Common Fixed Points of a Finite Family of I-Asymptotically Nonexpansive Mappings by S-Iteration Process in Banach Spaces

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Abstract : In this paper, we consider S iteration process to approximate the common fixed points of a finite family of I-asymptotically nonexpansive mappings in Banach spaces. We also study the different convergence criteria of S iteration process under some suitable conditions. It presents some new results in this paper.

Keywords : I-asymptotically nonexpansive mapping, Kadec-Klee property, Fréchet differentiable norm; Opial's condition; condition (B); common fixed point.
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1 Introduction

Let K be a nonempty subset of a real normed linear space X and let $T : K \to K$ be a mapping. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. Throughout this paper, we always assume that $F(T) \neq \emptyset$. $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in K$. The mapping $T : K \to K$ is said to be asymptotically nonexpansive [1]

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if there exists a sequence $\{u_n\} \subset [0, \infty)$, $\lim_{n \to \infty} u_n = 0$ such that

$$\|T^n x - T^n y\| \le (1 + u_n) \|x - y\|$$

for all $x, y \in K$ and $n \ge 1$. The mapping $T : K \to K$ is said to be uniformly Lipschitz with a Lipschitzian constant $L > 0$ if $\|T^n x - T^n y\| \le L \|x - y\|$ holds for all $x, y \in K$ and $n \ge 1$.

Note that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian with $L = \sup \{1 + u_n : n \ge 1\}$.

Recently, Temir [2, 3] introduced the following definitions which generalize notion of asymptotically nonexpansive mapping:

Definition 1.1. Let $T, I : K \to K$ be two mappings. T is said to be *I-asymptotically nonexpansive* [2, 3] if there exists a sequence $\{v_n\} \subset [0, \infty)$ with $\lim_{n \to \infty} v_n = 0$ such that

$$\|T^n x - T^n y\| \le (1 + v_n) \|I^n x - I^n y\| \tag{1.1}$$

for all $x, y \in K$ and $n \ge 1$. T is said to be *I-uniformly Lipschitz* if there exists $\Gamma > 0$ such that

$$\|T^n x - T^n y\| \le \Gamma \|I^n x - I^n y\|, \ x, y \in K \text{ and } n \ge 1. \tag{1.2}$$

It is obvious that, an *I*-asymptotically nonexpansive mapping is *I*-uniformly Lipschitz with Lipschitz constant $\Gamma = \sup \{1 + v_n : n \ge 1\}$.

We know that Picard and Mann iteration processes for a mapping $T : K \to K$ are defined as:
$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = T x_n, \ n \ge 1 \end{cases} \tag{1.3}$$

and

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \end{cases} \quad n \ge 1, \tag{1.4}$$

where $\{\alpha_n\}$ is in $(0, 1)$.

In 2007, Agarwal et al.[4] introduced the following iteration scheme:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n) T^n x_n + \alpha_n T^n y_n, \quad n \ge 1, \\ y_n = (1 - \beta_n) x_n + \beta_n T^n x_n, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. They showed that this scheme converges at a rate same as that of Picard iteration. Khan and Kim [5] continued to work in this direction and proved that this process also converges faster than Mann and Ishikawa iteration process.

Temir [2] introduced an iteration process for a finite family of I-asymptotically nonexpansive mappings as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n I_{i(n)}^{k(n)} y_n, & n \geq 1, \\ y_n = (1 - \beta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n, \end{cases} \qquad (1.6)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$ and $n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$.

We introduce the following iteration scheme to compute the common fixed points of a finite family of asymptotically I-nonexpansive mappings.

Let K be a nonempty subset of a Banach space X. Let $\{T_i\}_{i=1}^N$ be finite family of I_i-asymptotically nonexpansive self-mappings and $\{I_i\}_{i=1}^N$ be finite family of asymptotically nonexpansive self-mappings of K. Let $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. Then the sequence $\{x_n\}$ is generated as follows:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)I_i^n x_n + \alpha_n T_i^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n I_i^n x_n, \end{cases} \qquad n \geq 1, \qquad (1.7)$$

where $n = (k-1)N + i$, $i = i(n) \in I_0 := \{1, 2, \ldots, N\}$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Thus, (1.7) can be expressed in the following form:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)I_{i(n)}^{k(n)} x_n + \alpha_n T_{i(n)}^{k(n)} y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n I_{i(n)}^{k(n)} x_n, & n \geq 1. \end{cases}$$

Our purpose in the rest of the paper is to use the scheme (1.7) to prove some weak and strong convergence results for approximating common fixed points of a finite family of I-asymptotically nonexpansive mappings.

