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Differential Subordination for Subclasses of Analytic Functions Involving Srivastava-Attiya Operator

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Abstract : We introduce and investigate several new classes of analytic functions involving Srivastava-Attiya operator, and derive various useful properties and characteristics of these function classes by using the techniques of differential subordination. Several results are presented exhibiting relevant connections to some of the results proved here and those obtained in earlier works.

Keywords : analytic functions; Hadamard product; differential subordination; Srivastava-Attiya operator.
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1 Introduction

Let $\mathcal{A}$ denote the class of the functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

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which are analytic in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Given two functions f and F, which are analytic in U, we say that the function f is subordinated to F, and write $f(z) \prec F(z)$, if there exists a function w analytic in U such that $|w(z)| < 1$, $z \in \mathrm{U}$, and $w(0) = 0$, with $f(z) = F(w(z))$ in U.

In particular, if F is univalent in U, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathrm{U}) \subset F(\mathrm{U})$.

Definition 1.1. (Miller and Mocanu [1]) Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the following differential subordination

$$\psi(p(z), zp'(z)) \prec h(z), \tag{1.1}$$

then p is called a *solution of the differential subordination* (1.1). A univalent function q is called a *dominant of the solutions of the differential subordination* (1.1) or, more simply, a dominant if $p(z) \prec q(z)$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants q of (1.1) is said to be the *best dominant of* (1.1).

Definition 1.2. Let $\mathcal{P}$ denote the class of all functions ϕ which are analytic and univalent in U, with $\phi(0) = 1$.

Definition 1.3. (i) The *generalized hypergeometric function* ${}_qF_s$ is defined by

$${}_qF_s(z) = {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_q)_n}{(\beta_1)_n \cdot \ldots \cdot (\beta_s)_n} \frac{z^n}{n!},\ z \in \mathrm{U},$$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \ldots, q)$, $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathbb{Z}_0^- = \{0, -1, \ldots\}$ $(j = 1, \ldots, s)$, $q \leq s + 1$, $q, s \in \mathbb{N}_0$, and $(\alpha)_k$ is the *Pochhammer symbol* defined by

$$(\alpha)_0 = 1,\ (\alpha)_k = \alpha(\alpha + 1) \cdot \ldots \cdot (\alpha + k - 1),\ k \in \mathbb{N}.$$

(ii) The *general Hurwitz-Lerch Zeta function* $\phi(z, s, b)$ is defined by (cf., e.g. (Srivastava and Choi [2, p. 21 *et seq.*])

$$\phi(z, s, b) = \sum_{n=0}^{\infty} \frac{z^n}{(b + n)^s} = \frac{1}{b^s} + \frac{z}{(1 + b)^s} + \frac{z^2}{(2 + b)^s} + \ldots,$$

with $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, and $\operatorname{Re} s > 1$ when $|z| = 1$.

These general Hurwitz-Lerch Zeta function $\phi(z, s, a)$ is also contains as its special cases, well-known functions as the Riemann and Hurwitz (or generalized) Zeta function, Lerch Zeta function, the Polylogarithmic function and the Lipschitz-Lerch Zeta function. One may refer to the Srivastava and Choi [2] (see also Srivastava and Attiya [3]) for further details and references to these functions.

Srivastava and Attiya in [3] (see also Prajapat and Goyal [4]), introduced the following family of linear operator

$$J_{s,b} : \mathcal{A} \to \mathcal{A},$$

which is defined by

$$J_{s,b}f(z) = G_{s,b}(z) * f(z) \quad (b \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ s \in \mathbb{C}), \tag{1.2}$$

where the symbol "$*$" denotes the *Hadamard product (or convolution)* of analytic functions, and function $G_{s,b}$ is given by

$$G_{s,b}(z) = (b+1)^s \left[\phi(z,s,b) - b^{-s}\right] = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^s z^n. \tag{1.3}$$

From (1.2) and (1.3), we get

$$J_{s,b}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^s a_n z^n,\ z \in \mathrm{U}. \tag{1.4}$$

The Srivastava-Attiya operator $J_{s,b}$ contains among its special cases, the integral operators introduced and investigated by Alexander [5], Libera [6] and Jung et al. [7].

Using the relation (1.4) it can be easily verified that the linear operator $J_{s,b}$ satisfies the following differentiation formula:

$$z\left(J_{s+1,b}f(z)\right)' = (b+1)J_{s,b}f(z) - bJ_{s+1,b}f(z),\ z \in \mathrm{U}. \tag{1.5}$$

Now we will introduce the following subclasses of $\mathcal{A}$, involving the operator $J_{s,b}$.

