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Basic Hybrid Fixed Point Theorems for Contractive Mappings in Partially Ordered Metric Spaces

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Abstract : We present two basic hybrid fixed point theorems for the contractive mappings in partially ordered metric spaces and derive some interesting fixed point theorems as special cases. Our main result is illustrated with some examples.

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1 Introduction

The study of hybrid fixed point theorems for the contraction mappings in partially ordered metric spaces is initiated in Ran and Reuring [1] which is further continued by Nieto and Rodringuez-Lopez in [2] under different applicable conditions. Nieto and Rodringuez-Lopez [2] applied the established fixed point theorems to boundary value problems of nonlinear first order ordinary differential equations for proving the existence results under certain monotonic conditions. Since then many mathematicians have established several hybrid fixed point theorems (FPTs) for different classes of contraction mappings in partially ordered metric spaces. For the details, a reader is referred to a recent paper of Dhage *et al.* [3] and the references cited therein. The main advantage of these hybrid fixed point theorems is that the qualitative information of the fixed points of the mappings in question may be obtained under certain additional conditions. So the hybrid FPTs are useful for proving the existence as well as uniqueness theorems for some nonlinear problems under certain monotonic conditions (see Dhage [4] and the references therein). In the present paper we generalize the class of contraction mappings to contractive mappings and prove some basic hybrid FPTs in partially ordered metric spaces. In the following section we prove our main results of this paper.

2 Basic Hybrid FPTs

An order relation $\preceq$ on a non-empty set X is a reflexive, antisymmetric and transitive relation and the non-empty set X together with the order relation $\preceq$ is a partially ordered set denoted by $(X, \preceq)$. We need the following definitions in what follows.

Definition 2.1. A mapping $T : X \to X$ is called **monotone nondecreasing** or simply **nondecreasing** if it preserves the order relation, that is, if $x \preceq y$ then $Tx \preceq Ty$ for all $x, y \in X$. Similarly, T is called **monotone nonincreasing** if it preserves the order relation $\preceq$ reversely, i.e., $x \preceq y$ implies $Tx \succeq Ty$ for all $x, y \in X$. T is simply called **monotonic** if it is either monotone nondecreasing or monotone nonincreasing on X.

Definition 2.2. A mapping $T : X \to X$ is said to dominate a point $x \in X$ from above (resp. from below) if $x \preceq Tx$ (resp. $Tx \preceq x$). T is called **dominating from above** (resp. dominating from below) on X if it dominates every point of X from above (resp. from below). T is called **dominating** on X if it is either dominating above or dominating below on X.

Definition 2.3. A mapping $T : X \to X$ is called **mixed dominating** if it dominates every point $x \in X$ from above or from below i.e. if one of $x \preceq Tx$ and $x \succeq Tx$ holds for every $x \in X$.

It is clear that mixed dominating is a weaker property of a mapping than dominating above and dominating below in a partially ordered metric space.

Example 2.4. Let $X = \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a mapping defined as

$$f(x) = 2x.$$

Then f is mixed dominating on $\mathbb{R}$ because $x \leq Tx$ if $x \geq 0$ and $x \geq Tx$ if $x \leq 0$.

We introduce a metric d on X so that $(X, \preceq, d)$ is now becomes a partially ordered metric space. An orbit $\mathcal{O}(x; T)$ of a mapping $T : X \to X$ at a point $x \in X$ is a set of points in X defined as

$$\mathcal{O}(x; T) = \{x, Tx, T^2x, \ldots.\}$$

The mapping T is called T-orbitally continuous if for any sequence $\{x_n\}$ in $\mathcal{O}(x;T)$,
$x_n \to x^*$ implies $Tx_n \to Tx^*$ for all $x \in X$.

Definition 2.5 (Dhage [4])**.** A mapping $T : X \to X$ is called **partially compact** if $T(C)$ is relatively compact subset of the metric space X for every totally ordered set or chain C in X.

Note that every compact set is partially compact, but the converse is not necessarily true. Now we are well equipped with all necessary details to prove the main results of this paper. Our key result of this paper is as follows.

