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Coupled Coincidence Points for Monotone Operators in S-Metric Spaces

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Abstract : The aim of this paper is to establish some coupled coincidence point results in S-metric space by using the notion of compatible mappings in the setting of a partially ordered S-metric space. We prove the existence and uniqueness of coupled coincidence points involving a (φ, ψ)-contractive condition for a mappings having the mixed g-monotone property.

Keywords : S-metric space; partially ordered S-metric space; coupled coincidence point; mixed g-monotone property.
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1 Introduction

Metric spaces have very wide applications in mathematics and applied sciences. For this, many authors tried to give definitions of metric spaces in many ways. In 1966, Gahler [1, 2] introduced the notion of 2-metric spaces and Dhage [3] introduced the notion of D-metric spaces. After the introduction of these metric spaces many authors proved some fixed point results related to these metric spaces. In 2006, Mustafa and Sims [4] proved that most of the results of Dhage's D-metric spaces are not valid. So, they introduced the new concept of generalized metric space called G-metric space and gave some remarkable results in G-metric spaces. In 2012, Sedghi et al. [5] introduced the notion of S-metric spaces as the generalization of G-metric and D^*-metric spaces.

The notion of a coupled fixed point was introduced and studied by Opoitsev [6–8] and then by Guo and Lakshmikantham [9]. Bhashkar and Lakshmikantham in [10] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. They also discussed an application of their result by investigating the existence and uniqueness of solution for a periodic boundary value problem. Choudhury and Kundu [11] obtained coupled coincidence point results in partially ordered metric spaces for compatible mappings.

Lakshmikantham and Ćirić [12] defined a mixed g-monotone mapping and prove coupled coincidence and coupled common fixed point theorems for such nonlinear contractive mappings in partially ordered complete metric spaces. In 2011 Alotaibi and Alsulami [13] proved the existence and uniqueness of coupled coincidence point involving a (φ, ψ)-contractive condition for a mappings having the mixed g-monotone property.

2 Preliminaries

We begin with the following definition:

Definition 2.1. [5] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$.

(i) $S(x, y, z) \geq 0$;

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$;

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then the pair (X, S) is called an *S-metric space*.

The following is an intuitive geometric example for S-metric spaces.

Example 2.2. [5] Let $X = \mathbb{R}^2$ and d be an ordinary metric on X. Put

$$S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$$

for all $x, y, z \in \mathbb{R}$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.

Lemma 2.3. [5] *Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Lemma 2.4. *Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z) \quad \textit{and} \quad S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$$

for all $x, y, z \in X$.

Proof. It is a direct consequence of Definition 2.1 and Lemma 2.4. □

Definition 2.5. [5] Let (X, S) be an S-metric space.

(i) A sequence $\{x_n\} \subset X$ is said to *converge to* $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.

(ii) A sequence $\{x_n\} \subset X$ is called a *Cauchy sequence* if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

(iii) The S-metric space (X, S) is said to be *complete* if every Cauchy sequence is a convergent sequence.

Lemma 2.6. [5] *Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$, then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

Definition 2.7 ([12], Mixed g-Monotone Property)**.** Let $(X, \leq)$ be a partially ordered set and $F : X^2 \to X$. We say that the mapping F has the *mixed g-monotone property* if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument. That is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, gx_1 \leq gx_2 \;&\Rightarrow\; F(x_1, y) \leq F(x_2, y) \\ y_1, y_2 \in X, gy_1 \leq gy_2 \;&\Rightarrow\; F(x, y_1) \geq F(x, y_2). \end{aligned} \tag{2.1}$$

Definition 2.8 (Coupled Coincidence Point)**.** Let $(x, y) \in X \times X$, $F : X^2 \to X$ and $g : X \to X$. We say that (x, y) is a *coupled coincidence point* of F and g if $F(x, y) = gx$ and $F(y, x) = gy$ for $x, y \in X$.

Definition 2.9. The mapping F and g where $F : X^2 \to X$and $g : X \to X$, are said to be *compatible* if

$$\lim_{n\to\infty} S\big(g(F(x_n, y_n)), g(F(x_n, y_n)), F(gx_n, gy_n)\big) = 0$$

and

$$\lim_{n\to\infty} S\big(g(F(y_n,x_n)), g(F(y_n,x_n)), F(gy_n,gx_n)\big) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim\limits_{n\to\infty} F(x_n,y_n) = \lim\limits_{n\to\infty} gx_n = x$ and $\lim\limits_{n\to\infty} F(y_n,x_n) = \lim\limits_{n\to\infty} gy_n = y$, for all $x,y\in X$ are satisfied.

