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G_μ-Closed Sets and G_m-Closed Sets in GTMS Spaces

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Abstract : The main purpose of this article is to introduce the concepts of G_μ-closed sets and G_m-closed sets, which are a weak froms of closed sets in a generalized topology and minimal structure space. Some of their properties are studied. In particular, the characterizations of μmG-closed sets and μmG-closed sets are obtained using G_μ-closed and G_m-closed. Moreover, the notions of GT_1-GTMS spaces and GT_2-GTMS spaces are introduced.

Keywords : GTMS space; G_μ-closed set; G_m-closed set; GT_1-GTMS space; GT_2-GTMS space.
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1 Introduction

Generalized topology and minimal structure, which were the generalizations of topology, were first studied by Császár [1] and Popa and Noiri [2], respectively. After that, Buadong et al. [3] introduced the concept of generalized topology and minimal structure space (briefly GTMS-space), which was a non-empty set with generalized topology and minimal structure on its. They studied closed sets,

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open stes and weak separation axioms, which were T_1-GTMS and T_2-GTMS, in a GTMS space. Later, Zakari [4] proposed the notions of μmG-closed sets, μmG-closed sets μG-closed sets, and mG-closed sets in GTMS spaces. Moreover, lower separation axioms, which is T_0-GTMS and R_0-GTMS, were studied in a GTMS space. Also, μm-continuity on GTMS spaces was inteoduced by Zakari [5].

In this paper, we introduce the concepts of G_μ-closed sets and G_m-closed sets in a GTMS space and study some properties of such sets. Moreover, we study some separation axioms in the GTMS space using G_μ-open and G_m-open.

2 Preliminaries

In this section, we shall begin by repeating the concepts of minimal structure, see in [2] or [6]. A subcollection m of subsets of a non-empty set X is called a *minimal structure* (briefly, *m-structure*) on X if $\emptyset \in m$ and $X \in m$. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. For a minimal structure m on X and $A \subset X$, $c_m(A) = \bigcap\{F : A \subset F \text{ and } X \setminus F \in m\}$ and $i_m(A) = \bigcup\{U : U \subset A \text{ and } U \in m\}$. Clearly, $i_m(A) \subset A \subset c_m(A)$. If $A, B \subset X$ and m is a minimal structure on X, the following properties hold:

1. $c_m(X \setminus A) = X \setminus i_m(A)$ and $i_m(X \setminus A) = X \setminus c_m(A)$.
2. If $X \setminus A \in m$, then $c_m(A) = A$ and if $A \in m$, then $i_m(A) = A$.
3. If $A \subset B$, then $c_m(A) \subset c_m(B)$ and $i_m(A) \subset i_m(B)$.
4. $c_m(c_m(A)) = c_m(A)$ and $i_m(i_m(A)) = i_m(A)$.

Moreover, if $x \in X$ and $A \subset X$, then $x \in c_m(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing x.

Next, we will recall the notions of generalized topology, see in [1] or [6]. A subfamily μ of subsets of a non-empty set X is called a *generalized topology* (briefly, *GT*) on X if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. A subset A of X is called *μ-open* if $A \in \mu$. The complement of a μ-open set is called a *μ-closed* set. For a GT μ on X and $A \subset X$, $c_\mu(A)$ is the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A, and $i_\mu(A)$ is the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A. Obviously, $i_\mu(A) \subset A \subset c_\mu(A)$. If $A, B \subset X$ and μ is a GT on X, then the following statements hold:

1. $c_\mu(X \setminus A) = X \setminus i_\mu(A)$. and $i_\mu(X \setminus A) = X \setminus c_\mu(A)$.
2. If $X \setminus A \in \mu$, then $c_\mu(A) = A$ and if $A \in \mu$, then $i_\mu(A) = A$.
3. If $A \subset B$, then $c_\mu(A) \subset c_\mu(B)$ and $i_\mu(A) \subset i_\mu(B)$.
4. $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(i_\mu(A)) = i_\mu(A)$.

In [7], $x \in c_\mu(A)$ if and only if $x \in V \in \mu$ implies $V \cap A \neq \emptyset$.