2   Preliminaries

Let X be a Banach space with its dimension greater than or equal to 2. The modulus of X is the function $\delta_X(\varepsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\| = 1, \ \|y\| = 1, \ \varepsilon = \|x - y\| \right\}.$$

A Banach space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

A Banach space X is said to have a Fréchet differentiable norm [6] if for all $x \in S_X = \{x \in X : \|x\| = 1\}$,

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in S_X$.

A mapping T with domain $D(T)$ and range $R(T)$ in X is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \to x^* \in D(T)$ and $Tx_n \rightharpoonup p$ then $Tx^* = p$.

A mapping $T : K \to K$ is said to be semicompact if, for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some x^* in K.

A mapping $T : K \to K$ is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{Tx_{n_j}\}$ converges to some element of the range T.

Lemma 2.1 ([7]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \ \ n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 2.2 ([8]). *Suppose that X is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \ge 1$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of X such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \ \ \limsup_{n\to\infty} \|y_n\| \le r \ \ and \ \ \lim_{n\to\infty} \|t_n x_n + (1 - t_n)\, y_n\| = r$$

hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3 ([9]). *Let X be a real uniformly convex Banach space, K a nonempty closed subset of X, and let $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$, then $(E - T)$ is demiclosed at zero, where E is an identity mapping.*

Lemma 2.4 ([10]). *Let X be a uniformly convex Banach space and K a convex subset of X. Then there exists a strictly increasing continuous convex function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ such that for each $S : K \to K$ with Lipschitz constant L,*

$$\|\alpha Sx + (1 - \alpha)Sy - S\left[\alpha x + (1 - \alpha)y\right]\| \le L\gamma^{-1}\left(\|x - y\| + \frac{1}{L}\|Sx - Sy\|\right)$$

for all $x, y \in K$ and $0 < \alpha < 1$.

Let $\omega_w \{x_n\} = \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak limit set of $\{x_n\}$.

A Banach space X is said to have the Kadec–Klee property if, for every sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $\|x_n - x\| \to 0$. Every locally uniformly convex space has the Kadec–Klee property. In particular, L_p spaces, $1 < p < \infty$ have this property.

Lemma 2.5 ([10]). *Let X be a real reflexive Banach space such that its dual X has Kadec–Klee property. Let $\{x_n\}$ be a bounded sequence in X and $q_1, q_2 \in \omega_w\{x_n\}$. Suppose $\lim_{n\to\infty}\|\alpha x_n + (1-\alpha)q_1 - q_2\| = 0$ exists for all $\alpha \in [0,1]$. Then $q_1 = q_2$.*

The mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy Condition (A) [11] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $n \geq 1$. Senter and Dotson [11] pointed out that every continuous and semi-compact mapping must satisfy Condition (A). Khan and Fukharuddin [12] modified the Condition (A) for two mappings as follows: Two mappings $T_1, T_2 : K \to K$ are said to satisfy Condition (A') [12] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$\text{either } \max\|x - T_1x\| \geq f(d(x,F)) \text{ or } \max\|x - T_2x\| \geq f(d(x,F))$$

for all $x \in K$, where $d(x, F) = \inf\{\|x - p\| : p \in F := F(T_1) \cap F(T_2)\}$.

Let $\{T_i : i \in I_0\}$ and $\{I_i : i \in I_0\}$ be two family of mappings of K with nonempty fixed points set F. These families are said to satisfy Condition (B) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$\text{either } \max_{i \in I_0}\|x - T_ix\| \geq f(d(x,F)) \text{ or } \max_{i \in I_0}\|x - I_ix\| \geq f(d(x,F)).$$

3 Main Results

Lemma 3.1. *Let X be a real Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in I_0\}$ be N I_i-asymptotically nonexpansive mappings with sequences $\left\{l_n^{(i)}\right\} \subset [0,\infty)$ and $\{I_i : i \in I_0\}$ be N asymptotically nonexpansive self-mappings of K with sequences $\left\{k_n^{(i)}\right\} \subset [0,\infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. Suppose that for any given $x_1 \in K$, the sequence $\{x_n\}$ is generated by (1.7) satisfying the conditions:*

1) $\sum_{n=1}^{\infty} l_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$, where $l_n = \max\{l_n^{(i)} : i \in I_0\}$, $k_n = \max\{k_n^{(i)} : i \in I_0\}$;

2) there exist constants $\tau_1, \tau_2 \in (0,1)$ such that $\tau_1 \leq (1 - \alpha_n), (1 - \beta_n) \leq \tau_2$, $\forall n \geq 1$.