Theorem 2.6. *Let $(X, \preceq, d)$ be a partially ordered metric space and let $T : X \to X$ be a nondecreasing mapping satisfying the contractive condition*

$$d(Tx, T^2x) < d(x, Tx) \tag{2.1}$$

for all elements $x \in X$ comparable to $Tx \in X$ with $x \neq Tx$. Suppose that there is an $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the sequence $\{T^n x_0\}$ of successive iterates of T at x_0 has a convergent subsequence $\{T^{n_k} x_0\}$ converging to a point $u \in X$. If T is continuous at u and u and Tu are comparable, then u is a fixed point of T.

Proof. Define a sequence $\{x_n\} = \{T^n x_0\}$ of iterates of T at x_0 as

$$x_{n+1} = T^{n+1}x_0 = Tx_n,\ n = 0, 1, 2, \ldots \tag{2.2}$$

Since T is nondecreasing, one has

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots, \tag{2.3}$$

or

$$T^m x_0 \succeq T^n x_0 \,\forall\, m \geq n \in \mathbb{N}.$$

Denote

$$C_n = d(x_n, x_{n+1}) > 0$$

for each $n = 0, 1, 2, \ldots$. Then from (2.1) we obtain

$$C_0 > C_1 > \cdots > C_n > \cdots. \tag{2.4}$$

Thus $\{C_n\}$ is a decreasing sequence of positive real numbers. Hence $\{C_n\}$ is convergent, and there is a positive real number $C > 0$ such that

$$\lim_{n\to\infty} C_n = C. \tag{2.5}$$

Now every subsequence of $\{C_n\}$ also converges to the same limit point. Therefore,

$$\lim_{k\to\infty} C_{n_k} = C = \lim_{k\to\infty} C_{n_k+1}. \tag{2.6}$$

Suppose that a subsequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$ is convergent and converges to the point, say $u \in X$, i.e.$\lim_{k\to\infty} T^{n_k}x_0 = u$. Since T is continuous at u, one has

$$\lim_{k\to\infty} T^{n_k+1}x_0 = \lim_{k\to\infty} TT^{n_k}x_0 = T\Big(\lim_{k\to\infty} T^{n_k}x_0\Big) = Tu$$

and

$$\lim_{k\to\infty} T^{n_k+2}x_0 = \lim_{k\to\infty} T^2T^{n_k}x_0 = T^2\Big(\lim_{k\to\infty} T^{n_k}x_0\Big) = T^2u.$$

Therefore,

$$\lim_{k\to\infty} d\big(T^{n_k}x_0, T^{n_k+1}x_0\big) = d(u, Tu)$$

and

$$\lim_{k\to\infty} d\big(T^{n_k+1}x_0, T^{n_k+2}x_0\big) = d(Tu, T^2u).$$

Now from (2.6) it follows that

$$d(Tu, T^2u) = d(u, Tu). \tag{2.7}$$

As u and Tu are comparable, either $u \preceq Tu$ or $Tu \preceq u$. If $u \neq Tu$, in both the cases, by (2.1) we obtain

$$d(Tu, T^2u) < d(u, Tu)$$

which is a contraction to (2.7). Hence $u = Tu$. □

Theorem 2.7. *Let $(X, \preceq, d)$ be a partially ordered metric space and let $T : X \to X$ be a nondecreasing mapping satisfying the contractive condition (2.1). Suppose that there is an $x_0 \in X$ such that $x_0 \succeq Tx_0$ and the sequence $\{T^n x_0\}$ of successive iterates of T at x_0 has a convergent subsequence $\{T^{n_k}x_0\}$ converging to a point $u \in X$. If T is continuous at u and u and Tu are comparable, then u is a fixed point of T.*

Proof. The proof is similar to Theorem 2.6 with appropriate modifications. We omit the details. □

Corollary 2.8. *Let $(X, \preceq)$ be a partially ordered set and let there exist a metric space d such that (X, d) is partially compact. Let $T : X \to X$ be a nondecreasing mapping satisfying (2.1). Suppose that T is continuous and there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$. If T is mixed dominating on the set of cluster points of $\mathcal{O}(x_0, T)$, then T has a fixed point which is further unique if X is totally ordered.*

