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Maximal Buttonings of Non-Tree Graphs

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Abstract : Let G be a finite connected graph of n vertices $v_1, v_2, \ldots, v_n$. A *buttoning* of G is a closed walk consisting of n shortest paths

$$[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1].$$

The buttoning is said to be *maximal* if it has a maximum length when compared with all other buttonings of G. The goal of this work is to find a length of a maximal buttoning of non-tree graphs: complete multipartite graphs, grid graphs and rooted products of graphs.

Keywords : maximal buttoning; multipartite graph; grid graph; rooted product; walk; graph metric; centroid.
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1 Introduction

A question on how one can button a shirt can turn to be interesting mathematically. If our shirt has n buttons in a vertical line with a spacing of one unit between each adjacent pair, most of us usually button the shirt from top to bottom. In these manners, we get the same distance of $n-1$ units. If we button them in a different order, the number of units may be changed. The study by I. Short [1] addresses the question on what the maximum number of units our hands

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travel. He turned the more general setting for a finite tree T with a graph metric d. The problem is then identifying the maximum value of the sum

$$d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n) + d(v_n, v_1), \tag{1.1}$$

where $v_1, v_2, \ldots, v_n$ are vertices of T and, for the distinct pair i and j, the value $d(v_i, v_j)$ is defined to be the length of the shortest path between v_i and v_j. The shirt buttoning is a special case of this problem for a path tree with removing the final term of the sum. Moreover, the author extended the definition of a buttoning to the case of finite connected graphs in the last section.

As remarked in [1], the problem is a special case of the maximum travelling salesman problem which is an NP-hard problem in combinatorial optimization. The maximum travelling salesman problem was studied in [2, 3, 4]. In particular, a given graph in finding maximal buttoning will be first transformed to a complete graph with vertices $v_1, v_2, \ldots, v_n$ such that for each pair $i \neq j$, the edge incident to v_i and v_j is weighted by the value $d(v_i, v_j)$. Consequently, the summation (1.1) is the length of a Hamilton cycle in the induced graph.

In this article, every graph is a finite connected graph with a graph metric d, and we prefer to use d_1 if all edges in G are weighted by length one. This includes the case of edge-unweighted graphs. For any graph G, we denote V_G the set of all vertices, and E_G the family of all edges of G. We let $[u, v]$ denote the shortest path, (which need not be unique), from the vertex u to the vertex v in G. A *buttoning* of a graph G is defined to be a closed walk consisting of n shortest paths $[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1]$ where $v_1, v_2, \ldots, v_n$ are all distinct vertices of G. The length of the buttoning is the sum of the lengths of all paths in the buttoning. The buttoning is said to be *maximal* if it has a maximum length when compared with all other buttonings of G.

Next we recall a definition for a centroid of a graph. The definition of a centroid was first introduced for a tree. Let u be any vertex of T, the *branch weight* of u is the maximum number of vertices in any branch of u. See more detail in [5, 6]. A vertex c of T is called a *centroid* of T if its branch weight is minimum over all vertices of T. Equivalently,

$$\sum_{u \in V_T} d_1(c, u) \leq \sum_{u \in V_T} d_1(v, u)$$

for every $v \in V_T$, see [6, Theorem 2]. We call the value $\sum_{u \in V_T} d_1(v, u)$ the *distance of the vertex* v and denote it by $d_1(v)$. The result in [5] shows that every tree has either a single centroid or two adjacent centroids.

By using the definition of distance of a vertex, we can generalize the definition of a centroid of a tree to any metric connected graph G. A *centroid* of G is a vertex that has the minimum distance when the distance of a vertex u is defined by

$$d(u) = \sum_{x \in V_G} d(u, x).$$

Moreover, we define

$$\Phi(G) = 2d(c),$$

where c is a centroid of G. This includes the case of the graph metric d_1. The following lemma is the useful result of tree obtained in [1]. We know from this lemma when a considered buttoning is maximal.

Lemma 1.1. *Let $[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1]$ be a buttoning of a tree T. Then*

$$d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n) + d(v_n, v_1) \leq \Phi(T),$$

with equality if and only if each centroid of T is contained in every path $[v_i, v_{i+1}]$ (including $[v_n, v_1]$).

