta : X \times X \rightarrow [0, +\infty)$ as

$$\beta(x, y) = \begin{cases} 1 & \text{whenever } \alpha(x, y) \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then Corollary 3.7 directly follows from the Theorem 3.2.

Corollary 3.9. *Let (X, d) be a complete metric space and $S, T : X \rightarrow \mathcal{P}(X)$ be two compact valued multifunctions. Suppose there exists a function $\eta : X \times X \rightarrow [0, +\infty)$ such that the pair (S, T) is η_* -subadmissible. Assume that*

$$\begin{aligned}\eta(x, y) \leq 1 &\Rightarrow H(Tx, Sy) \leq \psi(d(x, y)); \\ &H(Sx, Ty) \leq \psi(d(x, y)).\end{aligned}$$

Suppose the following assertions hold:

1. there exists some x_0 such that $\eta(x_0, y) \leq 1$ whenever $y \in Tx_0$ or $y \in Sx_0$;
2. for any sequence (x_n) such that $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow z$ as $n \rightarrow +\infty$ then $\eta(x_n, z) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$

then S and T have a common fixed point.

Remark 3.10. In similar fashion if we consider the function $\beta : X \times X \rightarrow [0, +\infty)$ as

$$\beta(x, y) = \begin{cases} 1 & \text{whenever } \eta(x, y) \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then Corollary 3.9 directly follows from the Theorem 3.2.

Remark 3.11. *If we add the same additional hypothesis of Theorem 3.4 to the above theorem we get the uniqueness of common fixed point.*

Note: Theorem 3.5 implies the Result 2.1 of [2] if we set $S = T$.

Now we construct examples to validate the theorems.

Example 3.12. Let $X = [0, \infty)$ be endowed with the usual metric $d(x, y) = |x - y|$. Now we define two mappings S and T on (X, d) as $S : X \rightarrow \mathcal{P}(X)$ defined by

$$Sx = \begin{cases} \{0, \frac{1}{4x}\} & \text{for all } x \geq 1; \\ \{0, \frac{x}{12}\} & 0 \leq x \leq 1 \end{cases}$$

and $T : X \rightarrow \mathcal{P}(X)$ defined by

$$Tx = \begin{cases} [x - \frac{1}{3}, 100] & \text{for all } x \geq \frac{1}{2}; \\ \{0, \frac{x}{12}\} & 0 \leq x \leq \frac{1}{2}; \\ \{0\} & x = \infty. \end{cases}$$

We define $\beta : X \times X \rightarrow [0, \infty)$ by

$$\begin{aligned} \beta(x, y) &= 2 \quad \forall x, y \in [0, \frac{1}{2}]; \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and $\psi : [0, \infty) \rightarrow [0, \infty)$ is defined by

$$\psi(t) = \frac{t}{2}.$$

Now for all $x, y \in [0, \frac{1}{2}]$,

$$\beta(x, y) \geq 1, \quad Tx = \{0, \frac{x}{12}\} \quad \text{and} \quad Sy = \{0, \frac{y}{12}\}.$$

Therefore,

$$\begin{aligned} \max_{x_0 \in Tx} d(x_0, Sy) &= \max\{0, |\frac{x_0}{12} - \frac{y}{12}|\} = |\frac{x_0}{12} - \frac{y}{12}|; \\ \max_{y_0 \in Sy} d(y_0, Tx) &= \max\{0, |\frac{x}{12} - \frac{y_0}{12}|\} = |\frac{x}{12} - \frac{y_0}{12}|. \end{aligned}$$

So we have, $H(Tx, Sy) = |\frac{x_0}{12} - \frac{y}{12}|$ for all $x, y \in [0, \frac{1}{2}]$. Now $\beta(x, y) = 2$ for all $x, y \in [0, \frac{1}{2}]$ which implies

$$\beta_*(Tx, Sy) = 2 > 1 \quad \text{and} \quad \beta_*(Sx, Ty) = 2 > 1$$

i.e. the pair (S, T) is β_* -admissible mapping. Therefore

$$\begin{aligned} \beta_*(Tx, Sy)H(Tx, Sy) &= 2|\frac{x}{12} - \frac{y}{12}| \\ &= |\frac{x}{6} - \frac{y}{6}| \\ &< \frac{|x - y|}{2} \\ &< \psi(d(x, y)). \end{aligned}$$

All the conditions of the theorem are satisfied. So S and T have a common fixed point that is $x = 0$ and obviously this is the unique common fixed point of S and T .