Then

i) $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F$;

ii) $\lim_{n\to\infty} d(x_n, F)$ exists for $p \in F$, where $d(x_n, F) = \inf_{p\in F}\|x_n - p\|$.

Proof. Let $p \in F$. From (1.7), we have

$$
\begin{aligned}
\|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n I_i^n x_n - p\| \leq (1 - \beta_n)\|x_n - p\| + \beta_n \|I_i^n x_n - p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n(1 + k_n)\|x_n - p\| \\
&\leq \|x_n - p\| + \beta_n k_n \|x_n - p\| \\
&\leq (1 + \beta_n k_n)\|x_n - p\|
\end{aligned}
\tag{3.1}
$$

By (1.7) and (3.1), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n)I_i^n x_n + \alpha_n T_i^n y_n - p\| \\
&\leq (1 - \alpha_n)\|I_i^n x_n - p\| + \alpha_n \|T_i^n y_n - p\| \\
&\leq (1 - \alpha_n)(1 + k_n)\|x_n - p\| + \alpha_n(1 + l_n)\|I_i^n y_n - p\| \\
&\leq (1 - \alpha_n)(1 + k_n)\|x_n - p\| + \alpha_n(1 + l_n)(1 + k_n)\|y_n - p\| \\
&\leq (1 + k_n)[(1 - \alpha_n)\|x_n - p\| + \alpha_n(1 + l_n)(1 + \beta_n k_n)\|x_n - p\|] \\
&\leq (1 + k_n)[(1 + \alpha_n l_n + \alpha_n \beta_n k_n + \alpha_n \beta_n k_n l_n)\|x_n - p\|] \\
&\leq (1 + \delta_n)\|x_n - p\|,
\end{aligned}
\tag{3.2}
$$

where $\delta_n = \left\{ k_n + \alpha_n \beta_n k_n + \alpha_n l_n + (\alpha_n \beta_n + \alpha_n) k_n l_n + \alpha_n \beta_n k_n^2 + \alpha_n \beta_n k_n^2 l_n \right\}$.

Since $\sum_{n=1}^{\infty} l_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$, we obtain $\sum_{n=1}^{\infty} \delta_n < \infty$. Thus by Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists. Taking the infimum over all $p \in F$ in the inequalities (3.2), we get

$$
d(x_{n+1}, F) \leq (1 + \delta_n) d(x_n, F)
\tag{3.3}
$$

Now applying Lemma 2.1 to (3.3) we get the existence of the limit $\lim_{n\to\infty} d(x_n, F)$. This completes the proof of Lemma. $\square$

We first prove a strong convergence theorem of the sequence $\{x_n\}$ which defined by (1.7) in a real Banach space.

Theorem 3.2. *Let X be a real Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in I_0\}$ be N I_i-asymptotically nonexpansive mappings with sequences $\left\{ l_n^{(i)} \right\} \subset [0, \infty)$ and $\{I_i : i \in I_0\}$ be N asymptotically nonexpansive self-mappings of K with sequences $\left\{ k_n^{(i)} \right\} \subset [0, \infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. Suppose that for any given $x_1 \in K$, the sequence $\{x_n\}$ is generated by (1.7) satisfying the conditions:*

1) $\sum_{n=1}^{\infty} l_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$, where $l_n = \max\{l_n^{(i)} : i \in I_0\}$, $k_n = \max\{k_n^{(i)} : i \in I_0\}$;

2) there exist constants $\tau_1, \tau_2 \in (0, 1)$ such that $\tau_1 \leq (1 - \alpha_n), (1 - \beta_n) \leq \tau_2$, $\forall n \geq 1$.

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in I_0\}$ and $\{I_i : i \in I_0\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity is obvious.