This paper is organized into four sections. We examine maximal buttonings of complete multipartite graphs, two-dimensional grid graphs, rooted products of graphs in Section 2 to Section 4, respectively. With the metric d_1, we found that the length of a maximal buttoning of a complete multipartite graph depends on only the number of its vertices and partite set. Certainly, the results of incomplete multipartite graphs are different from the complete case as we show in Section 3 for the grid graphs, examples of incomplete bipartite graphs. In the last section, we study a maximal buttoning of the rooted product of graphs $G \circ_r H$ in the case that G is a tree. The length of its maximal buttoning can be written in terms of the length of a maximal buttoning on G, the distance of the root of H, and the numbers of vertices of these two graphs.

2 Buttonings of Complete Multipartite Graphs

A *complete multipartite graph* is a simple graph whose vertices can be partitioned into different independent sets, called *partite sets*, such that any two vertices are adjacent if and only if they are in different partite sets. The complete multipartite graph with m partite sets is called a *complete m-partite graph* and denoted by $K_{n_1,n_2,\ldots,n_m}$ where $n_1, n_2, \ldots, n_m$ are the sizes of the partite sets.

In this section, we provide some results of buttonings of complete multipartite graphs with the metric d_1 where every edge is weighted by one.

Proposition 2.1. *Let K be a complete m-partite graph of n vertices with the graph metric d_1. Then K contains a maximal buttoning of length $2n - m$.*

Proof. Let $K = K_{n_1,n_2,\ldots,n_m}$ and $V_1, V_2, \ldots, V_m$ be all partite sets of K such that for every $j = 1, 2, \ldots, m$, $V_j = \{v_1^{(j)}, v_2^{(j)}, \ldots v_{n_j}^{(j)}\}$. We see that the distance between two distinct vertices of K is one if they are in different partite sets, and two if they are in the same partite set. That is,

$$d_1(v_r^{(i)}, v_s^{(j)}) = \begin{cases} 1 & \text{if } i \neq j, \\ 2 & \text{if } i = j \text{ and } r \neq s. \end{cases}$$

To obtain a maximal buttoning, shortest paths of length two should be included in the buttoning as much as possible. Moreover, the shortest paths of length one have to be added to connect between two vertices in different partite sets. One way to do this is to button all vertices in the same partite first, then move to the next partite, and so on. For example, the following buttoning is a maximal one;

$$\begin{aligned}
&[v_1^{(1)}, v_2^{(1)}], [v_2^{(1)}, v_3^{(1)}], \ldots, [v_{n_1-1}^{(1)}, v_{n_1}^{(1)}], [v_{n_1}^{(1)}, v_1^{(2)}],\\
&[v_1^{(2)}, v_2^{(2)}], [v_2^{(2)}, v_3^{(2)}], \ldots, [v_{n_2-1}^{(2)}, v_{n_2}^{(2)}], [v_{n_1}^{(2)}, v_1^{(3)}],\\
&\qquad\vdots\\
&[v_1^{(m)}, v_2^{(m)}], [v_2^{(m)}, v_3^{(m)}], \ldots, [v_{n_m-1}^{(m)}, v_{n_m}^{(m)}], [v_{n_1}^{(m)}, v_1^{(1)}].
\end{aligned}$$

The maximum length is $m + 2\sum_{j=1}^{m}(n_j - 1) = 2n - m$. □

For any given positive integer n, a complete m-partite graphs K of n vertices is not unique. However, Proposition 2.1 also implies that the length of maximal buttonings of all K is invariant under this non-uniqueness.

Corollary 2.2. *Every complete m-partite graph of n vertices with the graph metric d_1 contains maximal buttonings of the same length.*

3 Buttonings of Two-Dimensional Grid Graphs

Let P_m and P_n be path graphs of m and n vertices, respectively. A *two-dimensional grid graph*, or *grid graph*, is constructed as the graph Cartesian product $P_m \square P_n$, denoted by $G_{m,n}$. The set of all vertices of $G_{m,n}$ is $V = V_{P_m} \times V_{P_n}$ and the family of all edges is

$$E = \{(x,y)(x,z) \mid x \in V_{P_m} \text{ and } yz \in E_{P_n}\} \cup \{(x,y)(w,y) \mid y \in V_{P_n} \text{ and } xw \in E_{P_m}\}.$$

The graph $G_{m,n}$ can be viewed as an array of an $m \times n$ matrix where every row is a copy of P_n and column is a copy of P_m. See Figure 1, for example, the grid graph $G_{5,6}$. We then denote by g_{ij} a vertex at the i^{th} row and the j^{th} column of $G_{m,n}$.