Proof. Define a sequence $\{x_n\} = \{T^n x_0\}$ of successive iteration of T at x_0. Since T is nondecreasing and $x_0 \preceq Tx_0$, we have that $\{T^n x_0\}$ is totally ordered set in X. As X is partially compact, the sequence $\{T^n x_0\}$ has a convergent subsequence, say $\{T^{n_k} x_0\}$ converging to some point $u \in X$. Further since T dominates every cluster point of $\{T^n x_0\}$ from above or below, it follows that u and Tu are comparable. Now the desired conclusion follows by an application of Theorem 2.6 or 2.7. □

Notice that Theorems 2.6 and 2.7 and Corollary 2.8 are useful to derive several interesting fixed point theorems in the literature for the contractive mappings in partially ordered metric spaces.

3 Contractive Mappings and Hybrid FPTs

In this section we obtain several hybrid fixed point theorems for partially contractive mappings in a partially ordered metric space which is not necessarily complete.

Theorem 3.1. *Let $(X, \preceq, d)$ be a partially ordered metric space and let $T : X \to X$ be a nondecreasing mapping satisfying the contractive condition*

$$d(Tx, Ty) < d(x, y) \tag{3.1}$$

for all comparable elements $x, y \in X$ with $x \neq y$. Suppose that there is an $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the sequence $\{T^n x_0\}$ of successive iterates of T at x_0 has a convergent subsequence $\{T^{n_k} x_0\}$ converging to a point $u \in X$. If T is continuous at u and u and Tu are comparable, then u is a fixed point of T which is further unique if X is totally ordered.

Proof. Set $y = Tx$ in (3.1). Then T satisfies the contractive condition(2.1). Hence an application of Theorem 2.6 yields that u is a fixed point of T.

To prove uniqueness, assume that X is totally ordered and let $v(\neq u)$ be another fixed point of T. Since X is totally ordered, either $u \preceq v$ or $v \preceq u$. Then in both the cases, from (3.1) we obtain a contradiction. This completes the proof. □

Theorem 3.2. *Let $(X, \preceq, d)$ be a partially ordered metric space and let $T : X \to X$ be a nondecreasing mapping satisfying the contractive condition* (3.1)*. Suppose that there is an $x_0 \in X$ such that $x_0 \succeq Tx_0$ and the sequence $\{T^n x_0\}$ of successive iterates of T at x_0 has a convergent subsequence $\{T^{n_k} x_0\}$ converging to a point $u \in X$. If T is continuous at u and u and Tu are comparable, then u is a fixed point of T which is further unique if X is totally ordered.*

Remark 3.3. We note the continuity of the mapping T in above Theorems 3.1, and 3.2 at every cluster point of $\mathcal{O}(x; T)$ may be replaced with the condition of T-orbitally continuity of T on the metric space X. Again, Theorems 3.1 and 3.2 are the extensions of well-known Edelstein's fixed point theorem [5] for contractive mappings to partially ordered metric spaces.

As a consequence of Theorem 3.1 we obtain the following interesting corollary. Before stating this applicable fixed point result, we require the following definition.

Definition 3.4 (Dhage [6]). The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(X, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Corollary 3.5. *Let $(X, \preceq)$ be a partially ordered set and let there exist a metric space d such that (X, d) is partially compact. Let $T : X \to X$ be a nondecreasing mapping satisfying (3.1). Suppose that T is continuous and there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$. If T is mixed dominating on the set of cluster points of $\mathcal{O}(x_0, T)$, then T has a fixed point x^* and the sequence of successive iterations $\{T^n x_0\}$ converges to x^*. Further if X is totally ordered, then the fixed point of T is unique.*