3 Existence of Coupled Coincidence Points

Let Φ denote all functions $\varphi:[0,\infty)\to[0,\infty)$ which satisfy

(1) φ continuous and non-decreasing;

(2) $\varphi(t)=0$ if and only if $t=0$;

(3) $\varphi(t+s)\le\varphi(t)+\varphi(s), \forall t,s\in[0,\infty)$

and let Ψ denote all the functions $\psi:[0,\infty)\to(0,\infty)$ which satisfy $\lim\limits_{t\to r}\psi(t)>0$ for all $r>0$ and $\lim\limits_{t\to 0^+}\psi(t)=0$.

Now, let us start proving our main results.

Theorem 3.1. *Let $(X,\le,S)$ be a partially ordered complete S-metric space. Let $F:X^2\to X$ is such that F has the mixed g-monotone property such that there exists $x_0,y_0\in X$ with*

$$gx_0\le F(x_0,y_0) \quad \text{and} \quad gy_0\ge F(y_0,x_0).$$

Suppose there exist $\varphi\in\Phi$ and $\psi\in\Psi$ such that

$$\varphi\Big(S\big(F(x,y),F(x,y),F(u,v)\big)\Big) \le \frac{1}{2}\varphi\big(S(gx,gx,gu)+S(gy,gy,gv)\big)-\psi\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right), \tag{3.1}$$

for all $x,y,u,v\in X$ with $gx\ge gu$ and $gy\le gv$. Suppose $F(X\times X)\subseteq g(X)$, g is continuous and compatible with F and also suppose either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a non-decreasing sequence $\{x_n\}\to x$, then $x_n\le x$, for all n,*

(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \geq y$, *for all* n.

Then there exists $x, y \in X$ *such that*

$$gx = F(x, y) \quad and \quad gy = F(y, x),$$

i.e., F *and* g *have a coupled coincidence point in* X.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$. Using $F(X \times X) \subseteq g(X)$, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X as

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n) \qquad \text{for all } n \geq 0. \tag{3.2}$$

We are going to prove that

$$gx_n \leq gx_{n+1} \qquad \text{for all } n \geq 0 \tag{3.3}$$

and

$$gy_n \geq gy_{n+1} \qquad \text{for all } n \geq 0. \tag{3.4}$$

To prove these, we are going to use the mathematical induction.

Let $n = 0$. Since $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$ and as $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \leq gx_1$ and $gy_0 \geq gy_1$. Thus (3.3) and (3.4) hold for $n = 0$.

Suppose now that (3.3) and (3.4) hold for some fixed $n \geq 0$, Then, since $gx_n \leq gx_{n+1}$ and $gy_n \geq gy_{n+1}$, and by mixed g-monotone property of F, we have

$$gx_{n+2} = F(x_{n+1}, y_{n+1}) \geq F(x_n, y_{n+1}) \geq F(x_n, y_n) = gx_{n+1} \tag{3.5}$$

and

$$gy_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_{n+1}) \leq F(y_n, x_n) = gy_{n+1}. \tag{3.6}$$

Using (3.5) and (3.6), we get

$$gx_{n+1} \leq gx_{n+2} \quad \text{and} \quad gy_{n+1} \geq gy_{n+2}.$$

Hence by the mathematical induction we conclude that (3.3) and (3.4) hold for all $n \geq 0$.

Therefore,

$$gx_0 \leq gx_1 \leq gx_2 \leq \cdots \leq gx_n \leq gx_{n+1} \leq \cdots \tag{3.7}$$

and

$$gy_0 \geq gy_1 \geq gy_2 \geq \cdots \geq gy_n \geq gy_{n+1} \geq \cdots . \tag{3.8}$$

Since $gx_n \geq gx_{n-1}$ and $gy_n \leq gy_{n-1}$, using (3.1) and (3.2), we have

$$\begin{aligned}\varphi\big(S(gx_{n+1}, gx_{n+1}, gx_n)\big) &= \varphi\Big(S\big(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})\big)\Big)\\ &\leq \frac{1}{2}\varphi\big(S(gx_n, gx_n, gx_{n-1}) + S(gy_n, gy_n, gy_{n-1})\big)\\ &\quad - \psi\bigg(\frac{S(gx_n, gx_n, gx_{n-1}) + S(gy_n, gy_n, gy_{n-1})}{2}\bigg).\end{aligned} \tag{3.9}$$

Similarly, since $gy_{n-1} \geq gy_n$ and $gx_{n-1} \leq gx_n$, using (3.1) and (3.2), we have

$$\begin{aligned}\varphi\big(S(gy_n, gy_n, gy_{n+1})\big) &= \varphi\Big(S\big(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big)\Big)\\ &\leq \frac{1}{2}\varphi\big(S(gy_{n-1}, gy_{n-1}, gy_n) + S(gx_{n-1}, gx_{n-1}, gx_n)\big)\\ &\quad -\psi\bigg(\frac{S(gy_{n-1}, gy_{n-1}, gy_n) + S(gx_{n-1}, gx_{n-1}, gx_n)}{2}\bigg).\end{aligned} \tag{3.10}$$