In this section, we give a maximal buttoning for a grid graph, a non-complete bipartite graph, so to contrast with a result from the previous section. We again endow the graph $G_{m,n}$ with the metric d_1. For any two vertices g_{ij} and g_{kl} of $G_{m,n}$, the distance between them can be realized as

$$d_1(g_{ij}, g_{kl}) = |i - k| + |j - l|. \tag{3.1}$$

Any pair of vertices g_{ij}, g_{kl} induces a grid subgraph $A_{ij,kl}$ where vertices at its four outermost corners are g_{ij}, g_{il}, g_{kl} and g_{kj}. It is quite obvious that a vertex g_{st} of $G_{m,n}$ belongs to $A_{ij,kl}$ precisely when

$$d_1(g_{ij}, g_{kl}) = d_1(g_{ij}, g_{st}) + d_1(g_{st}, g_{kl}).$$

We have the following lemma.

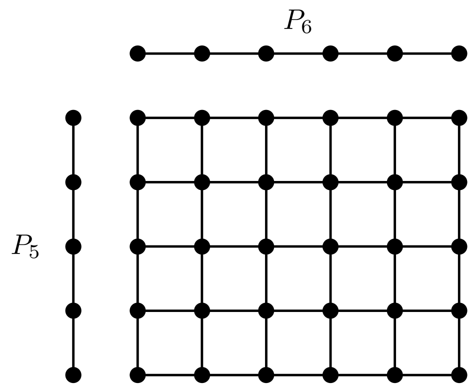

Figure 1: Example of grid graph, $G_{5,6}$

Lemma 3.1. *Let g_{ij}, g_{kl} and g_{st} be any vertices of a grid graph $G_{m,n}$. Then*

$$d_1(g_{ij}, g_{kl}) = d_1(g_{ij}, g_{st}) + d_1(g_{st}, g_{kl}) \tag{3.2}$$

if and only if s lies between i and k, and t lies between j and l.

Proof. A proof of this lemma relies on the distance formula for any two vertices in $G_{m,n}$ defined in (3.1), and the fact that for any positive integers i, k, and s, $|i-k| = |i-s| + |s-k|$ if and only if $(i-s)(s-k) \geq 0$. □

By the definition of $G_{m,n}$, the number of its centroids truly relates to the number of centroids of the inducing path graphs P_m and P_n. To clarify this relation, consider a path graph P_n with the set of all ordered vertices $V_{P_n} = \{p_1, p_2, \ldots, p_n\}$. One can see that P_n contains either one centroid at $p_{\frac{n+1}{2}}$, or two adjacent centroids at $p_{\frac{n}{2}}$ and $p_{\frac{n+2}{2}}$, depending on whether n is odd or even, respectively. This implies that $G_{m,n}$ contains one, two, or four centroids depending on the integers m and n. More precisely, a vertex g_{ij} of $G_{m,n}$ is a centroid if and only if p_i and p_j are centroids of P_m and P_n, respectively. We thus have the following proposition which includes the computation to verify this fact for the sake of completeness.