Next, we will recall some concepts of GTMS spaces in [3] and [4]. A nonempty set X equipped with a GT μ and a minimal structure m on its is called a *generalized topology and minimal structure space* or simply a *GTMS space*, is denoted by (X, μ, m). For a GTMS space (X, μ, m), a subset A of X is said to be *μm-closed* (resp. *$m\mu$-closed*) [3] if $c_\mu(c_m(A)) = A$ (resp. $c_m(c_\mu(A)) = A$). The complement of a μm-closed (resp. $m\mu$-closed) set is said to be *μm-open* (resp. *$m\mu$-open*) [3]. Then the following are equivalent:

1. A is μm-closed.
2. $c_\mu(A) = A$ and $c_m(A) = A$.
3. A is $m\mu$-closed.

In a GTMS space (X, μ, m), a subset A of X is said to be *closed* (resp. *s-closed*, *c-closed*) [3] if A is μm-closed (resp. $c_\mu(A) = c_m(A)$, $c_\mu(c_m(A)) = c_m(c_\mu(A))$). The complement of a closed (resp. s-closed, c-closed) set is said to be *open* (resp. *s-open*, *c-open*) [3]. Clearly, A is open in a GTMS space (X, μ, m) if and only if $i_\mu(A) = A$ and $i_m(A) = A$. A subset A of X is said to be *$\mu m G$-closed* (resp. *$m\mu G$-closed*, *μG-closed*, *mG-closed*) [4] in a GTMS space (X, μ, m) if $c_\mu(c_m(A)) \subset U$ (resp. $c_m(c_\mu(A)) \subset U$, $c_m(c_\mu(A)) \subset U$, $c_\mu(c_m(A)) \subset U$) whenever $A \subset U$ and U is open (resp. U is open, U is μ-open, U is m-open). Also, a subset A of X is said to be *G-closed* (resp. *G^*-closed*) [4] in a GTMS space (X, μ, m) if A is $m\mu$G-closed and μmG-closed (resp. μG-closed and mG-closed) .

Now, we recall some separation axioms in a GTMS space.

Definition 2.1 ([4]). A GTMS space (X, μ, m) is called a *T_0-GTMS space* if for any pair of distinct points x and y in X, there exist a subset U which is either μ-open or m-open such that $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$.

Definition 2.2 ([3]). A GTMS space (X, μ, m) is called a *T_1-GTMS space* if for any pair of distinct points x and y in X, there exist a μ-open set U and a m-open set V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Definition 2.3 ([3]). A GTMS space (X, μ, m) is called a *T_2-GTMS space* if for any pair of distinct points x and y in X, there exist a μ-open set U and a m-open set V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 2.4 ([4]). A GTMS space (X, μ, m) is called a *R_0-GTMS space* if $\{x\}$ is G^*-closed set for each $x \in X$.

Theorem 2.5 ([4]). *Let (X, μ, m) be a GTMS space. Then the following are equivalent:*

1. *X is a T_1-GTMS space.*
2. *X is a T_0-GTMS space and R_0-GTMS space.*

3 G_μ-Closed Sets and G_m-Closed Sets

In this section, we shall start by introducing the notion of G_μ-closed sets and investigate some of their properties.

Definition 3.1. A subset A of a GTMS space (X, μ, m) is said to a *G_μ-closed set* if $c_\mu(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a G_μ-closed set is called a *G_μ-open set.*

Proposition 3.2. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is a μmG-closed set, then A is a G_μ-closed set.*

Proof. Assume that A is μmG-closed and let U be open such that $A \subset U$. Then $c_\mu(c_m(A)) \subset U$. From $c_\mu(A) \subset c_\mu(c_m(A))$, we have $c_\mu(A) \subset U$. Therefore, A is G_μ-closed. □

Proposition 3.3. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is a $m\mu$G-closed set, then A is a G_μ-closed set.*

Proof. The proof is similar to the proof of Proposition 3.2. □

Proposition 3.4. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is a μ-closed set, then A is a G_μ-closed set.*

Proof. It follows from the fact that if A is a μ-closed set, then $c_\mu(A) = A$. □

Remark 3.5. *The converse of Proposition 3.2, 3.3 and 3.4 may not be true as the following example.*

Example 3.6. Consider the GTMS space (X, μ, m), where $X = \{1, 2, 3, 4\}$,

$$\mu = \{\emptyset, \{3, 4\}, \{1, 2, 3\}, X\} \text{ and } m = \{\emptyset, \{2, 4\}, \{1, 2, 3\}, X\}.$$

Then $\{2\}$ is G_μ-closed but it is not μmG-closed, $m\mu$G-closed and μ-closed. Moreover, $\{1, 3, 4\}$ is G_μ-open.