Proof. Define a sequence $\{x_n\} = \{T^n x_0\}$ of successive iteration of T at x_0. Since T is nondecreasing and $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, we have that $\{T^n x_0\}$ is totally ordered set in X. As X is partially compact, the sequence $\{T^n x_0\}$ has a convergent subsequence, say $\{T^{n_k} x_0\}$ converging to some point $u \in X$. Further since T dominates every cluster point of $\{T^n x_0\}$, it follows that u and Tu are comparable. Now an application of Theorem 3.1 yields that T has a fixed point x^*. Since the order relation $\leq$ and the metric d are compatible, the whole sequence $\{T^n x_0\}$ converges to x^*. This completes the proof. □

Theorem 3.6. *Let $(X, \preceq, d)$ be a totally ordered metric space and let $T : X \to X$ be a nondecreasing mapping. Suppose that there exists a positive integer p such that*

$$d(T^p x, T^p y) < d(x, y) \tag{3.2}$$

for all comparable $x, y \in X$ with $x \neq y$. Suppose that there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the sequence $\{T^n x_0\}$ of iteration of T at x_0 has a convergent subsequence $\{T^{pn_k} x_0\}$ converging to the point u. If T is continuous at u, then u is a unique fixed point of T.

Proof. Let $S = T^p$. Suppose that a subsequence $\{T^{pn_k} x_0\}$ of the sequence $\{T^n x_0\}$ is convergent and converges to u. Then S is continuous at u, because T is continuous at u. Now, an application of Theorem 3.1 yields that S has a unique fixed point, that is, it is a point $u \in X$ such that $S(u) = T^p(u) = u$. Now $T(u) = T(T^p u) = S(Tu)$, showing that Tu is again a fixed point of S. By the uniqueness of u, we get $Tu = u$. The proof is complete. □

Theorem 3.7. *Let $(X, \preceq, d)$ be a totally ordered metric space and let $T : X \to X$ be a nondecreasing mapping. Suppose that there exists a positive integer p such* *that the contraction condition* (3.2) *holds. Suppose further that there exists an* $x_0 \in X$ *such that* $x_0 \succeq Tx_0$ *and the sequence* $\{T^n x_0\}$ *of iteration of* T *at* x_0 *has a convergent subsequence* $\{T^{pn_k} x_0\}$ *converging to the point* u*. If* T *is continuous at* u*, then* u *is a unique fixed point of* T*.*

Corollary 3.8. *Let* $(X, \preceq)$ *be a totally ordered set and there exists a metric* d *such that* (X, d) *is partially compact. Let* $T : X \to X$ *be a nondecreasing mapping satisfying* (3.2) *and there exists an* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *or* $x_0 \succeq Tx_0$*. If* T *is mixed dominating and continuous on the set of cluster points of* $\mathcal{O}(x_0, T)$*, then* T *has a unique fixed point.*

Proof. The proof is similar to Corollary 3.5 and now the conclusion follows by the direct application of Theorem 3.6 or 3.7. □

The classes of fixed point mappings characterized by the inequalities (3.1) and (3.2) can be generalized in two ways. The first way is to generalize the term present on the left hand sides of the inequalities (3.1) and (3.2) called the generalization in left direction. The second way of generalization is in the right direction which is obtained by generalizing the term present on the right hand sides of the inequalities (3.1) and (3.2). The fixed point theorems for generalizations of contraction mappings of later type may be proved in partially ordered metric spaces are as follows.

Theorem 3.9. *Let* $(X, \preceq, d)$ *be a partially ordered metric space and let* $T : X \to X$ *be a nondecreasing mapping satisfying the contractive condition*

$$d(Tx, Ty) < \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\Big[d(x, Ty) + d(y, Tx)\Big] \right\} \tag{3.3}$$

for all comparable elements $x, y \in X$ *for which the right hand side of the inequality is not zero. Suppose that there is an* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *and the sequence* $\{T^n x_0\}$ *of successive iterates of* T *at* x_0 *has a convergent subsequence* $\{T^{n_k} x_0\}$ *converging to a point* $u \in X$*. If* T *is continuous at* u *and* u *and* Tu *are comparable, then* u *is a fixed point of* T *which is further unique if* X *is totally ordered.*