Proposition 3.2. *A vertex g_{ij} is a centroid of a grid graph $G_{m,n}$ if and only if p_i and p_j are centroids of P_m and P_n respectively. In particular,*

1. *$G_{m,n}$ has only one centroid if and only if m and n are odd,*
2. *$G_{m,n}$ has two centroids if and only if either m or n are odd,*
3. *$G_{m,n}$ has four centroids if and only if m and n are even.*

Proof. Let p_i and p_j be centroids of P_m and P_n, respectively. Then the distances of them are minimum on the graph on which they are. Thus, for every $k \in \{1, 2 \ldots, m\}$ and every $l \in \{1, 2, \ldots, n\}$, we have

$$\sum_{t=1}^{n} d_1(g_{kj}, g_{kt}) \leq \sum_{t=1}^{n} d_1(g_{kl}, g_{kt}) \tag{3.3}$$

and

$$\sum_{s=1}^{m} d_1(g_{il}, g_{sl}) \leq \sum_{s=1}^{m} d_1(g_{kl}, g_{sl}). \tag{3.4}$$

This implies directly that $d_1(g_{ij})$ is minimum in $G_{m,n}$, and so g_{ij} is a centroid of $G_{m,n}$.

Conversely, if either p_k is not a centroid of P_m, or p_l is not a centroid of P_n, then (3.3) or (3.4) will be a strict inequality. Hence, $d_1(g_{kl})$ cannot be the minimum in $G_{m,n}$. Therefore, g_{kl} is not a centroid of $G_{m,n}$. □

We are now ready to find the length of a maximal buttoning of $G_{m,n}$. The two lemmas below provide upper bounds of buttoning lengths of $G_{m,n}$. Theorem 3.5 provides the length of a maximal buttoning of $G_{m,n}$ in the case of four centroids which is different from the two other cases in Theorem 3.6.

Lemma 3.3. *Let G be a graph with any metric d. Then $2d(u)$ is an upper bound of lengths of all buttonings of G where u is any vertex of G.*

Proof. It is immediate from the metric triangle inequality. □

Lemma 3.4. *Let $G_{m,n}$ be a grid graph containing four centroids and u be any vertex of $G_{m,n}$. Then the length of a buttoning of $G_{m,n}$ is less than $2d_1(u)$.*

Proof. Since $G_{m,n}$ has four centroids, Proposition 3.2 implies that the integer m and n are even. To prove this lemma, we first partition the graph into four grid subgraphs A, B, D, and E with

$$\begin{aligned}
V_A &= \{g_{jk} \in V_{G_{m,n}} \mid 1 \leq j \leq m/2 \text{ and } 1 \leq k \leq n/2\}, \\
V_B &= \{g_{jk} \in V_{G_{m,n}} \mid m/2 < j \leq m \text{ and } n/2 < k \leq n\}, \\
V_D &= \{g_{jk} \in V_{G_{m,n}} \mid m/2 < j \leq m \text{ and } 1 \leq k \leq n/2\}, \\
V_E &= \{g_{jk} \in V_{G_{m,n}} \mid 1 \leq j \leq m/2 \text{ and } n/2 < k \leq n\}.
\end{aligned}$$

See Figure 2 for an example of the grid graph $G_{6,6}$.

Let $\mathcal{M}$ be a maximal buttoning of $G_{m,n}$. We construct a multiple graph G, of four vertices, $V_G = \{a, b, d, e\}$, from the graph $G_{m,n}$ corresponding to $\mathcal{M}$. Suppose that $[w, v]$ is a path in $\mathcal{M}$ such that $w \in V_A$ and $v \in V_B$. In this case we add an edge ab to the graph G. Similarly, if $w \in V_B$ and $v \in V_A$, an edge ba, (which can be written as ab), is added to the graph G. Thus the number of edges ab of G is the number of paths in $\mathcal{M}$ between two vertices in V_A and V_B.

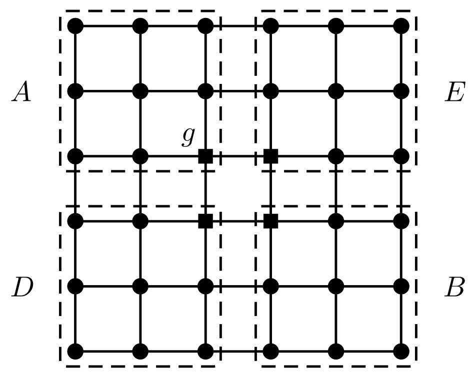

Figure 2: $G_{6,6}$

Other edges of G are obtained in the same way. Certainly, the degree of the vertex a of G is not greater than $mn/2$.