Proposition 3.7. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is open and G_μ-closed, then A is μ-closed.*

Proof. Assume that A is open and G_μ-closed. Then $c_\mu(A) \subset A$. This implies $c_\mu(A) = A$. Thus A is μ-closed. □

Proposition 3.8. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is G_μ-closed, then $c_\mu(A) \setminus A$ does not contain any nonempty closed set.*

Proof. Assume that A is G_μ-closed. Suppose to the contrary that $c_\mu(A) \setminus A$ contains a nonempty closed set, say F. Then $F \subset c_\mu(A) \setminus A = c_\mu(A) \cap (X \setminus A)$. Thus $A \subset X \setminus F$. Since A is G_μ-closed and $X \setminus F$ is open, $c_\mu(A) \subset X \setminus F$. This implies $F \subset X \setminus c_\mu(A)$. From $F \subset c_\mu(A), F = \emptyset$ which contradicts with $F \neq \emptyset$. □

Theorem 3.9. *Let (X, μ, m) be a GTMS space with $X \notin \mu$ and $A, B \subset X$. If A is G_μ-closed and $A \subset B$, then B is G_μ-closed.*

Proof. Assume that A is G_μ-closed and $A \subset B$. Suppose B is not G_μ-closed. Thus there exists an open set U such that $B \subset U$ and $c_\mu(B) \not\subset U$. Since A is G_μ-closed, $c_\mu(A) \subset U$, and so $X \setminus U \subset X \setminus c_\mu(A)$. This implies $X = (X \setminus c_\mu(A)) \cup U$ is μ-open which contradicts with $X \notin \mu$. Thus B is G_μ-closed. □

Theorem 3.10. *Let (X, μ, m) be a GTMS space and $A \subset X$. Then A is G_μ-open if and only if $F \subset i_\mu(A)$ whenever F is closed and $F \subset A$.*

Proof. ($\Rightarrow$) Let F be closed such that $F \subset A$. Then $X \setminus F$ is open and $X \setminus A \subset X \setminus F$. By assumption, we obtain that $X \setminus A$ is G_μ-closed, and so $c_\mu(X \setminus A) \subset X \setminus F$. Since $X \setminus i_\mu(A) = c_\mu(X \setminus A)$, $F \subset i_\mu(A)$.

($\Leftarrow$) Let U be open such that $X \setminus A \subset U$. Then $X \setminus U$ is closed and $X \setminus U \subset A$. By assumption, $X \setminus U \subset i_\mu(A)$. Thus $X \setminus i_\mu(A) \subset U$, and so $c_\mu(X \setminus A) \subset U$. Hence $X \setminus A$ is G_μ-closed, and so A is G_μ-open. □

Proposition 3.11. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is G_μ-closed, then $c_\mu(A) \setminus A$ is G_μ-open.*

Proof. Assume that A is G_μ-closed. Suppose to the contrary that $c_\mu(A) \setminus A$ is not G_μ-open. By Theorem 3.10, there exists a closed set F such that $F \subset c_\mu(A) \setminus A$ and $F \not\subset i_\mu(c_\mu(A) \setminus A)$. This implies $\emptyset \neq F \subset c_\mu(A) \setminus A$. It is a contradiction with Proposition 3.8. Hence $c_\mu(A) \setminus A$ is G_μ-open. □

Proposition 3.12. *Let (X, μ, m) be a GTMS space and $A, B \subset X$. If A is G_μ-open and $i_\mu(A) \subset B \subset A$, then B is G_μ-open.*

Proof. It follows from Theorem 3.10 and the fact that if $B \subset A \subset X$, then $i_\mu(B) \subset i_\mu(A)$ and $i_\mu(i_\mu(A)) \subset i_\mu(A)$. □

Next, we will introduce the concept of G_m-closed sets and investigate some of their properties.