Theorem 3.10. *Let* $(X, \preceq, d)$ *be a partially ordered metric space and let* $T : X \to X$ *be a nondecreasing mapping satisfying the contractive condition (3.3). Suppose that there is an* $x_0 \in X$ *such that* $x_0 \succeq Tx_0$ *and the sequence* $\{T^n x_0\}$ *of successive iterates of* T *at* x_0 *has a convergent subsequence* $\{T^{n_k} x_0\}$ *converging to a point* $u \in X$*. If* T *is continuous at* u *and* u *and* Tu *are comparable, then* u *is a fixed point of* T *which is further unique if* X *is totally ordered.*

Corollary 3.11. *Let* $(X, \preceq)$ *be a partially ordered set and let there exist a metric space* d *such that* (X, d) *is partially compact. Let* $T : X \to X$ *be a nondecreasing mapping satisfying (3.4). Suppose that* T *is continuous and there exists an* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *or* $x_0 \succeq Tx_0$*. If* T *is mixed dominating on the set of cluster points of* $\mathcal{O}(x_0, T)$*, then* T *has a fixed point which is further unique if* X *is totally ordered.*

Theorem 3.12. *Let $(X, \preceq, d)$ be a totally ordered metric space and let $T : X \to X$ be a nondecreasing mapping. Suppose that there exists a positive integer p such that*

$$d(T^p x, T^p y) < \max\left\{d(x,y)d(x,T^p x), d(y,T^p y), \frac{1}{2}\Big[d(x,T^p y) + d(y,T^p x)\Big]\right\} \tag{3.4}$$

for all comparable $x, y \in X$ for which the right hand side of the inequality is not zero. Suppose that there exists a $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the sequence $\{T^n x_0\}$ of iteration of T at x_0 has a convergent subsequence $\{T^{pn_k} x_0\}$ converging to the point u. If T is continuous at u, then u is a unique fixed point of T.

Theorem 3.13. *Let $(X, \preceq, d)$ be a totally ordered metric space and let $T : X \to X$ be a nondecreasing mapping. Suppose that there exists a positive integer p such that the contractive condition* (3.4) *holds. Suppose that there exists a $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the sequence $\{T^n x_0\}$ of iteration of T at x_0 has a convergent subsequence $\{T^{pn_k} x_0\}$ converging to the point u. If T is continuous at u, then u is a unique fixed point of T.*

Corollary 3.14. *Let $(X, \preceq)$ be a partially ordered set and there exists a metric space d such that (X, d) is partially compact. Let $T : X \to X$ be a nondecreasing mapping satisfying (3.5). Suppose that T is continuous and there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$. If T is mixed dominating on the set of cluster points of $\mathcal{O}(x_0, T)$, then T has a fixed point which is further unique if X is totally ordered.*

Next, we state some fixed point results for generalized contractive mappings of left direction in partially ordered metric spaces. A nonunique fixed point theorem for generalized Ciric [7] type contractive mappings in partially ordered metric spaces which is not necessarily complete can be stated as follows.

Theorem 3.15. *Let $(X, \preceq, d)$ be a partially ordered metric space and let $T : X \to X$ be a nondecreasing mapping satisfying the contractive condition*

$$\begin{aligned} &\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} \\ &\quad - \min\{d(x,Ty), d(y,Tx)\} < d(x,y) \end{aligned} \tag{3.5}$$

for all comparable elements $x, y \in X$ with $x \neq y$. Suppose that there is an $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the sequence $\{T^n x_0\}$ of successive iterates of T at x_0 has a convergent subsequence $\{T^{n_k} x_0\}$ converging to a point $u \in X$. If T is continuous at u and u and Tu are comparable, then u is a fixed point of T.