Using Lemma 2.3, we have

$$\begin{aligned}\varphi\big(S(gy_{n+1}, gy_{n+1}, gy_n)\big) &\leq \frac{1}{2}\varphi\big(S(gy_n, gy_n, gy_{n-1}) + S(gx_n, gx_n, gx_{n-1})\big)\\ &\quad -\psi\bigg(\frac{S(gy_n, gy_n, gy_{n-1}) + S(gx_n, gx_n, gx_{n-1})}{2}\bigg).\end{aligned} \tag{3.11}$$

Using (3.9) and (3.11), we have

$$\begin{aligned}&\varphi\big(S(gx_{n+1}, gx_{n+1}, gx_n)\big) + \varphi\big(S(gy_{n+1}, gy_{n+1}, gy_n)\big)\\ &\quad \leq \varphi\big(S(gx_n, gx_n, gx_{n-1}) + S(gy_n, gy_n, gy_{n-1})\big)\\ &\quad\quad - 2\psi\left(\frac{S(gx_n, gx_n, gx_{n-1}) + S(gy_n, gy_n, gy_{n-1})}{2}\right).\end{aligned} \tag{3.12}$$

By property (3) of φ, we have

$$\begin{aligned}&\varphi\big(S(gx_{n+1}, gx_{n+1}, gx_n) + S(gy_{n+1}, gy_{n+1}, gy_n)\big)\\ &\quad \leq \varphi\big(S(gx_{n+1}, gx_{n+1}, gx_n)\big) + \varphi\big(S(gy_{n+1}, gy_{n+1}, gy_n)\big).\end{aligned} \tag{3.13}$$

Using (3.12) and (3.13), we have

$$\begin{aligned}\varphi\big(S(gx_{n+1},gx_{n+1},gx_n)+S(gy_{n+1},gy_{n+1},gy_n)\big)\\ \le \varphi\big(S(gx_n,gx_n,gx_{n-1})+S(gy_n,gy_n,gy_{n-1})\big)\\ -2\psi\left(\frac{S(gx_n,gx_n,gx_{n-1})+S(gy_n,gy_n,gy_{n-1})}{2}\right),\end{aligned}\tag{3.14}$$

which implies, since ψ is a non-negative function,

$$\begin{aligned}\varphi\big(S(gx_{n+1},gx_{n+1},gx_n)+S(gy_{n+1},gy_{n+1},gy_n)\big)\\ \le \varphi\big(S(gx_n,gx_n,gx_{n-1})+S(gy_n,gy_n,gy_{n-1})\big).\end{aligned}$$

Using the fact that φ is non-decreasing, we get

$$\begin{aligned}S(gx_{n+1},gx_{n+1},gx_n)+S(gy_{n+1},gy_{n+1},gy_n)\\ \le S(gx_n,gx_n,gx_{n-1})+S(gy_n,gy_n,gy_{n-1}).\end{aligned}$$

Set

$$\delta_n = S(gx_{n+1},gx_{n+1},gx_n)+S(gy_{n+1},gy_{n+1},gy_n).$$

Now we would like to show that $\delta_n \to 0$ as $n \to \infty$. It is clear that the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \ge 0$ such that

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\Big[S(gx_{n+1},gx_{n+1},gx_n)+S(gy_{n+1},gy_{n+1},gy_n)\Big]=\delta. \tag{3.15}$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\delta_n \to \delta$) of both sides of (3.14) and remembering $\lim\limits_{t\to r}\psi(t) > 0$ for all $r > 0$ and φ is continuous, we have

$$\begin{aligned}\varphi(\delta) = \lim_{n\to\infty}\varphi(\delta_n) &\le \lim_{n\to\infty}\left[\varphi(\delta_{n-1}) - 2\psi\left(\frac{\delta_{n-1}}{2}\right)\right]\\ &= \varphi(\delta) - 2\lim_{\delta_{n-1}\to\delta}\psi\left(\frac{\delta_{n-1}}{2}\right) < \varphi(\delta),\end{aligned}$$

this is a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\Big[S(gx_{n+1},gx_{n+1},gx_n)+S(gy_{n+1},gy_{n+1},gy_n)\Big]=0. \tag{3.16}$$

Now, we will prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences

$\{gx_{n(k)}\}, \{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$S\big(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}\big) + S\big(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}\big) \geq \varepsilon. \tag{3.17}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (3.17). Then

$$S\big(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}\big) + S\big(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}\big) < \varepsilon. \tag{3.18}$$

Using (3.17), (3.18), Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\varepsilon \leq r_k &:= S\big(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}\big) + S\big(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}\big) \\
&\leq 2S\big(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}\big) + S\big(gx_{m(k)}, gx_{m(k)}, gx_{n(k)-1}\big) \\
&\quad + 2S\big(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}\big) + S\big(gy_{m(k)}, gy_{m(k)}, gy_{n(k)-1}\big) \\
&\leq 2S\big(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}\big) + 2S\big(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}\big) + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ and using (3.16), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \Big[S\big(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}\big) + S\big(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}\big)\Big] = \varepsilon. \tag{3.19}$$