Definition 3.13. A subset A of a GTMS space (X, μ, m) is said to a *G_m-closed set* if $c_m(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a G_m-closed set is called a *G_m-open set.*

Proposition 3.14. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is a $\mu m G$-closed set, then A is a G_m-closed set.*

Proof. The proof is similar to the proof of Proposition 3.2. □

Proposition 3.15. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is a $m \mu G$-closed set, then A is a G_m-closed set.*

Proof. The proof is similar to the proof of Proposition 3.2. □

Proposition 3.16. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is a m-closed set, then A is a G_m-closed set.*

Proof. It follows from the fact that if A is a m-closed set, then $c_m(A) = A$. □

Remark 3.17. *The converse of Proposition 3.14, 3.15 and 3.16 may not be true as the following example.*

Example 3.18. In Example 3.6, we see that $\{3\}$ is G_m-closed but it is not μmG-closed, $m\mu$G-closed and m-closed. Moreover, $\{1, 2, 4\}$ is G_m-open.

Proposition 3.19. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is open and G_m-closed, then $c_m(A) = A$.*

Proof. Assume that A is open and G_m-closed. Then $c_m(A) \subset A$. This implies $c_m(A) = A$. □

Proposition 3.20. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is G_m-closed, then $c_m(A) \setminus A$ does not contain any nonempty closed set.*

Proof. The proof is similar to the proof of Proposition 3.8. □

Theorem 3.21. *Let (X, μ, m) be a GTMS space and $A \subset X$. Then A is G_m-open if and only if $F \subset i_m(A)$ whenever F is closed and $F \subset A$.*

Proof. The proof is similar to the proof of Theorem 3.10. □

Theorem 3.22. *Let (X, μ, m) be a GTMS space and $A \subset X$. If A is G_m-closed, then $c_m(A) \setminus A$ is G_m-open.*

Proof. It follows from Theorem 3.21 and Proposition 3.20. □

Proposition 3.23. *Let (X, μ, m) be a GTMS space and $A, B \subset X$. If A is G_m-open and $i_m(A) \subset B \subset A$, then B is G_m-open.*

Proof. It follows from Theorem 3.21 and the fact that if $B \subset A \subset X$, then $i_m(B) \subset i_m(A)$ and $i_m(i_m(A)) \subset i_m(A)$. □

Now, we will give a characterization of $m\mu G$-closed sets and $\mu m G$-closed sets using G_μ-closed sets and G_m-closed sets.

Theorem 3.24. *Let (X, μ, m) be a GTMS space and $A \subset X$. Then A is $m\mu G$-closed if and only if A is G_μ-closed and $c_\mu(A)$ is G_m-closed.*

Proof. ($\Rightarrow$) Assume that A is $m\mu$G-closed. By Proposition 3.3, we have A is G_μ-closed. Next, we shall prove that $c_\mu(A)$ is G_m-closed. Let U be an open set such that $c_\mu(A) \subset U$. Then $A \subset U$. Since A is $m\mu$G-closed, $c_m(c_\mu(A)) \subset U$. Then $c_\mu(A)$ is G_m-closed.

($\Leftarrow$) Assume that A is G_μ-closed and $c_\mu(A)$ is G_m-closed. To show that A is $m\mu$G-closed, let U be an open set such that $A \subset U$. Since A is G_μ-closed, $c_\mu(A) \subset U$. Since $c_\mu(A)$ is G_m-closed, $c_m(c_\mu(A)) \subset U$. Then A is $m\mu$G-closed. □

Theorem 3.25. *Let (X, μ, m) be a GTMS space and $A \subset X$. Then A is μmG-closed if and only if A is G_m-closed and $c_m(A)$ is G_μ-closed.*

Proof. The proof is similar to the proof of Theorem 3.24. □

Finally, we will discuss a relation of G_μ-closed sets and G_m-closed sets under some conditions.

Theorem 3.26. *Let (X, μ, m) be a GTMS space such that $X \notin \mu$ and $A \subset X$. If A is G_μ-closed, then A is G_m-closed.*

Proof. Assume that A is G_μ-closed. Suppose to the contrary that A is not G_m-closed. Then there exists an open set U such that $A \subset U$ and $c_m(A) \not\subset U$. Since A is G_μ-closed, $c_\mu(A) \subset U$. From $c_\mu(A)$ is μ-closed, $X \setminus c_\mu(A)$ is μ-open. Since U is open, U is μ-open. This implies $X = (X \setminus c_\mu(A)) \cup U$ is μ-open. Thus $X \in \mu$ which contradicts with $X \notin \mu$. Thus A is G_m-closed. □

Corollary 3.27. *Let (X, μ, m) be a GTMS space such that $X \notin \mu$ and $A \subset X$. If A is G_μ-open, then A is G_m-open.*

Proof. It follows from Theorem 3.26. □

4 GT_1-GTMS Spaces and GT_2-GTMS Spaces

In this section, we shall introduce the notions of GT_1-GTMS spaces and GT_2-GTMS spaces and investigate some of their characterization. We start by defining the G_μ-closure and G_m-closure of a set in GTMS spaces.