Proof. Set $y = Tx$ in (3.5). Then T satisfies the contractive condition (2.1). Hence an application of Theorem 2.6 yields that u is a fixed point of T. □

Theorem 3.16. *Let $(X, \preceq, d)$ be a partially ordered metric space and let $T : X \to X$ be a nondecreasing mapping satisfying the contractive condition* (3.5)*. Suppose* *that there is an $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the sequence $\{T^n x_0\}$ of successive iterates of T at x_0 has a convergent subsequence $\{T^{n_k} x_0\}$ converging to a point $u \in X$. If T is continuous at u and u and Tu are comparable, then u is a fixed point of T.*

Corollary 3.17. *Let $(X, \preceq)$ be a partially ordered set and let there exist a metric space d such that (X, d) is partially compact. Let $T : X \to X$ be a nondecreasing mapping satisfying (3.5). Suppose that T is continuous and there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$. If T is mixed dominating on the set of cluster points of $\mathcal{O}(x_0, T)$, then T has a fixed point.*

Proof. Define a sequence $\{x_n\} = \{T^n x_0\}$ of successive iteration of T at x_0. Since T is nondecreasing and $x_0 \preceq Tx_0$, we have that $\{T^n x_0\}$ is totally ordered set in X. As X is partially compact, the sequence $\{T^n x_0\}$ has a convergent subsequence, say $\{T^{n_k} x_0\}$ converging to some point $u \in X$. Further since T dominates every cluster point of $\{T^n x_0\}$, it follows that u and Tu are comparable. Now the desired conclusion follows by an application of Theorem 3.15 or 3.16. □

Again, further generalizations of Theorems 3.1, 3.2, 3.15 and 3.16 for classes of generalized contractive mappings in both the directions considered by Dhage [6] may be stated as follows in a partially ordered metric space.

Theorem 3.18. *Let $(X, \preceq, d)$ be a partially ordered metric space and let $T : X \to X$ be a nondecreasing mapping satisfying the contractive condition: there exists a number $b \in \mathbb{R}$ such that*

$$\begin{aligned} 0 \le \min\Big\{ & d(Tx,Ty), d(x,Tx), d(y,Ty), \\ & \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}, \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} \Big\} \\ & + b \min\big\{ d(x,Ty), d(y,Tx) \big\} < \max\big\{ d(x,y), [\min\{d(x,Tx), d(y,Ty)\}] \big\} \end{aligned} \tag{3.6}$$

for all comparable elements $x, y \in X$ for which the right hand side expression $\max\big\{d(x,y), [\min\{d(x,Tx), d(y,Ty)\}]\big\} \neq 0$. Suppose that there is an $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the sequence $\{T^n x_0\}$ of successive iterations of T at x_0 has a convergent subsequence $\{T^{n_k} x_0\}$ converging to a point $u \in X$. If T is continuous at u and u and Tu are comparable, then u is a fixed point of T.

Proof. Set $y = Tx$ in (3.6). Then T satisfies the contractive condition (2.1). Hence an application of Theorem 2.6 yields that u is a fixed point of T. □

Theorem 3.19. *Let $(X, \preceq, d)$ be a partially ordered metric space and let $T : X \to X$ be a nondecreasing mapping satisfying the contractive condition* (3.5). *Suppose that there is an $x_0 \in X$ such that $x_0 \succeq Tx_0$ and the sequence $\{T^n x_0\}$ of successive iterates of T at x_0 has a convergent subsequence $\{T^{n_k} x_0\}$ converging to a point $u \in X$. If T is continuous at u and u and Tu are comparable, then u is a fixed point of T.*

Corollary 3.20. *Let $(X, \preceq)$ be a partially ordered set and let there exist a metric space d such that (X, d) is partially compact. Let $T : X \to X$ be a nondecreasing mapping satisfying (3.6). Suppose that T is continuous and there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$. If T is mixed dominating on the set of cluster points of $\mathcal{O}(x_0, T)$, then T has a fixed point which is further unique if X is totally ordered.*

4 Examples

Let $J = [a, b]$ be a closed and bounded interval in $\mathbb{R}$, the set of real numbers, where $a, b \in \mathbb{R},\ a < b$. Define the usual order relation $\leq$ and usual standard metric d on $\mathbb{R}$ by $d(x, y) = |x - y|$, where $|\cdot|$ is the usual absolute value function defined on $\mathbb{R}$.