By Lemma 2.4, we have

$$\begin{aligned}
r_k &= S\big(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}\big) + S\big(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}\big) \\
&\leq 2S\big(gx_{n(k)}, gx_{n(k)}, gx_{n(k)+1}\big) + S\big(gx_{m(k)}, gx_{m(k)}, gx_{n(k)+1}\big) \\
&\quad + 2S\big(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}\big) + S\big(gy_{m(k)}, gy_{m(k)}, gy_{n(k)+1}\big) \\
&\leq 2S\big(gx_{n(k)}, gx_{n(k)}, gx_{n(k)+1}\big) + S\big(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)}\big) \\
&\quad + 2S\big(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}\big) + S\big(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)}\big) \\
&\leq 2S\big(gx_{n(k)}, gx_{n(k)}, gx_{n(k)+1}\big) + \Big[2S\big(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}\big) \\
&\quad + S\big(gx_{m(k)+1}, gx_{m(k)+1}, gx_{m(k)+1}\big)\Big] + 2S\big(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}\big) \\
&\quad + \Big[2S\big(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}\big) + S\big(gy_{m(k)+1}, gy_{m(k)+1}, gy_{m(k)+1}\big)\Big] \\
&\leq 2\delta_{n(k)} + \delta_{m(k)} + S\big(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}\big) \\
&\quad + S\big(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}\big).
\end{aligned}$$

Using the property of φ, we have

$$\begin{aligned}\varphi(r_k) &= \varphi\Big(\delta_{n(k)} + \delta_{m(k)} + S\big(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}\big)\\ &\quad + S\big(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}\big)\Big)\\ &\le \varphi\big(\delta_{n(k)} + \delta_{m(k)}\big) + \varphi\Big(S\big(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}\big)\Big)\\ &\quad + \varphi\Big(S\big(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}\big)\Big).\end{aligned} \tag{3.20}$$

Since $n(k) > m(k)$, hence $gx_{n(k)} \ge gx_{m(k)}$ and $gy_{n(k)} \ge gy_{m(k)}$. Using (3.1) and (3.2), we get

$$\begin{aligned}&\varphi\Big(S\big(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}\big)\Big)\\ &\quad = \varphi\Big(S\big(F(x_{n(k)}, y_{n(k)}), F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})\big)\Big)\\ &\quad \le \frac{1}{2}\varphi\Big(S\big(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}\big) + S\big(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}\big)\Big)\\ &\qquad - \psi\left(\frac{S\big(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}\big) + S\big(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}\big)}{2}\right)\\ &\quad = \frac{1}{2}\varphi(r_k) - \psi\left(\frac{r_k}{2}\right).\end{aligned} \tag{3.21}$$

By the same way, we also have

$$\begin{aligned}&\varphi\Big(S\big(gy_{m(k)+1}, gy_{m(k)+1}, gy_{n(k)+1}\big)\Big)\\ &\quad = \varphi\Big(S\big(F(y_{m(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)})\big)\Big)\\ &\quad \le \frac{1}{2}\varphi\Big(S\big(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}\big) + S\big(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\big)\Big)\\ &\qquad - \psi\left(\frac{S\big(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}\big) + S\big(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\big)}{2}\right)\\ &\quad = \frac{1}{2}\varphi(r_k) - \psi\left(\frac{r_k}{2}\right).\end{aligned} \tag{3.22}$$

Inserting (3.21) and (3.22) in (3.20), we have

$$\varphi(r_k) \ \le\ \varphi\big(\delta_{n(k)} + \delta_{m(k)}\big) + \varphi(r_k) - 2\psi\left(\frac{r_k}{2}\right).$$

Letting $k \to \infty$ and using (3.16) and (3.19), we get

$$\varphi(\varepsilon) \leq \varphi(0) + \varphi(\varepsilon) - 2 \lim_{k\to\infty} \psi\left(\frac{r_k}{2}\right) = \varphi(\varepsilon) - 2 \lim_{r_k\to\infty} \psi\left(\frac{r_k}{2}\right) < \varphi(\varepsilon),$$

this is a contradiction. This shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Since X is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y. \tag{3.23}$$

Since F and g are compatible mappings, we have

$$\lim_{n\to\infty} S\Big(g\big(F(x_n, y_n)\big), g\big(F(x_n, y_n)\big), F\big(gx_n, gy_n\big)\Big) = 0 \tag{3.24}$$

and

$$\lim_{n\to\infty} S\Big(g\big(F(y_n, x_n)\big), g\big(F(y_n, x_n)\big), F\big(gy_n, gx_n\big)\Big) = 0. \tag{3.25}$$

We now show that $gx = F(x, y)$ and $gy = F(y, x)$. Suppose that the assumption (a) holds. For all $n \geq 0$, we have

$$\begin{aligned} S\big(gx, gx, F(gx_n, gy_n)\big) &\leq S\big(gx, gx, g(F(x_n, y_n))\big) \\ &\quad + S\big(g(F(x_n, y_n)), g(F(x_n, y_n)), F(gx_n, gy_n)\big). \end{aligned}$$

Taking the limit as $n \to \infty$, using (3.2), (3.23), (3.24) and the fact that F and g are continuous, we have $S(gx, gx, F(x, y)) = 0$.