Definition 4.1. Let (X, μ, m) be a GTMS space and $A \subset X$. Defined the *G_μ-closure* and *G_m-closure* of A as follows:

$$c_{G_\mu}(A) = \bigcap\{K \ : \ K \text{ is } G_\mu\text{-closed and } A \subset K\}$$

and

$$c_{G_m}(A) = \bigcap\{K \ : \ K \text{ is } G_m\text{-closed and } A \subset K\},$$

respectively.

Lemma 4.2. *Let (X, μ, m) be a GTMS space and $A \subset X$. Then $x \in c_{G_\mu}(A)$ if and only if $A \cap U \neq \emptyset$ for all G_μ-open U containing x.*

Proof. ($\Rightarrow$) Assume that there exists a G_μ-open set U containing x such that $A \cap U = \emptyset$. Then $X \setminus U$ is G_μ-closed and $A \subset X \setminus U$. Since $x \notin X \setminus U, x \notin c_{G_\mu}(A)$.

($\Leftarrow$) Assume that $x \notin c_{G_\mu}(A)$. Then there exists a G_μ-closed set K such that $A \subset K$ and $x \notin K$. Thus $X \setminus K$ is G_μ-open and $x \in X \setminus K$. Moreover, $A \cap (X \setminus K) = \emptyset$. □

Lemma 4.3. *Let (X, μ, m) be a GTMS space and $A \subset X$. Then $x \in c_{G_m}(A)$ if and only if $A \cap U \neq \emptyset$ for all G_m-open U containing x.*

Proof. The proof is similar to the proof of Lemma 4.2. □

Now, we shall give definition of GT_1-GTMS spaces.

Definition 4.4. A GTMS space (X, μ, m) is said to be *GT_1-GTMS* if for pair of distinct points x and y in X, there exist a G_μ-open set U and a G_m-open set V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Proposition 4.5. *If (X, μ, m) is T_1-GTMS, then (X, μ, m) is GT_1-GTMS.*

Proof. It follows from the fact that every μ-open set is G_μ-open and every m-open set is G_m-open. □

Remark 4.6. *The converse of Proposition 4.5 may not be true as the following example.*

Example 4.7. Consider the GTMS space (X, μ, m), where $X = \{1, 2, 3\}$,

$$\mu = \{\emptyset, \{1, 2\}, \{1, 3\}, X\} \text{ and } m = \{\emptyset, X\}.$$

Then (X, μ, m) is GT_1-GTMS but it is not T_1-GTMS.

Now, we will give a characterization of GT_1-GTMS spaces.

Theorem 4.8. *Let (X, μ, m) be a GTMS space. Then the following are equivalent:*

1. *(X, μ, m) is GT_1-GTMS.*
2. *$c_{G_\mu}(\{x\}) = \{x\}$ and $c_{G_m}(\{x\}) = \{x\}$ for all $x \in X$.*

Proof. (1) ⇒ (2) Assume that X is GT_1-GTMS. We will show that $c_{\mathrm{G}_\mu}(\{x\}) = \{x\}$ and $c_{\mathrm{G}_m}(\{x\}) = \{x\}$ for all $x \in X$. Let $x \in X$. It is clear that $\{x\} \subset c_{\mathrm{G}_\mu}(\{x\})$. Let $y \in X$ be such that $y \neq x$. By assumption, there exists a G_μ-open set U such that $y \in U$ but $x \notin U$. Then $U \cap \{x\} = \emptyset$. Thus $y \notin c_{\mathrm{G}_\mu}(\{x\})$. Hence $c_{\mathrm{G}_\mu}(\{x\}) \subset \{x\}$. Then $c_{\mathrm{G}_\mu}(\{x\}) = \{x\}$. Similarly, we can prove that $c_{\mathrm{G}_m}(\{x\}) = \{x\}$.