Now we prove the sufficiency of Theorem. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, F)$ exists. It follows from $\liminf_{n\to\infty} d(x_n, F) = 0$ that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. In fact, since $\sum\limits_{n=1}^{\infty} \delta_n < \infty$, $1 + t \leq \exp\{t\}$ for all $t > 0$, and (3.3), therefore we have

$$\|x_{n+1} - p\| \leq \exp\{\delta_n\} \|x_n - p\|. \tag{3.4}$$

Hence, for any positive integers n, m, from (3.4) it follows that

$$
\begin{aligned}
\|x_{n+m} - p\| &\leq \exp\{\delta_{n+m-1}\} \|x_{n+m-1} - p\| \\
&\leq \exp\{\delta_{n+m-1}\} [\exp\{\delta_{n+m-2}\} \|x_{n+m-2} - p\|] \\
&= \exp\{\delta_{n+m-1} + \delta_{n+m-2}\} \|x_{n+m-2} - p\| \leq \cdots \\
&\leq \exp\left\{ \sum_{j=n}^{n+m-1} \delta_j \right\} \|x_n - p\| \leq M \|x_n - p\|,
\end{aligned}
$$

where $M = \sum\limits_{j=1}^{\infty} \delta_j < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that $d(x_n, F) < \frac{\varepsilon}{1+M}$, $\forall n \geq n_0$. Therefore there exists $p_1 \in F$ such that $d(x_n, p_1) < \frac{\varepsilon}{1+M}$, $\forall n \geq n_0$.

Consequently, for any $n \geq n_0$ and for all $m \geq 1$ we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \leq (1+M) \|x_n - p\| \leq (1+M) \frac{\varepsilon}{1+M} = \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ is convergent. Assume that $\lim_{n\to\infty} x_n = x^*$. Then $x^* \in K$, because K is closed subset of X and $\lim_{n\to\infty} d(x_n, F) = 0$ implies that $d(x^*, F) = 0$. F is closed, thus $x_n \to x^*$. This completes the proof. $\square$

Lemma 3.3. *Let X be a real uniformly convex Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in I_0\}$ be N I_i-asymptotically nonexpansive mappings with sequences $\left\{l_n^{(i)}\right\} \subset [0, \infty)$ and $\{I_i : i \in I_0\}$ be N asymptotically nonexpansive self-mappings of K with sequences $\left\{k_n^{(i)}\right\} \subset [0, \infty)$ such that $F = \bigcap\limits_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. Suppose that for any given $x_1 \in K$, the sequence $\{x_n\}$ is generated by (1.7) satisfying the conditions:*

1) $\sum\limits_{n=1}^{\infty} l_n < \infty$ and $\sum\limits_{n=1}^{\infty} k_n < \infty$, where $l_n = \max\{l_n^{(i)} : i \in I_0\}$, $k_n = \max\{k_n^{(i)} : i \in I_0\}$;

2) there exist constants $\tau_1, \tau_2 \in (0, 1)$ such that $\tau_1 \leq (1 - \alpha_n), (1 - \beta_n) \leq \tau_2$, $\forall n \geq 1$.

Then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - I_i x_n\| = 0$ for all $i \in I_0$.

Proof. For any $p \in F$, by Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. Assume $\lim_{n\to\infty} \|x_n - p\| = d$ for some $d \geq 0$. By (3.1), we have

$$\|y_n - p\| \leq (1 + \beta_n k_n) \|x_n - p\|.$$

Taking limsup on both sides in the above inequality, we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq d. \tag{3.5}$$

Also

$$\|I_i^n x_n - p\| \leq (1 + k_n) \|x_n - p\|$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty} \|I_i^n x_n - p\| \leq d. \tag{3.6}$$

Next,

$$\|T_i^n y_n - p\| \leq (1 + l_n) \|I_i^n y_n - p\| \leq (1 + l_n)(1 + k_n) \|y_n - p\|$$

gives by (3.5) that

$$\limsup_{n\to\infty} \|T_i^n y_n - p\| \leq d. \tag{3.7}$$

Moreover, we have

$$d = \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|(1 - \alpha_n)(I_i^n x_n - p) + \alpha_n (T_i^n y_n - p)\|.$$