Similarly, using (3.2), (3.23), (3.25) and the fact that F and g are continuous, we have $S(gy, gy, F(y, x)) = 0$.

Combining the above two results, we get

$$gx = F(x, y) \quad \text{and} \quad gy = F(y, x).$$

Finally, suppose that (b) holds. By (3.3), (3.4) and (3.23), we have $\{gx_n\}$ is a non-decreasing sequence, $gx_n \to x$ and $\{gy_n\}$ is a non-increasing sequence, $gy_n \to y$ as $n \to \infty$. Hence, by assumption (b), we have for all $n \geq 0$,

$$gx_n \leq x \quad \text{and} \quad gy_n \leq y. \tag{3.26}$$

Since F and g are compatible mappings and g is continuous, by (3.24) and (3.25), we have

$$\lim_{n\to\infty} g(gx_n) = gx = \lim_{n\to\infty} g\big(F(x_n, y_n)\big) = \lim_{n\to\infty} F(gx_n, gy_n) \tag{3.27}$$

and

$$\lim_{n\to\infty} g(gy_n) = gy = \lim_{n\to\infty} g\big(F(y_n, x_n)\big) = \lim_{n\to\infty} F(gy_n, gx_n). \tag{3.28}$$

Now we have

$$S\big(gx, gx, F(x,y)\big) \le S\big(gx, gx, g(gx_{n+1})\big) + S\big(g(gx_{n+1}), g(gx_{n+1}), F(x,y)\big).$$

Taking $n \to \infty$ in the above inequality, using (3.2) and (3.20) we have,

$$\begin{aligned} S\big(gx, gx, F(x,y)\big) &\le \lim_{n\to\infty} S\big(gx, gx, g(gx_{n+1})\big) \\ &\quad + \lim_{n\to\infty} S\big(g(F(x_n,y_n)), g(F(x_n,y_n)), F(x,y)\big) \\ &\le \lim_{n\to\infty} S\big(F(gx_n, gy_n), F(gx_n, gy_n), F(x,y)\big). \end{aligned} \tag{3.29}$$

Using the property of φ, we get

$$\varphi\Big(S\big(gx, gx, F(x,y)\big)\Big) \le \lim_{n\to\infty} \varphi\Big(S\big(F(gx_n, gy_n), F(gx_n, gy_n), F(x,y)\big)\Big).$$

Since the mapping g is monotone increasing, using (3.1), (3.26) and (3.29), we have for all $n \ge 0$,

$$\begin{aligned} \varphi\Big(S\big(gx, gx, F(x,y)\big)\Big) &\le \lim_{n\to\infty} \frac{1}{2}\varphi\Big(S\big(ggx_n, ggx_n, gx\big) + S\big(gy_n, gy_n, ggy\big)\Big) \\ &\quad - \lim_{n\to\infty} \left(\frac{S\big(ggx_n, ggx_n, gx\big) + S\big(gy_n, gy_n, ggy\big)}{2}\right). \end{aligned}$$

Using the above inequality, using (3.23) and the property of ψ, we get $\psi\Big(S\big(gx, gx, F(x,y)\big)\Big) = 0$, thus $S(gx, gx, F(x,y)) = 0$. Hence $gx = F(x,y)$.

Similarly, we can show that $gy = F(y,x)$. Thus we proved that F and g have a coupled coincidence point. □

Corollary 3.2. *Let $(X, \le, S)$ be a partially ordered complete S-metric space. Let $F : X^2 \to X$ is such that F has the mixed monotone property such that there exists $x_0, y_0 \in X$ with*

$$x_0 \le F(x_0, y_0) \quad \text{and} \quad y_0 \ge F(y_0, x_0).$$

Suppose there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\begin{aligned} &\varphi\Big(S\big(F(x,y), F(x,y), F(u,v)\big)\Big) \\ &\quad \le \frac{1}{2}\varphi\big(S(x,x,u) + S(y,y,v)\big) - \psi\left(\frac{S(x,x,u) + S(y,y,v)}{2}\right), \end{aligned} \tag{3.30}$$

for all $x, y, u, v \in X$ with $x \ge u$ and $y \le v$. Suppose either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, for all n,*

(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \geq y$, for all n.*

Then there exists $x, y \in X$ such that

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x),$$

that is, F has a coupled fixed point in X.