Example 3.13. Let $X = [0, +\infty)$ and we define a metric $d : [0, +\infty) \rightarrow \mathbb{R}^+$ by

$$d(x, y) = \begin{cases} \max\{x, y\} & \text{whenever } x \neq y; \\ 0 & \text{for } x = y. \end{cases}$$

Now we consider two mappings S and T on X where $S : X \rightarrow \mathcal{P}(X)$ is defined by

$$Sx = \begin{cases} [x - 1, x + 1] & \text{for all } x \geq 1; \\ \{0, \frac{x}{16}\} & 0 \leq x \leq 1; \\ \{4\} & \text{for } x = +\infty \end{cases}$$

and $T : X \rightarrow \mathcal{P}(X)$ is defined by

$$Tx = \begin{cases} [1, x + 2] & \text{for all } x \geq 1; \\ \{0, \frac{x}{8}\} & 0 \leq x \leq 1; \\ \{0\} & \text{for } x = +\infty. \end{cases}$$

We define $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 + x^2 + y^2 & \text{for all } x, y \in [0, 1]; \\ 0 & \text{otherwise} \end{cases}$$

and $\eta(x, y) = x^2 + y^2$ for all $x, y \in X$. Let $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is defined by

$$\psi(t) = \frac{t}{2}.$$

Clearly $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in [0, 1]$.

Again for all $x, y \in [0, 1]$, we have,

$$Sx = \{0, \frac{x}{16}\} \text{ and } Ty = \{0, \frac{y}{8}\}.$$

For all $x, y \in [0, 1]$, we get,

$$\begin{aligned} \alpha_*(Sx, Ty) &= \inf\{1 + a^2 + b^2 : a \in Sx, b \in Ty\} \\ &= 1. \end{aligned}$$

$$\begin{aligned} \eta_*(Sx, Ty) &= \sup\{a^2 + b^2 : a \in Sx, b \in Ty\} \\ &= \frac{1}{16^2} + \frac{1}{8^2}. \end{aligned}$$

So it is clear that for all $x, y \in [0, 1]$

$$\begin{aligned} \alpha_*(Sx, Ty) &\geq \eta_*(Sx, Ty); \\ \alpha_*(Tx, Sy) &\geq \eta_*(Tx, Sy), \end{aligned}$$

i.e. the pair (S, T) is α_* -admissible function with respect to η_* .

Now

$$\begin{aligned} d(0, Ty) &= \inf\{d(0, 0), d(0, \frac{y}{8})\} \\ &= \inf\{0, \frac{y}{8}\} \\ &= 0; \end{aligned}$$

$$\begin{aligned} d(\frac{x}{16}, Ty) &= \inf\{d(0, \frac{x}{16}), d(\frac{x}{16}, \frac{y}{8})\} \\ &= \inf\{\frac{x}{16}, d(\frac{x}{16}, \frac{y}{8})\}; \end{aligned}$$

$$\begin{aligned} d(0, Sx) &= \inf\{d(0, 0), d(0, \frac{x}{16})\} \\ &= \inf\{0, \frac{x}{16}\} \\ &= 0; \end{aligned}$$

$$\begin{aligned} d(\frac{y}{8}, Sx) &= \inf\{d(0, \frac{y}{8}), d(\frac{x}{16}, \frac{y}{8})\} \\ &= \inf\{\frac{y}{8}, d(\frac{x}{16}, \frac{y}{8})\}. \end{aligned}$$

Case-I: whenever $x > 2y$,

$$\begin{aligned} d(\frac{x}{16}, Ty) &= \frac{x}{16}; \\ d(\frac{y}{8}, Sx) &= \frac{y}{8}. \end{aligned}$$

Therefore

$$\begin{aligned} H(Sx, Ty) &= \max\{\sup_{a \in Sx} d(a, Ty), \sup_{a \in Ty} d(a, Sx)\} \\ &= \max\{\frac{x}{16}, \frac{y}{8}\} \\ &= \frac{x}{16} \\ &< \frac{\max\{x, y\}}{2} \\ &= \psi(d(x, y)). \end{aligned} \tag{3.1}$$

Case-II: whenever $x \leq 2y$,

$$\begin{aligned} d(\frac{x}{16}, Ty) &= \frac{x}{16}; \\ d(\frac{y}{8}, Sx) &= \frac{y}{8}. \end{aligned}$$

Therefore

$$\begin{aligned}
 H(Sx, Ty) &= \max\{\sup_{a \in Sx} d(a, Ty), \sup_{a \in Ty} d(a, Sx)\} \\
 &= \max\{\frac{x}{16}, \frac{y}{8}\} \\
 &= \frac{y}{8} \\
 &< \frac{\max\{x, y\}}{2} \\
 &= \psi(d(x, y)).
 \end{aligned} \tag{3.2}$$

From equations (3.1) and (3.2), we have,

$$H(Sx, Ty) \leq \psi(d(x, y)) \text{ for all } x, y \in [0, 1].$$

All the conditions of the above theorem are satisfied. So S and T have common fixed points. Also notice that S and T have infinite number of common fixed points.

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