This together with (3.6), (3.7) and Lemma 2.2 imply that

$$\lim_{n\to\infty} \|I_i^n x_n - T_i^n y_n\| = 0. \tag{3.8}$$

Now

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n)I_i^n x_n + \alpha_n T_i^n y_n - p\| \\
&= \|(I_i^n x_n - p) + \alpha_n (T_i^n y_n - I_i^n x_n)\| \\
&\leq \|I_i^n x_n - p\| + \alpha_n \|T_i^n y_n - I_i^n x_n\|
\end{aligned}
$$

yields that

$$d \leq \liminf_{n\to\infty} \|I_i^n x_n - p\|. \tag{3.9}$$

Combining (3.6) and (3.9), we obtain

$$\lim_{n\to\infty} \|I_i^n x_n - p\| = d.$$

Observe that

$$
\begin{aligned}
\|I_i^n x_n - p\| &\leq \|I_i^n x_n - T_i^n y_n\| + \|T_i^n y_n - p\| \\
&\leq \|I_i^n x_n - T_i^n y_n\| + (1 + l_n)\|I_i^n y_n - p\| \\
&\leq \|I_i^n x_n - T_i^n y_n\| + (1 + l_n)(1 + k_n)\|y_n - p\|.
\end{aligned}
$$

Hence, we have

$$d \leq \lim_{n \to \infty} \inf \|y_n - p\|. \tag{3.10}$$

By (3.5) and (3.10), we obtain

$$\lim_{n \to \infty} \|y_n - p\| = d. \tag{3.11}$$

On the other hand, $\|I_i^n x_n - p\| \leq (1 + k_n) \|x_n - p\|$ gives that

$$\lim_{n \to \infty} \sup \|I_i^n x_n - p\| \leq d.$$

Thus $d = \lim_{n \to \infty} \|y_n - p\| = \lim_{n \to \infty} \|(1 - \beta_n)(x_n - p) + \beta_n(I_i^n x_n - p)\|$ gives by Lemma 2.2 that

$$\lim_{n \to \infty} \|I_i^n x_n - x_n\| = 0. \tag{3.12}$$

Next,

$$\|y_n - x_n\| = \beta_n \|x_n - I_i^n x_n\|$$

gives by (3.12) that

$$\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{3.13}$$

From (3.8) and (3.12), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1 - \alpha_n)I_i^n x_n + \alpha_n T_i^n y_n - x_n\| \\
&\leq \|I_i^n x_n - x_n\| + \alpha_n \|I_i^n x_n - T_i^n y_n\| \\
&\to 0, \ n \to \infty.
\end{aligned} \tag{3.14}
$$

Thus from $\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - y_n\|$, we get

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. \tag{3.15}$$

Furthermore, we have

$$\|x_n - T_i^n y_n\| \leq \|x_n - I_i^n x_n\| + \|I_i^n x_n - T_i^n y_n\| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

By the triangle inequality,

$$\|x_{n+1} - T_i^n y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - T_i^n y_n\|.$$

Therefore, by (3.14) and (3.16), we obtain

$$\lim_{n \to \infty} \|x_{n+1} - T_i^n y_n\| = 0. \tag{3.17}$$

We shall now make use of the fact that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian. Then

$$
\begin{aligned}
\|x_{n+1} - I_i x_{n+1}\| &\leq \|x_{n+1} - I_i^{n+1} x_{n+1}\| + \|I_i^{n+1} x_{n+1} - I_i^{n+1} x_n\| \\
&\quad + \|I_i^{n+1} x_n - I_i x_{n+1}\| \\
&\leq \|x_{n+1} - I_i^{n+1} x_{n+1}\| + L \|x_{n+1} - x_n\| + L \|I_i^n x_n - x_{n+1}\| \\
&= \|x_{n+1} - I_i^{n+1} x_{n+1}\| + L \|x_{n+1} - x_n\| \\
&\quad + L\alpha_n \|I_i^n x_n - T_i^n y_n\|,
\end{aligned}
$$

yields

$$\lim_{n\to\infty} \|x_n - I_i x_n\| = 0. \tag{3.18}$$

From (3.13), (3.14) and (3.17) we have

$$
\begin{aligned}
\|x_n - T_i^n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^n y_n\| + \|T_i^n y_n - T_i^n x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^n y_n\| + (1 + l_n)(1 + k_n)\|y_n - x_n\|
\end{aligned}
$$

and so

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0. \tag{3.19}$$

Finally, from

$$
\begin{aligned}
\|x_{n+1} - T_i x_{n+1}\| &\leq \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + \left\|T_i^{n+1} x_{n+1} - T_i x_{n+1}\right\| \\
&\leq \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + \Gamma \left\|I_i^{n+1} x_{n+1} - I_i x_{n+1}\right\| \\
&\leq \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + \Gamma L \left\|I_i^n x_{n+1} - x_{n+1}\right\| \\
&\leq \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| \\
&\quad + \Gamma L \left(\|I_i^n x_{n+1} - I_i^n x_n\| + \|I_i^n x_n - x_{n+1}\|\right) \\
&\leq \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + \Gamma L^2 \|x_{n+1} - x_n\| \\
&\quad + \Gamma L \alpha_n \|I_i^n x_n - T_i^n y_n\|,
\end{aligned}
$$

with (3.8), (3.14) and (3.19), we obtain

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0. \tag{3.20}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

Applying Theorem 3.2, we obtain a strong convergence of the scheme (1.7) under the Condition (B) as follows.