Corollary 3.3. *Let $(X, \leq, S)$ be a partially ordered complete S-metric space. Let $F : X^2 \to X$ is such that F has the mixed monotone property such that there exists $x_0, y_0 \in X$ with*

$$x_0 \leq F(x_0, y_0) \quad \textit{and} \quad y_0 \geq F(y_0, x_0).$$

Suppose there exist $\psi \in \Psi$ such that

$$S\big(F(x,y), F(x,y), F(u,v)\big) \leq \frac{S(x,x,u) + S(y,y,v)}{2} - \psi\left(\frac{S(x,x,u) + S(y,y,v)}{2}\right), \tag{3.31}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, for all n,*

(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \geq y$, for all n.*

Then there exists $x, y \in X$ such that

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x),$$

that is, F has a coupled fixed point in X.

Proof. Take $\varphi(t) = t$ in Corollary 3.2. □

Corollary 3.4. *Let $(X, \leq, S)$ be a partially ordered complete S-metric space. Let $F : X^2 \to X$ is such that F has the mixed monotone property such that there exists $x_0, y_0 \in X$ with*

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0).$$

Suppose there exists a real number $k \in [0, 1)$ such that

$$S\big(F(x,y), F(x,y), F(u,v)\big) \;\leq\; \frac{k}{2}\Big(S(x,x,u) + S(y,y,v)\Big), \tag{3.32}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either

(a) *F is continuous or*

(b) *X has the following property:*

  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, for all n,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \geq y$, for all n.*

Then there exists $x, y \in X$ such that

$$x = F(x, y) \quad \text{and} \quad y = F(y, x),$$

that is, F has a coupled fixed point in X.

Proof. Take $\psi(t) = (1 - k)t$ in Corollary 3.3. □

4 Uniqueness of Coupled Coincidence Point

In this section, we will prove the uniqueness of the coupled coincidence point. Note that if $(X; \leq)$ is a partially ordered set, then we endow the product X^2 with the following partial order relation, for all $(x, y), (u, v) \in X^2$,

$$(x, y) \leq (u, v) \quad \Leftrightarrow \quad x \leq u \ , \ \ y \geq v.$$

Theorem 4.1. *In addition to hypotheses of Theorem 3.1, suppose that for every $(x, y), (z, t)$ in X^2, if there exists a (u, v) in X^2 that is comparable to (x, y) and (z, t), then F has a unique coupled coincidence point.*

Proof. From Theorem 3.1, the set of coupled coincidence points of F and g is nonempty. Suppose (x, y) and (z, t) are coupled coincidence points of F and g, that is $gx = F(x, y), gy = F(y, x), gz = F(z, t)$ and $gt = F(t, z)$. We are going to show that $gx = gz$ and $gy = gt$. By assumption, there exists

$(u,v) \subset X^2$ that is comparable to (x,y) and (z,t). We define sequences $\{gu_n\}, \{gv_n\}$ as follows,

$$u_0 = u\,, \quad v_0 = v. \quad gu_{n+1} = F(u_n, v_n) \quad \text{and} \quad gv_{n+1} = F(v_n, u_n) \text{ for all } n.$$

Since (u,v) is comparable with (x,y), we may assume that $(x,y) \geq (u,v) = (u_0, v_0)$. Using the mathematical induction, it is easy to prove that

$$(x,y) \geq (u_n, v_n) \qquad \text{for all } n. \tag{4.1}$$

Using (3.1) and (4.1), we have

$$\begin{aligned}\varphi\Big(S\big(gx, gx, gu_{n+1}\big)\Big) &= \varphi\Big(S\big(F(x,y), F(x,y), F(u_n, v_n)\big)\Big) \\ &\leq \frac{1}{2}\varphi\big(S(gx, gx, gu_n) + S(gy, gy, gv_n)\big) \\ &\quad - \psi\left(\frac{S(gx, gx, gu_n) + S(gy, gy, gv_n)}{2}\right).\end{aligned} \tag{4.2}$$

Similarly,

$$\begin{aligned}\varphi\Big(S\big(gv_{n+1}, gy, gy\big)\Big) &= \varphi\Big(S\big(F(v_n, u_n), F(y,x), F(y,x)\big)\Big) \\ &\leq \frac{1}{2}\varphi\Big(S(gv_n, gy, gy) + S(gu_n, gx, gx)\Big) \\ &\quad - \psi\left(\frac{S(gv_n, gy, gy) + S(gu_n, gx, gx)}{2}\right).\end{aligned} \tag{4.3}$$