(2) ⇒ (1) Assume that $c_{\mathrm{G}_\mu}(\{x\}) = \{x\}$ and $c_{\mathrm{G}_m}(\{x\}) = \{x\}$ for all $x \in X$. To show that X is GT_1-GTMS, let $x, y \in X$ with $x \neq y$. By assumption, $c_{\mathrm{G}_m}(\{x\}) = \{x\}$ and $c_{\mathrm{G}_\mu}(\{y\}) = \{y\}$. Thus $x \notin c_{\mathrm{G}_\mu}(\{y\})$ and $y \notin c_{\mathrm{G}_m}(\{x\})$. Then there exist a G_μ-open set U and G_m-open set V such that $x \in U$, $\{y\} \cap U = \emptyset$ and $y \in V$, $\{x\} \cap V = \emptyset$. Hence X is GT_1-GTMS. □

Theorem 4.9. *Let (X, μ, m) be a GTMS space such that X has at least two elements and $X \notin \mu$. Then X is GT_1-GTMS if and only if $\{a\}$ is G_μ-open in X for all $a \in X$.*

Proof. ($\Rightarrow$) Assume that X is GT_1-GTMS. To show that $\{a\}$ is G_μ-open in X for all $a \in X$, let $a \in X$. Suppose $\{a\}$ is not G_μ-open. Then there exists a closed set F such that $F \subset \{a\}$ and $F \not\subset i_\mu(\{a\})$. This implies $\{a\} = F$ is closed. Thus $X \setminus \{a\}$ is open. Since X has a least two elements, $X \setminus \{a\} \neq \emptyset$, say $b \in X \setminus \{a\}$. By assumption, there exist a G_μ-open set U and a G_m-open set V such that $a \in U, b \notin U$ and $b \in V, a \notin V$. Since U is G_μ-open and $\{a\}$ is closed such that $\{a\} \subset U$, $\{a\} \subset i_\mu(U)$. Then $X = (X \setminus \{a\}) \cup i_\mu(U) \in \mu$ which contradicts with $X \notin \mu$. Hence $\{a\}$ is G_μ-open.

($\Leftarrow$) Assume that $\{a\}$ is GT_μ-open in X for all $a \in X$. To show that X is GT_1-GTMS, let $x, y \in X$ be such that $x \neq y$. By assumption, $\{x\}$ and $\{y\}$ is G_μ-open. Since $X \notin \mu$, by Corollary 3.27, $\{y\}$ is G_m-open. Set $U = \{x\}$ and $V = \{y\}$. Then U is G_μ-open and V is G_m-open. Moreover, $x \in U, y \notin U$ and $y \in V, x \notin V$. Hence X is GT_1-GTMS. □

Next, we will introduce the concepts of GT_0-GTMS spaces and GR_0-GTMS spaces.

Definition 4.10. A GTMS space (X, μ, m) is called *GT_0-GTMS* if for any pair of distinct points x and y in X, there exists a subset U of X which is G_μ-open or G_m-open such that $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$.

Lemma 4.11. *If (X, μ, m) is GT_1-GTMS, then (X, μ, m) is GT_0-GTMS.*

Proof. Assume that (X, μ, m) is GT_1-GTMS. To show that (X, μ, m) is GT_0-GTMS, let $x, y \in X$ with $x \neq y$. Since (X, μ, m) is GT_1-GTMS, there exist a G_μ-open set U and a G_m-open set V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Hence (X, μ, m) is GT_0-GTMS. □

Remark 4.12. *The converse of the previous Lemma 4.11 need not be true as the following example.*

Example 4.13. Consider the GTMS space (X, μ, m), where $X = \{1, 2, 3\}$,

$$\mu = \{\emptyset, \{1, 2\}\} \text{ and } m = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, X\}.$$

Then (X, μ, m) is GT_0-GTMS but it is not GT_1-GTMS.

Definition 4.14. A GTMS space (X, μ, m) is said to be *GR_0-GTMS* if for each $x, y \in X$ if $x \in c_{G_\mu}(c_{G_m}(\{y\}))$, then $y \in c_{G_\mu}(\{x\})$ and if $x \in c_{G_m}(c_{G_\mu}(\{y\}))$, then $y \in c_{G_m}(\{x\})$.

Example 4.15. Consider the GTMS space (X, μ, m), where $X = \{1, 2, 3\}$,

$$\mu = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\} = m.$$

Then (X, μ, m) is GR_0-GTMS.