Theorem 3.4. *Let X be a real uniformly convex Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in I_0\}$ be N I_i-asymptotically nonexpansive mappings with sequences $\left\{l_n^{(i)}\right\} \subset [0, \infty)$ and $\{I_i : i \in I_0\}$ be N asymptotically nonexpansive self-mappings of K with sequences $\left\{k_n^{(i)}\right\} \subset [0, \infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. Suppose that for any given $x_1 \in K$, the sequence $\{x_n\}$ is generated by (1.7) satisfying the conditions:*

1) $\sum_{n=1}^{\infty} l_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$, where $l_n = \max\{l_n^{(i)} : i \in I_0\}$, $k_n = \max\{k_n^{(i)} : i \in I_0\}$;

2) there exist constants $\tau_1, \tau_2 \in (0,1)$ such that $\tau_1 \leq (1 - \alpha_n), (1 - \beta_n) \leq \tau_2$, $\forall n \geq 1$.

If $\{T_i : i \in I_0\}$ and $\{I_i : i \in I_0\}$ satisfy Condition (B) then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in I_0\}$ and $\{I_i : i \in I_0\}$.

Proof. From Lemma 3.3, we know that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - I_i x_n\| = 0$$

for all $i \in I_0$.

Since $\{T_i : i \in I_0\}$ and $\{I_i : i \in I_0\}$ satisfy Condition (B), we get that

$$\text{either } f\left(d(x_n, F)\right) \le \max_{i \in I_0} \|x_n - T_i x_n\| \text{ or } f\left(d(x_n, F)\right) \le \max_{i \in I_0} \|x_n - I_i x_n\|.$$

In both cases, we get $\lim_{n\to\infty} f\left(d(x_n, F)\right) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$, therefore we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Now all the conditions of Theorem 3.2 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of F. This completes the proof. $\square$

Next, we prove a weak convergence of the iteration (1.7) in a uniformly convex Banach space X whose dual X^* has the Kadec–Klee property. Most weak convergence theorems are proved in a uniformly convex Banach space and the presence of Opial's condition or the Fréchet differentiability of the norm.

In [13], it is point out that there exist uniformly convex Banach spaces which have neither a Fréchet differentiable norm nor the Opial property but their duals do have the Kadec–Klee property. And the duals of reflexive Banach spaces with Fréchet differentiable norms or the Opial property have the Kadec–Klee property.

The following lemma is our main tool for proving the weak convergence theorem.

Lemma 3.5. *Let X be a real uniformly convex Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in I_0\}$ be N I_i-asymptotically nonexpansive mappings with sequences $\left\{l_n^{(i)}\right\} \subset [0, \infty)$ and $\{I_i : i \in I_0\}$ be N asymptotically nonexpansive self-mappings of K with sequences $\left\{k_n^{(i)}\right\} \subset [0, \infty)$ such that $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \ne \emptyset$. Suppose that for any given $x_1 \in K$, the sequence $\{x_n\}$ is generated by (1.7) satisfying the conditions:*

1) $\sum_{n=1}^{\infty} l_n < \infty$ *and* $\sum_{n=1}^{\infty} k_n < \infty$, *where* $l_n = \max\{l_n^{(i)} : i \in I_0\}$, $k_n = \max\{k_n^{(i)} : i \in I_0\}$;

2) *there exist constants* $\tau_1, \tau_2 \in (0, 1)$ *such that* $\tau_1 \le (1 - \alpha_n), (1 - \beta_n) \le \tau_2$, $\forall n \ge 1$.

Then $\lim_{n\to\infty} \|t x_n + (1 - t) p_1 - p_2\|$ *exists for all* $t \in [0, 1]$ *and* $p_1, p_2 \in F$.