We now contrarily assume that a maximal buttoning $\mathcal{M}$ of $G_{m,n}$ has the length of $2d_1(u)$. Thus, every shortest path $[w, v]$ in $\mathcal{M}$, we have $d(w, v) = d(w, u) + d(u, v)$. We may suppose that the vertex u is a centroid of $G_{m,n}$ belonging to V_A. By Lemma 3.1, every vertex of B can connect to only vertices of A. Now there are $mn/2$ edges incident to the vertex a of G. Certainly, d or e is adjacent to a since the definition of a buttoning. It is impossible that the degree of the vertex a is greater than $mn/2$. Hence, the length of $\mathcal{M}$ is less than $2d_1(u)$. □

Theorem 3.5. *A grid graph $G_{m,n}$ of four centroids contains a maximal buttoning of length $\Phi(G_{m,n}) - 2$.*

Proof. Let g_{st} be a centroid of $G_{m,n}$. To complete this proof, we first verify that $2d_1(g_{st})-2$ is an upper bound of lengths of buttonings of $G_{m,n}$, and then construct a buttoning whose length is $2d_1(g_{st}) - 2$.

Let $\mathcal{B}$ be a buttoning of $G_{m,n}$. Since $G_{m,n}$ contains four centroids, Lemma 3.4 implies that the length of $\mathcal{B}$ is less than $d_1(g_{st})$, that is, at least one of the shortest paths in $\mathcal{B}$ does not satisfy (3.2). We Suppose that the path $[g_{ij}, g_{kl}]$ in $\mathcal{B}$ lacks (3.2), so by Lemma 3.1 and (3.1) we have

$$|i-k| < |i-s| + |s-k| \qquad \text{or} \qquad |j-l| < |j-t| + |t-l|.$$

We may assume that $|i-k| = |i-s| + |s-k|$. Then $|j-l| < |j-t| + |t-l|$, so $|j-t| \neq 0 \neq |t-l|$ and $|j-t| + |t-l| - |j-l| = 2\min\{|j-t|, |t-l|\}$. We thus have

$$2 \leq |j-t| + |t-l| - |j-l|.$$

Hence,

$$2 \leq d_1(g_{ij}, g_{st}) + d_1(g_{st}, g_{kl}) - d_1(g_{ij}, g_{kl}).$$

This implies that the length of $\mathcal{B}$ is not greater than $2d_1(g_{st}) - 2$. Therefore, $2d_1(g_{st}) - 2$ is an upper bound of lengths of buttonings of $G_{m,n}$.

Next we show that the value $2d_1(g_{st}) - 2$ can be attained by the length of a buttoning of $G_{m,n}$. We first separate $G_{m,n}$ into four grid subgraphs A, B, D, and E as in Lemma 3.4. We write,

$$\begin{aligned} V_A &= \{a_1, a_2, \ldots, a_k = c_A\}, \quad V_D = \{d_1, d_2, \ldots, d_k = c_D\}, \\ V_B &= \{b_1, b_2, \ldots, b_k = c_B\}, \quad V_E = \{c_E = e_1, e_2, \ldots, e_k\}, \end{aligned}$$

where c_A, c_B, c_D, and c_E are centroids of $G_{m,n}$ contained in A, B, D, and E, respectively. We may, without loss of generality, assume that $c_A = g_{st}$. By Lemma 3.1, all paths in the buttoning

$$\begin{aligned} &[e_1, d_1], [d_1, e_2], [e_2, d_2], \ldots, [e_k, d_k], [d_k, a_1], \\ &[a_1, b_1], [b_1, a_2], [a_2, b_2], \ldots, [a_k, b_k], [b_k, e_1] \end{aligned}$$

satisfy the equation (3.2) except the path $[b_k, e_1] = [c_B, c_E]$. Then the buttoning has the length of

$$2d_1(c_A) - \big(d_1(c_B, c_A) + d_1(c_A, c_E) - d_1(c_B, c_E)\big) = 2d_1(c_A) - 2.$$

Consequently, every grid graph $G_{m,n}$ of four centroids contains a maximal buttoning of length $\Phi(G_{m,n}) - 2$. □

Theorem 3.6. *A grid graph $G_{m,n}$ of one or two centroids contains a maximal buttoning of length $\Phi(G_{m,n})$.*

Proof. We only show that the value $\Phi(G_{m,n})$ can be obtained as the length of some buttoning of $G_{m,n}$, and then it becomes the length of a maximal buttoning of $G_{m,n}$ by Lemma 3.3.