Lemma 4.16. *If (X, μ, m) is GT_1-GTMS, then (X, μ, m) is GR_0-GTMS.*

Proof. Assume that (X, μ, m) is GT_1-GTMS. To show that (X, μ, m) is GR_0-GTMS, let $x, y \in X$ with $x \in c_{\mathrm{G}_\mu}(c_{\mathrm{G}_m}(\{y\}))$. Since X is GT_1-GTMS, we obtain that $c_{\mathrm{G}_\mu}(c_{\mathrm{G}_m}(\{y\})) = c_{\mathrm{G}_\mu}(\{y\}) = \{y\}$. Then $x \in \{y\}$, and so $x = y$. Hence $y \in c_{\mathrm{G}_\mu}(\{x\})$. Similarly, we can prove that if $x \in c_{\mathrm{G}_m}(c_{\mathrm{G}_\mu}(\{y\}))$, then $y \in c_{\mathrm{G}_m}(\{x\})$. Therefore, (X, μ, m) is GR_0-GTMS. □

Theorem 4.17. *(X, μ, m) is GT_1-GTMS if and only if (X, μ, m) is GT_0-GTMS and GR_0-GTMS.*

Proof. (⇒) It follows from Lemma 4.11 and 4.16.

(⇐) Assume that (X, μ, m) is GT_0-GTMS and GR_0-GTMS. To show that (X, μ, m) is GT_1-GTMS, fix $x \in X$. Let $y \in X$ with $y \neq x$. Since X is GT_0-GTMS, there exists a subset U of X which is G_μ-open or G_m-open such that $x \in U, y \notin U$ or $y \in U, x \notin U$. Without loss of generality, we assume that U is G_μ-open. If $x \in U$ and $y \notin U$, then $x \notin c_{\mathrm{G}_\mu}(\{y\})$. Since X is GR_0-GTMS, $y \notin c_{\mathrm{G}_\mu}(c_{\mathrm{G}_m}(\{x\}))$, and so $y \notin c_{\mathrm{G}_\mu}(\{x\})$ and $y \notin c_{\mathrm{G}_m}(\{x\})$. On the other hand, if $y \in U$ and $x \notin U$, then $y \notin c_{\mathrm{G}_\mu}(\{x\})$. Since X is GR_0-GTMS, $x \notin c_{\mathrm{G}_\mu}(c_{\mathrm{G}_m}(\{y\}))$. Then $x \notin c_{\mathrm{G}_\mu}(\{y\})$, and so $y \notin c_{\mathrm{G}_\mu}(\{x\})$ and $y \notin c_{\mathrm{G}_m}(\{x\})$. This implies $c_{\mathrm{G}_\mu}(\{x\}) = \{x\}$ and $c_{\mathrm{G}_m}(\{x\}) = \{x\}$. Therefore, (X, μ, m) is GT_1-GTMS. □

Now, we will introduce the notion of GT_2-GTMS spaces.

Definition 4.18. A GTMS space (X, μ, m) is is said to be *GT_2-GTMS* if for any pair of distinct points x and y in X, there exist a G_μ-open set U and a G_m-open set V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Proposition 4.19. *If (X, μ, m) is T_2-GTMS, then (X, μ, m) is GT_2-GTMS.*

Proof. Assume that (X, μ, m) is T_2-GTMS. To show that (X, μ, m) is GT_2-GTMS, let $x, y \in X$ be such that $x \neq y$. Since X is T_2-GTMS, there exist μ-open U and m-open V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. By Proposition 3.4 and 3.16, U is G_μ-open and V is G_m-open. Therefore (X, μ, m) is GT_2-GTMS. □

Example 4.20. Consider the GTMS space (X, μ, m), where $X = \{1, 2, 3\}$,

$$\mu = \{\emptyset\} \text{ and } m = \{\emptyset, X\}.$$

Then (X, μ, m) is GT_2-GTMS but it is not T_2-GTMS.

Lemma 4.21. *If (X, μ, m) is GT_2-GTMS, then (X, μ, m) is GT_1-GTMS.*

Proof. Assume that (X, μ, m) is GT_2-GTMS. To show that (X, μ, m) is GT_1-GTMS, let $x, y \in X$ be such that $x \neq y$. Since X is GT_2-GTMS, there exist G_μ-open U and G_m-open V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $x \in U$, $y \notin U$ and $y \in V, x \notin V$. Hence X is GT_1-GTMS. □

Remark 4.22. *The converse of Lemma 4.21 need not be true as the following example.*

Example 4.23. In Example 4.15, we see that (X, μ, m) is GT_1-GTMS but it is not GT_2-GTMS.