Consider a continuous and nondecreasing function $f : J \to J$ satisfying $|f'(x)| < 1$ for all $x \in (a, b)$. Clearly, J is compact and f is a partially contractive on J. To see this, let $x, y \in J$ be such that $x < y$. Then, using the theory of derivative and Riemann integration, we obtain

$$\begin{aligned}
d(fx, fy) &= |fx - fy| \\
&= \left| \int_x^y f'(x)\, ds \right| \\
&\leq \left| \int_x^y |f'(x)|\, ds \right| \\
&< |x - y| \\
&= d(x, y).
\end{aligned}$$

Further if there is an element $x_0 \in J$ such that $x_0 \leq fx_0$ or $x_0 \geq fx_0$, an application of Corollary 3.5 yields that f has a unique fixed point in view of the fact that J is a compact set in $\mathbb{R}$. Hence has a unique fixed point x^* in $[a, b]$.

Now the order relation $\leq$ and the metric d in $\mathbb{R}$ are compatible, so the sequence of iterations $\{f^n(x_0)\}$ converges to x^*. Though existence of the fixed points of the mappings f can also be proved by using existing classical fixed point theorem in metric spaces, there we do not find any algorithm to obtain it. This is the advantage of our fixed point principles in partially ordered metric spaces over those already proved for contractive mappings in metric spaces.

Example 4.1. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the mapping f on J defined by $f(x) = \dfrac{x^2}{2}$ for $x \in [0, 1]$.

Clearly, f is continuous and nondecreasing on J into itself. Now, $|f'(x)| = x < 1$ for each $x \in (0, 1)$. Furthermore, there is an element $x_0 = 1$ in $[0, 1]$ such that $f(1) = \frac{1}{2} < 1$. Hence an application of above result yields that f has a unique fixed point in $[0, 1]$. In this case the unique fixed point of f is $x^* = 0$. Now the order relation $\leq$ and the metric d in $\mathbb{R}$ are compatible, so the sequence of iterations $\{f^n(\frac{1}{2})\}$ converges to 0.

Example 4.2. Given a closed and bounded interval $J = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in $\mathbb{R}$, Consider the function f on J defined by $f(x) = \tan^{-1} x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Define the usual order relation $\leq$ and usual standard metric d on $\mathbb{R}$ as in Example 4.1 above. By Heine-Borel theorem, J is a compact and totally ordered subset of $\mathbb{R}$. Clearly f is a continuous and nondecreasing mapping on J into itself. We show that f satisfies the contractive condition (3.1) on J. Let $x, y \in J$ be such that $x \geq y$, $x \neq y$. Then,

$$\begin{aligned} d(fx, fy) &= |fx - fy| \\ &= |\tan^{-1} x - \tan^{-1} y| \\ &= \tan^{-1} x - \tan^{-1} y \\ &= \frac{1}{1+\xi^2}(x-y) \\ &= \frac{1}{1+\xi^2}|x-y| \\ &< d(x,y) \quad [\because y < \xi < x] \end{aligned}$$

Again, we have a point $x_0 = -\frac{\pi}{2}$ such that $-\frac{\pi}{2} \leq f\left(-\frac{\pi}{2}\right)$. Now we apply Corollary 3.5 and conclude that f has a unique fixed point, namely $x^* = 0$. Now the order relation $\leq$ and the metric d in $\mathbb{R}$ are compatible, so the sequence of iterations $\{f^n(\frac{-\pi}{2})\}$ converges to the fixed point 0.

Remark 4.3. From the foregoing discussion and the examples it is clear that the fixed point theorems for contractive mappings in metric spaces on the lines of Edelstein [5] do not provide any constructive method for obtaining the fixed points of the mappings in question. This problem has been settled down to some extent by our approach of fixed point theorems in partially ordered metric spaces. The fixed point results of this paper may be extended to commuting and non-commuting pairs of mappings in partially ordered metric spaces satisfying the generalized contractive type conditions of the form of the inequalities (3.3) and (3.4) for proving the common fixed point results. These and other similar results form the further scope and open problems in the area of hybrid fixed point theory in the subject of nonlinear functional analysis and applications.

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