We start the proof with the case that $G_{m,n}$ has two centroids. By Proposition 3.2, we may suppose that m is odd and n is even. The case of $m = 1$ is trivial by [1, Lemma 4]. Thus we may assume that $m \geq 3$. To prove this case we partition the graph into four grid subgraphs A, B, D, and E with

$$\begin{aligned} V_A &= \{g_{jk} \in V_{G_{m,n}} \mid 1 \leq j \leq (m+1)/2 \text{ and } 1 \leq k \leq n/2\}, \\ V_B &= \{g_{jk} \in V_{G_{m,n}} \mid (m+1)/2 \leq j \leq m \text{ and } n/2 < k \leq n\}, \\ V_D &= \{g_{jk} \in V_{G_{m,n}} \mid (m+1)/2 < j \leq m \text{ and } 1 \leq k \leq n/2\}, \\ V_E &= \{g_{jk} \in V_{G_{m,n}} \mid 1 \leq j \leq (m-1)/2 \text{ and } n/2 < k \leq n\}. \end{aligned}$$

See Figure 3 for an example of the grid graph $G_{5,6}$.

Let c_A and c_B be centroids of $G_{m,n}$ contained in A and B, respectively. For $i = (m+1)/2$, we write

$$\begin{aligned} V_A &= \{g_{i1} = a_1, a_2, \ldots, a_k\}, \quad V_D = \{d_1, d_2, \ldots, d_j\}, \\ V_B &= \{b_1, b_2, \ldots, b_k = c_B\}, \quad V_E = \{e_1, e_2, \ldots, e_j\}. \end{aligned}$$

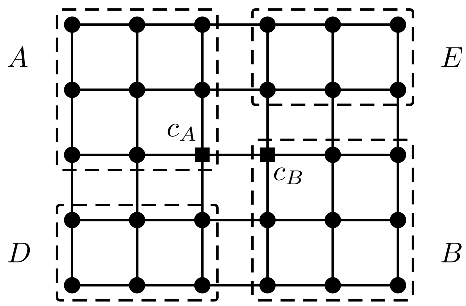

Figure 3: $G_{5,6}$

Thus Lemma 3.1 and Lemma 3.3 imply that the buttoning

$$[a_1, b_1], [b_1, a_2], [a_2, b_2], \ldots, [a_k, b_k], [b_k, d_1], [d_1, e_1], [e_1, d_2], \ldots, [d_j, e_j], [e_j, a_1]$$

is a maximal one with the length $\Phi(G_{m,n})$.

In the case that $G_{m,n}$ has only one centroid Proposition 3.2 implies that m and n are odd. The case of $m = 1$ or $n = 1$ are obvious by [1, Lemma 4], so we assume that $m \geq 3$ and $n \geq 3$. In this case we define grid subgraphs A, B, D, and E of $G_{m,n}$ by

$$\begin{aligned}
V_A &= \{g_{jk} \in V_{G_{m,n}} \mid 1 \leq j \leq (m+1)/2 \text{ and } 1 \leq k \leq (n+1)/2\},\\
V_B &= \{g_{jk} \in V_{G_{m,n}} \mid (m+1)/2 \leq j \leq m \text{ and } (n+1)/2 < k \leq n\},\\
V_D &= \{g_{jk} \in V_{G_{m,n}} \mid (m+1)/2 < j \leq m \text{ and } 1 \leq k \leq (n-1)/2\},\\
V_E &= \{g_{jk} \in V_{G_{m,n}} \mid 1 \leq j \leq (m-1)/2 \text{ and } (n+1)/2 < k \leq n\}.
\end{aligned}$$

Figure 4 shows an example of the grid graph $G_{5,5}$.