Using (4.2), (4.3) and the property of φ, we have

$$\begin{aligned}&\varphi\Big(S\big(gx, gx, gu_{n+1}\big) + S\big(gv_{n+1}, gy, gy\big)\Big) \\ &\qquad \leq \varphi\Big(S\big(gx, gx, gu_{n+1}\big)\Big) + \varphi\Big(S\big(gv_{n+1}, gy, gy\big)\Big) \\ &\qquad \leq \varphi\Big(S\big(gx, gx, gu_n\big) + S\big(gy, gy, gv_n\big)\Big) \\ &\qquad\quad - 2\psi\left(\frac{S\big(gv_n, gy, gy\big) + S\big(gu_n, gx, gx\big)}{2}\right),\end{aligned} \tag{4.4}$$

which implies, using the definition of ψ,

$$\begin{aligned}&\varphi\Big(S\big(gx, gx, gu_{n+1}\big) + S\big(gv_{n+1}, gy, gy\big)\Big) \\ &\qquad \leq \varphi\Big(S\big(gx, gx, gu_n\big)\Big) + \varphi\Big(S\big(gv_n, gy, gy\big)\Big).\end{aligned}$$

Thus, using the definition of φ,

$$S\big(gx, gx, gu_{n+1}\big) + S\big(gv_{n+1}, gy, gy\big) \le S\big(gx, gx, gu_n\big) + S\big(gv_n, gy, gy\big).$$

That is the sequence $\big\{S(gx, gx, gu_n)+S(gy, gy, gv_n)\big\}$ is decreasing. Therefore, there exists $\alpha \ge 0$ such that

$$\lim_{n\to\infty} \Big(S\big(gx, gx, gu_n\big) + S\big(gy, gy, gv_n\big)\Big) = \alpha. \tag{4.5}$$

We will show that $\alpha = 0$. Suppose to the contrary that $\alpha > 0$. Taking the limit as $n \to \infty$ in (4.4), we have, using the property of ψ,

$$\varphi(\alpha) \le \varphi(\alpha) - 2 \lim_{n\to\infty} \psi\left(\frac{S(gx, gx, gu_n) + S(gy, gy, gv_n)}{2}\right) < \varphi(\alpha),$$

this is a contradiction. Thus $\alpha = 0$, that is,

$$\lim_{n\to\infty} \Big(S\big(gx, gx, gu_n\big) + S\big(gy, gy, gv_n\big)\Big) = 0.$$

It implies

$$\lim_{n\to\infty} S\big(gx, gx, gu_n\big) = \lim_{n\to\infty} S\big(gy, gy, gv_n\big) = 0. \tag{4.6}$$

Similarly, we show that

$$\lim_{n\to\infty} S\big(gz, gz, gu_n\big) = \lim_{n\to\infty} S\big(gt, gt, gv_n\big) = 0. \tag{4.7}$$

Using (4.6) and (4.7), we have $gx = gz$ and $gy = gt$. □

Corollary 4.2. *In addition to hypotheses of Corollary 3.2, suppose that for every* $(x, y), (z, t)$ *in* X^2*, if there exists a* (u, v) *in* X^2 *that is comparable to* (x, y) *and* (z, t)*, then* F *has a unique coupled fixed point.*

5 Example

Example 5.1. Let $X = [0, 1]$. Then $(X, \le)$ is a partially ordered set with the natural ordering of real numbers. Let

$$S(x, y, z) = \frac{|x-y| + |x-z| + |y-z|}{2} \quad \text{for } x, y, z \in [0, 1].$$

Then $S(x, x, y) = |x - y|$ and (X, S) is a complete S-metric space.

Let $g : X \to X$ be defined as:

$$gx = x^2 \ , \quad \text{for all } x \in X,$$

and let $F : X^2 \to X$ be defined as:

$$F(x,y) = \begin{cases} \dfrac{x^2 - y^2}{3}, & \text{if } x \geq y; \\ 0, & \text{if } x < y. \end{cases}$$

F obeys the mixed g-monotone property. Let $\varphi : [0,\infty) \to [0,\infty)$ be defined as:

$$\varphi(t) = \frac{3}{4}t \ , \quad \text{for } t \in [0,\infty),$$

and let $\psi : [0,\infty) \to [0,\infty)$ be defined as:

$$\psi(t) = \frac{1}{4}t \ , \quad \text{for } t \in [0,\infty).$$

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = a$, $\lim_{n\to\infty} gx_n = a$, $\lim_{n\to\infty} F(y_n, x_n) = b$ and $\lim_{n\to\infty} gy_n = b$ Then obviously, $a = 0$ and $b = 0$. Now, for all $n \geq 0$,

$$gx_n = x_n^2 \ , \quad gy_n = y_n^2 \ , \quad F(x_n, y_n) = \begin{cases} \dfrac{x_n^2 - y_n^2}{3}, & \text{if } x_n \geq y_n; \\ 0, & \text{if } x_n < y_n, \end{cases}$$

and

$$F(y_n, x_n) = \begin{cases} \dfrac{y_n^2 - x_n^2}{3}, & \text{if } y_n \geq x_n; \\ 0, & \text{if } y_n < x_n. \end{cases}$$

Then it follows that,

$$\lim_{n\to\infty} S\Big(g\big(F(x_n, y_n)\big), g\big(F(x_n, y_n)\big), F(gx_n, gy_n)\Big) = 0$$

and

$$\lim_{n\to\infty} S\Big(g\big(F(y_n, x_n)\big), g\big(F(y_n, x_n)\big), F(gy_n, gx_n)\Big) = 0.$$

Hence, the mappings F and g are compatible in X. Also, $x_0 = 0$ and $y_0 = c \ (> 0)$ are two points in X such that

$$gx_0 = g(0) = 0 = F(0, c) = F(x_0, y_0) \qquad \text{and}$$

$$gy_0 = g(c) = c^2 \geq \frac{c^2}{3} = F(c, 0) = F(y_0, x_0).$$

We next verify the contraction (3.1). We take $x, y, u, v \in X$ such that $gx \geq gu$ and $gy \leq gv$, that is, $x^2 \geq u^2$ and $y^2 \leq v^2$.