Proof. Setting $a_n(t) = \|t x_n + (1 - t) p_1 - p_2\|$. Then

$$\lim_{n\to\infty} a_n(0) = \lim_{n\to\infty} \|p_1 - p_2\|,$$

and by Lemma 3.1, $\lim_{n\to\infty} a_n(t) = \lim_{n\to\infty} \|x_n - p_2\|$ exists. Let $t \in (0,1)$. Define the mapping $A_n, B_n : K \to K$ by

$$\begin{cases} A_n x = (1 - \beta_n)x + \beta_n I_i^n x, \\ B_n x = (1 - \alpha_n)I_i^n x + \alpha_n T_i^n A_n x, \end{cases} \quad n \geq 1, \tag{3.21}$$

for all $x \in K$. Then $B_n x_n = x_{n+1}$, $B_n p = p$ for all $p \in F$. We have

$$\begin{aligned} \|A_n x - A_n y\| &\leq \|((1-\beta_n)x + \beta_n I_i^n x) - ((1-\beta_n)y + \beta_n I_i^n y)\| \\ &\leq \|(1-\beta_n)(x-y) + \beta_n (I_i^n x - I_i^n y)\| \\ &\leq (1-\beta_n)\|x-y\| + \beta_n (1+k_n)\|x-y\| \\ &\leq (1+k_n)\|x-y\|. \end{aligned} \tag{3.22}$$

for all $x, y \in K$. By (3.22), we obtain

$$\begin{aligned} \|B_n x - B_n y\| &\leq \|((1-\alpha_n)I_i^n x + \alpha_n T_i^n A_n x) - ((1-\alpha_n)I_i^n y + \alpha_n T_i^n A_n y)\| \\ &= \|(1-\alpha_n)(I_i^n x - I_i^n y) + \alpha_n (T_i^n A_n x - T_i^n A_n y)\| \\ &\leq (1-\alpha_n)\|I_i^n x - I_i^n y\| + \alpha_n \|T_i^n A_n x - T_i^n A_n y\| \\ &\leq (1-\alpha_n)(1+k_n)\|x-y\| + \alpha_n (1+l_n)(1+k_n)\|A_n x - A_n y\| \\ &\leq (1-\alpha_n)(1+k_n)\|x-y\| + \alpha_n (1+l_n)(1+k_n)(1+k_n)\|x-y\| \\ &= (1+k_n)[1+\alpha_n k_n + \alpha_n l_n + \alpha_n l_n k_n]\|x-y\| \\ &\leq (1+k_n)[1 + k_n + l_n + l_n k_n]\|x-y\| \\ &= (1+k_n)^2 (1+l_n)\|x-y\|. \end{aligned} \tag{3.23}$$

For the sake of simplicity, set $h_n = max\{k_n, l_n\}$, then obviously $\lim_{n\to\infty} h_n = 0$. Thus we have

$$\|B_n x - B_n y\| \leq (1+h_n)^3 \|x-y\|.$$

This implies that $B_n : K \to K$ is Lipschitz with the Lipschitz constant $(1+h_n)^3$ and $x_{n+1} = B_n x_n$. Setting

$$H_n = \prod_{j=n}^{\infty} (1+h_j)^3, \quad R_{n,m} = B_{n+m-1}B_{n+m-2}\cdots B_n, \ n, m \geq 1,$$

then $H_n \to 1$ (as $n \to \infty$) and $R_{n,m} : K \to K$ is Lipschitz with the Lipschitz constant H_n. Moreover, $R_{n,m} x_n = x_{n+m}$ and $R_{n,m} p = p$ for each $p \in F$.

Letting

$$b_{n,m} = \|t R_{n,m} x_n + (1-t) R_{n,m} p_1 - R_{n,m}(t x_n + (1-t)p_1)\| \tag{3.24}$$

From (3.24) and Lemma 2.4, we have

$$
\begin{aligned}
b_{n,m} &\leq \prod_{j=n}^{n+m-1}(1+h_j)^3\,\gamma^{-1}\Bigg(\|x_n-p_1\| \\
&\quad -\left(\prod_{j=n}^{n+m-1}(1+h_j)^3\right)^{-1}\|R_{n,m}x_n-R_{n,m}p_1\|\Bigg) \\
&\leq \prod_{j=n}^{\infty}(1+h_j)^3\,\gamma^{-1}\left(\|x_n-p_1\|-\left(\prod_{j=n}^{\infty}(1+h_j)^3\right)^{-1}\|R_{n,m}x_n-R_{n,m}p_1\|\right) \\
&=H_n\gamma^{-1}\left(\|x_n-p_1\|-H_n^{-1}\|x_{n+m}-p_1\|\right).
\end{aligned}
$$