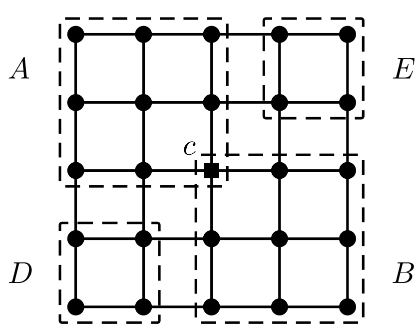

Figure 4: $G_{5,5}$

In this case we let c be a centroid of $G_{m,n}$. For $i = 1$ we write

$$V_A = \{g_{i1} = a_1, a_2, \ldots, a_k = c\}, \quad V_D = \{d_1, d_2, \ldots, d_j\},$$
$$V_B = \{b_1, b_2, \ldots, b_k = c\}, \quad V_E = \{e_1, e_2, \ldots, e_j\}.$$

Hence, Lemma 3.1 and Lemma 3.3 imply that

$$[a_1, b_1], [b_1, a_2], \ldots, [a_{k-1}, b_{k-1}], [b_{k-1}, b_k], [b_k, d_1], [d_1, e_1], [e_1, d_2], \ldots, [d_j, e_j], [e_j, a_1]$$

is a maximal buttoning of length $\Phi(G_{m,n})$. □

4 Buttonings of Rooted Product of Graphs

A *rooted graph* is a graph in which one vertex, called the *root*, is distinguished from others. Let H be a rooted graph with the root r, the rooted product of graphs G and H at r, denoted by $G \circ_r H$, is the graph obtained by identifying each vertex of G by the root r of $|V_G|$ copies of H. This product was studied regarding its metric dimension in [7, 8]. The rooted product of graphs was first introduced by Godsil and McKay in [9] that the rooted graphs at every vertex of G may be different.

In formal defining, the vertex set of $G \circ_r H$ is the set $V = V_G \times V_H$, and the family of all edges therein is the set

$$E = \big\{(g,h)(g',h) \mid gg' \in E_G\big\} \cup \big\{(g,h)(g,h') \mid g \in V_G \text{ and } hh' \in E_H\big\}.$$

The metric on $G \circ_r H$ is induced from the given metrics d_G and d_H on G and H, respectively. For any $(g,h), (g',h') \in V$,

$$d((g,h),(g',h')) = \begin{cases} d_H(h,r) + d_G(g,g') + d_H(r,h') & \text{if } g \neq g', \\ d_H(h,h') & \text{if } g = g'. \end{cases}$$

If G has a single vertex, $G \circ_r H$ is simply H. Thus the maximal buttonings of $G \circ_r H$ follows from those of H. We will now assume that G has more than one vertex.

If T and S are both trees, then $T \circ_r S$ ia also a tree and its maximal buttoning is obtained by Lemma 1.1. Here we give the length of a maximal buttoning of $T \circ_r S$ in the form of $\Phi(T)$ and $d_s(r)$, the distance of r in S.

Proposition 4.1. *Let T be a tree, and S be a rooted tree with the root r. Then $T \circ_r S$ contains a maximal buttoning of length $|V_S|\Phi(T) + 2|V_T|d_S(r)$.*

Proof. Let c be a centroid of T and $[t_1, t_2], [t_2, t_3], \ldots, [t_{n-1}, t_n], [t_n, t_1]$ be a maximal buttoning of T. By Lemma 1.1, each of these shortest paths $[t_i, t_j]$ contains the centroid c of T. Let $V_S = \{s_1, s_2, \ldots, s_m\}$ be the set of all vertices of S and $r = s_1$ be the root. At the rooted product, we see that (c, r) becomes a centroid of $T \circ_r S$. Therefore, a maximal buttoning of a tree $T \circ_r S$ can be constructed as follows:

$$[(t_1, s_1), (t_2, s_1)], [(t_2, s_1), (t_3, s_1)], \ldots, [(t_{n-1}, s_1), (t_n, s_1)], [(t_n, s_1), (t_1, s_2)],$$
$$[(t_1, s_2), (t_2, s_2)], [(t_2, s_2), (t_3, s_2)], \ldots, [(t_{n-1}, s_2), (t_n, s_2)], [(t_n, s_2), (t_1, s_3)],$$
$$\vdots$$
$$[(t_1, s_m), (t_2, s_m)], [(t_2, s_m), (t_3, s_m)], \ldots, [(t_{n-1}, s_m), (t_n, s_m)], [(t_n, s_m), (t_1, s_1)].$$