We consider the following cases:

Case 1. $x \geq y, u \geq v$. Then

$$\begin{aligned}
&\varphi\Big(S\big(F(x,y), F(x,y), F(u,v)\big)\Big)\\
&= \frac{3}{4}\Big(S\big(F(x,y), F(x,y), F(u,v)\big)\Big)\\
&= \frac{3}{4}\left(S\left(\frac{x^2-y^2}{3}, \frac{x^2-y^2}{3}, \frac{u^2-v^2}{3}\right)\right)\\
&= \frac{3}{4}\left|\frac{(x^2-y^2)-(u^2-v^2)}{3}\right|\\
&= \frac{3}{4}\left(\frac{|x^2-u^2|-|y^2-v^2|}{3}\right)\\
&= \frac{1}{2}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)\\
&= \frac{3}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right) - \frac{1}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)\\
&= \frac{3}{8}\Big(S(gx,gx,gu)+S(gy,gy,gv)\Big) - \frac{1}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)\\
&= \frac{1}{2}\varphi\Big(S(gx,gx,gu)+S(gy,gy,gv)\Big) - \psi\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right).
\end{aligned}$$

Case 2. $x \geq y, u < v$. Then

$$\begin{aligned}
\varphi\Big(S\big(F(x,y), F(x,y), F(u,v)\big)\Big) &= \frac{3}{4}\Big(S\big(F(x,y), F(x,y), F(u,v)\big)\Big)\\
&= \frac{3}{4}\left(S\left(\frac{x^2-y^2}{3}, \frac{x^2-y^2}{3}, 0\right)\right)\\
&= \frac{3}{4}\frac{|x^2-y^2|}{3} = \frac{3}{4}\left(\frac{|u^2+x^2-y^2-u^2|}{3}\right)\\
&< \frac{3}{4}\left(\frac{|v^2+x^2-y^2-u^2|}{3}\right)\\
&= \frac{3}{4}\left(\frac{|(v^2-y^2)-(u^2-x^2)|}{3}\right)
\end{aligned}$$

$$\leq \frac{3}{4}\left(\frac{|u^2-x^2|+|y^2-v^2|}{3}\right)$$
$$=\frac{1}{2}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)$$
$$=\frac{3}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)-\frac{1}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)$$
$$=\frac{3}{8}\Big(S(gx,gx,gu)+S(gy,gy,gv)\Big)-\frac{1}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)$$
$$=\frac{1}{2}\varphi\Big(S(gx,gx,gu)+S(gy,gy,gv)\Big)-\psi\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right).$$

Case 3. $x<y, u\geq v$. Then,

$$\varphi\Big(S\big(F(x,y),F(x,y),F(u,v)\big)\Big)$$
$$=\frac{3}{4}\left(S\left(0,0,\frac{u^2-v^2}{3}\right)\right)$$
$$=\frac{3}{4}\frac{|u^2-v^2|}{3}=\frac{3}{4}\left(\frac{|u^2+x^2-v^2-x^2|}{3}\right)$$
$$<\frac{3}{4}\left(\frac{|u^2+y^2-v^2-x^2|}{3}\right)$$
$$\leq \frac{3}{4}\left(\frac{|u^2-x^2|+|y^2-v^2|}{3}\right)$$
$$=\frac{1}{2}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)$$
$$=\frac{3}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)-\frac{1}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)$$
$$=\frac{3}{8}\Big(S(gx,gx,gu)+S(gy,gy,gv)\Big)-\frac{1}{4}\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right)$$
$$=\frac{1}{2}\varphi\Big(S(gx,gx,gu)+S(gy,gy,gv)\Big)-\psi\left(\frac{S(gx,gx,gu)+S(gy,gy,gv)}{2}\right).$$

Case 4. $x<y$ and $u<v$ with $x^2\leq u^2$ and $y^2\geq v^2$. Then, $F(x,y)=0$ and $F(u,v)=0$, that is,

$$\varphi\Big(S\big(F(x,y),F(x,y),F(u,v)\big)\Big)=\varphi\big(S(0,0,0)\big)=\varphi(0)=0.$$

Therefore all conditions of Theorem 3.1 are satisfied. Thus the conclusion follows. □

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