It follows from Lemma 3.1 and $\lim_{n\to\infty}H_n=1$ that $\lim_{n\to\infty}b_{n,m}=0$ uniformly for all m. Observe that

$$
\begin{aligned}
a_{n+m}(t) &\leq \|tx_{n+m}+(1-t)\,p_1-p_2 \\
&\quad +(R_{n,m}(tx_n+(1-t)\,p_1)-tR_{n,m}x_n-(1-t)\,R_{n,m}p_1)\| \\
&\quad +\|-(R_{n,m}(tx_n+(1-t)\,p_1)-tR_{n,m}x_n-(1-t)\,R_{n,m}p_1)\| \\
&= \|R_{n,m}(tx_n+(1-t)\,p_1)-p_2\|+b_{n,m} \\
&= \|R_{n,m}(tx_n+(1-t)\,p_1)-R_{n,m}p_2\|+b_{n,m} \\
&\leq \prod_{j=n}^{n+m-1}(1+h_j)^3\,\|tx_n+(1-t)\,p_1-p_2\|+b_{n,m} \\
&\leq \prod_{j=n}^{\infty}(1+h_j)^3\,\|tx_n+(1-t)\,p_1-p_2\|+b_{n,m}=H_na_n(t)+b_{n,m}.
\end{aligned}
$$

Consequently, $\limsup_{n\to\infty}a_n(t)\leq\liminf_{n\to\infty}a_n(t)$. That is

$$
\lim_{n\to\infty}\|tx_n+(1-t)\,p_1-p_2\|
$$

exists for all $t\in(0,1)$. This completes the proof. $\qquad\square$

Theorem 3.6. *Let X be a real uniformly convex Banach space such that its dual X^* has the Kadec–Klee property, K be a nonempty closed convex subset of X, $\{T_i:i\in I_0\}$ be N I_i-asymptotically nonexpansive mappings with sequences $\left\{l_n^{(i)}\right\}\subset[0,\infty)$ and $\{I_i:i\in I_0\}$ be N asymptotically nonexpansive self-mappings of K with sequences $\left\{k_n^{(i)}\right\}\subset[0,\infty)$ such that $F=\bigcap_{i=1}^{N}F(T_i)\cap F(I_i)\neq\emptyset$. Suppose that for any given $x_1\in K$, the sequence $\{x_n\}$ is generated by (1.7) satisfying the conditions:*

1) $\sum_{n=1}^{\infty} l_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$, where $l_n = \max\{l_n^{(i)} : i \in I_0\}$, $k_n = \max\{k_n^{(i)} : i \in I_0\}$;

2) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $\tau_1 \leq (1-\alpha_n), (1-\beta_n) \leq \tau_2$, $\forall n \geq 1$.

Then $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i : i \in I_0\}$ *and* $\{I_i : i \in I_0\}$.

Proof. Let $p \in F$. Then $\lim_{n \to \infty} \|x_n - p\|$ exists from Lemma 3.1 and so $\{x_n\}$ is bounded. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. For, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 3.3, $\lim_{n \to \infty} \|x_n - T_i x_n\| = 0$ and $\lim_{n \to \infty} \|x_n - I_i x_n\| = 0$ for all $i \in I_0$. Lemma 2.3 guarantees that $I_i u = u$ and $T_i u = u$. Again in the same fashion, we can prove that $v \in F$. Next, we prove the uniqueness. Since $\lim_{n \to \infty} \|tx_n + (1-t)u - v\|$ exists for all $t \in [0,1]$ by Lemma 3.5, therefore $u = v$ by Lemma 2.5. Consequently, $\{x_n\}$ converges weakly to a point of F and this completes the proof. $\square$

Remark 3.7. *Since iteration scheme (1.5) converges faster than Ishikawa iteration process, therefore our results improve and generalize corresponding results of Temir [2] and many other in the contemporary literature.*

Remark 3.8. *If we choose $I_i = E$ for all $i \in I_0$, we obtain related results of the previously known results for Mann Iteration in this area.*

Remark 3.9. *Under suitable conditions, the sequence $\{x_n\}$ defined by (1.7) can also be generalized to the iterative sequences with errors. That is, if the error terms are added in (1.7) and assumed to be bounded, then the results of this paper still hold.*

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