This is a buttoning of length $|V_S|\Phi(T)+2|V_T|d_S(r)$, and its maximality comes from the fact that the centroid (c, r) lies in each of the shortest paths $[(t_i, s_k), (t_j, s_l)]$ in the buttoning. □

Next we extend the result to the case when S is a rooted graph instead of a tree. The lemma below shows the technique of constructing a spanning tree S' of S preserving the distance of the root r. That is, $d_{S'}(r) = d_S(r)$ where $d_{S'}$ is a metric on S' induced from the same edges' length of S.

Lemma 4.2. *Let S be a rooted graph with the root r. Then there exists a spanning tree S' of S such that $d_{S'}(r) = d_S(r)$.*

Proof. Let $0 = k_0 < k_1 < k_2 < \ldots < k_n$ be the list in ascending order of all different distances from r to every vertex of S. We partition V_S into $n+1$ subsets $\{r\} = V_0, V_1, \ldots, V_n$, where $V_i = \{v \in V_S \mid d_S(r, v) = k_i\}$, corresponding to the list of distances as above.

For $m \in \{1, 2, \ldots, n\}$, $v \in V_m$, we see that v is adjacent to some vertex $u \in \cup_{j=0}^{m-1} V_j$ where $d_S(r, u) + d_S(u, v) = k_m$. We use this fact to construct a subgraph S' of S by connecting the vertex v with the vertex u. Hence, the subgraph S' is a spanning tree of S and $d_{S'}(r) = d_S(r)$. □

Corollary 4.3. *Let T be a tree, and S be a rooted graph with the root r. Then $T \circ_r S$ contains a maximal buttoning of length $|V_S|\Phi(T) + 2|V_T|d_S(r)$.*

Proof. Let S' be a spanning tree of S constructed as in the above lemma. We know that any pair of vertices s_i, and s_j of S, the length of the shortest path $[s_i, s_j]$ in S' is greater than or equal to that in S. By Proposition 4.1, $2|V_S|d_T(c)+2|V_G|d_{S'}(r)$ is an upper bound of lengths of buttonings of $T \circ_r S$. We see that the maximal buttoning of $T \circ_r S$ obtained by the same way of Proposition 4.1 is also a maximal buttoning of $T \circ_r S$. Now the corollary is proved. □

From the corollary above, we see that the length of a maximal buttoning of the rooted product $T \circ_r S$ depends on the choice of the root r of S.

Lastly, we note that computing a maximal buttoning of the rooted product between graphs can be subtle. The construction in Proposition 4.1 cannot be extended to the rooted product in general.

Consider the rooted product $C_4 \circ P_2$ with the metric d_1. In this case the product graph does not depend on the choice of the root r of P_2, so we omit the root r from the notation and use the simple notation of vertices, without using oder pairs, see in Figure 5.

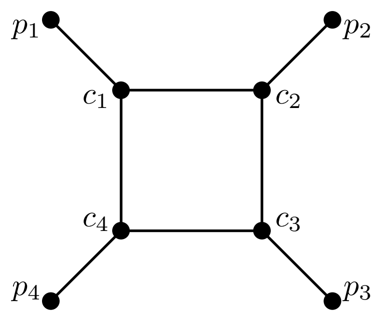

Figure 5: the rooted product $C_4 \circ P_2$

By Theorem 3.5, a buttoning $[c_1, c_3], [c_3, c_2], [c_2, c_4], [c_4, c_1]$ of C_4 is a maximal one. By the way of Proposition 4.1 we obtain the buttoning

$$[c_1, c_3], [c_3, c_2], [c_2, c_4], [c_4, p_1], [p_1, p_3], [p_3, p_2], [p_2, p_4], [p_4, c_1].$$

However, it is not maximal since its length is less than the length of the following buttoning

$$[c_1, c_3], [c_3, p_1], [p_1, p_3], [p_3, c_2], [c_2, c_4], [c_4, p_2], [p_2, p_4], [p_4, c_1].$$

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