Next, we will introduce the concept of GR_1-GTMS spaces.

Definition 4.24. A GTMS space (X, μ, m) is is said to be *GR_1-GTMS* if for all $x, y \in X$ with $x \neq y$ if $c_{\mathrm{G}_\mu}(\{x\}) \neq c_{\mathrm{G}_m}(\{y\})$, then there exist disjoint a G_μ-open set U and a G_m-open set V such that $c_{\mathrm{G}_\mu}(\{x\}) \subset U$ and $c_{\mathrm{G}_m}(\{y\}) \subset V$.

Example 4.25. Consider the GTMS space (X, μ, m), where $X = \{1, 2\}$,

$$\mu = \{\emptyset, \{1\}, \{2\}, X\} = m.$$

Then (X, μ, m) is GR_1-GTMS.

Lemma 4.26. *If (X, μ, m) is GR_1-GTMS, then (X, μ, m) is GR_0-GTMS.*

Proof. Assume that (X, μ, m) is GR_1-GTMS. To show that (X, μ, m) is GR_0-GTMS, let $x, y \in X$ with $y \notin c_{G_\mu}(\{x\})$. It is clear that $x \neq y$ and $c_{\mathrm{G}_\mu}(\{x\}) \neq c_{\mathrm{G}_m}(\{y\})$. By assumption, there exist disjoint a G_μ-open set U and a G_m-open set V such that $c_{\mathrm{G}_\mu}(\{x\}) \subset U$ and $c_{\mathrm{G}_m}(\{y\}) \subset V$. Hence $c_{\mathrm{G}_m}(\{y\}) \cap U = \emptyset$. Thus $x \notin c_{\mathrm{G}_\mu}(c_{\mathrm{G}_m}(\{y\}))$. Similarly, we can prove that if $y \notin c_{\mathrm{G}_m}(\{x\})$, then $x \notin c_{\mathrm{G}_m}(c_{\mathrm{G}_\mu}(\{y\}))$. Therefore, (X, μ, m) is GR_0-GTMS. □

Remark 4.27. *The converse of Lemma 4.26 may not be true as the following example.*

Example 4.28. In Example 4.15, we see that (X, μ, m) is GR_0-GTMS but it is not GR_1-GTMS.

Lemma 4.29. *If (X, μ, m) is GT_2-GTMS, then (X, μ, m) is GR_1-GTMS.*

Proof. Assume that (X, μ, m) is GT_2-GTMS. To show that (X, μ, m) is GR_1-GTMS, let $x, y \in X$ with $x \neq y$ and $c_{\mathrm{G}_\mu}(\{x\}) \neq c_{\mathrm{G}_m}(\{y\})$. By assumption and Lemma 4.21, (X, μ, m) is GT_1-GTMS. Then $c_{\mathrm{G}_\mu}(\{x\}) = \{x\}$ and $c_{\mathrm{G}_m}(\{y\}) = \{y\}$. Since (X, μ, m) is GT_2-GTMS, there exist disjoint G_μ-open U and G_m-open V such that $c_{\mathrm{G}_\mu}(\{x\}) = \{x\} \subset U$ and $c_{\mathrm{G}_m}(\{y\}) = \{y\} \subset V$. Therefore, (X, μ, m) is GR_1-GTMS. □

Theorem 4.30. *(X, μ, m) is GT_2-GTMS if and only if (X, μ, m) is GT_0-GTMS and GR_1-GTMS.*

Proof. ($\Rightarrow$) It follows from Lemma 4.21, 4.11 and 4.29.

($\Leftarrow$) Assume that (X, μ, m) is GT_0-GTMS and GR_1-GTMS. By Lemma 4.26 and Theorem 4.17, (X, μ, m) is GT_1-GTMS. To show that (X, μ, m) is GT_2-GTMS, let $x, y \in X$ with $x \neq y$. Since X is GT_1-GTMS, thus $c_{G\mu}(\{x\}) = \{x\} \neq \{y\} = c_{G_m}(\{y\})$. Since X is GR_1-GTMS, there exist disjoint a G_μ-open set U and a G_m-open set V such that $c_{\mathrm{G}_\mu}(\{x\}) = \{x\} \subset U$ and $c_{\mathrm{G}_m}(\{y\}) = \{y\} \subset V$. Therefore, (X, μ, m) is GT_2-GTMS. □

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