Definition 1.4. Let $\phi \in \mathcal{P}$, $s \in \mathbb{C}$, and $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then the function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{M}(s,b;\phi)$, if it satisfies

$$\frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} \prec \phi(z).$$

Further, we set

$$\mathcal{M}\left(s,b;1+\frac{A-B}{b+1}\frac{z}{1+Bz}\right) =: \widetilde{\mathcal{M}}\left(s,b;A,B\right),$$

and

$$\mathcal{M}\left(s,b;1+\frac{2(1-\alpha)}{b+1}\frac{z}{1-z}\right) =: \widetilde{\mathbb{M}}\left(s,b,\alpha\right).$$

We observe that the classes $\mathcal{M}(s,b;\phi)$, $\widetilde{\mathcal{M}}\left(s,b;A,B\right)$, and $\widetilde{\mathbb{M}}\left(s,b,\alpha\right)$ generalize several previously studied classes, and we will show some of the interesting cases as follows:

(i) The classes

$$\mathcal{M}(-1,0;\phi) = S^*(\phi) \quad \text{and} \quad \mathcal{M}(-2,0;\phi) = C(\phi)$$

has been studied by Ma and Minda [8];

(ii) The class

$$\widetilde{\mathcal{M}}(-1,0;A,B) = S[A,B]$$

was studied by Janowski [9];

(iii) The classes

$$\mathcal{M}\left(-1,0;\frac{1+(1-2\alpha)z}{1-z}\right) = \widetilde{\mathcal{M}}(-1,0;1-2\alpha,-1) = \widetilde{\mathbb{M}}(-1,0,\alpha) = \mathcal{S}^*(\alpha)$$

and

$$\mathcal{M}\left(-2,0;\frac{1+(1-2\alpha)z}{1-z}\right) = \widetilde{\mathcal{M}}(-2,0;1-2\alpha,-1) = \widetilde{\mathbb{M}}(-2,0,\alpha) = \mathcal{K}(\alpha),$$

are the familiar subclasses of $\mathcal{A}$ of starlike and convex functions of order α ($0 \le \alpha < 1$), respectively (see Srivastava and Owa [10]).

In the present paper we derive various useful and interesting properties and characteristics of the above defined function classes by using the subordination principle.

The following lemmas will be required in our present investigation.

Lemma 1.5. (Miller and Mocanu [1, p. 132]) *Let q be analytic and univalent in* U. *Also, let θ and ϕ be analytic in a domain D containing $q(\mathrm{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathrm{U})$. Set*

$$Q(z) = zq'(z)\phi(q(z)), \quad h(z) = \theta(q(z)) + Q(z),$$

and suppose that

$$(i) \quad Q \quad \textit{is univalent and starlike in } \mathrm{U},$$

$$(ii) \quad \operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right] > 0,\ z \in \mathrm{U}.$$

If p is analytic in U, *with $p(0) = q(0)$, $p(\mathrm{U}) \subset D$, and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.6. (Miller and Mocanu [11]) *If $-1 \le B < A \le 1$, $\beta > 0$ and the complex number γ satisfies $\operatorname{Re}\gamma \ge -\beta(1-A)/(1-B)$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz},\ z \in \mathrm{U},$$

has a univalent solution in U *given by*

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}\,\mathrm{d}\,t} - \dfrac{\gamma}{\beta}, & \textit{if} \quad B \neq 0 \\ \dfrac{z^{\beta+\gamma}\ \exp(\beta Az)}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}\ \exp(\beta At)\,\mathrm{d}\,t} - \dfrac{\gamma}{\beta}, & \textit{if} \quad B = 0. \end{cases}$$

If $\phi(z) = 1 + c_1 z + c_2 z^2 + \dots$ *is analytic in* U *and satisfies*

$$\phi(z) + \frac{z\phi'(z)}{\beta\phi(z) + \gamma} \prec \frac{1+Az}{1+Bz}, \tag{1.6}$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

and q *is the best dominant of* (1.6).

Lemma 1.7. (Wilken and Feng [12]) *Let* ν *be a positive measure on* $[0,1]$*, and let* h *be a complex valued function defined on* $\mathrm{U} \times [0,1]$*, such that* $h(\cdot, t)$ *is analytic in* U *for each* $t \in [0,1]$*, and* $h(z,\cdot)$ *is* ν*-integrable on* $[0,1]$ *for all* U*. In addition, suppose that* $\operatorname{Re} h(z,t) > 0$*,* $h(-r,t)$ *is real, and*

$$\operatorname{Re}\frac{1}{h(z,t)} \geq \frac{1}{h(-r,t)}, \quad \textit{for} \quad |z| \leq r < 1 \quad \textit{and} \quad t \in [0,1].$$

If the function $\mathcal{H}$ *is defined by*

$$\mathcal{H}(z) = \int_0^1 h(z,t)\, \mathrm{d}\,\nu(t),$$

then

$$\operatorname{Re}\frac{1}{\mathcal{H}(z)} \geq \frac{1}{h(-r)}, \quad \textit{for} \quad |z| \leq r < 1.$$

Lemma 1.8. (Whittaker and Watson [13]) *For real or complex numbers* a_1*,* b_1 *and* c_1 $(c_1 \neq 0, -1, -2, \dots)$*, the following identities hold:*

$$\int_0^1 t^{b_1-1}(1-t)^{c_1-b_1-1}(1-zt)^{-a_1}\,\mathrm{d}\,t = \frac{\Gamma(b_1)\Gamma(c_1-b_1)}{\Gamma(c_1)}\ {}_2F_1(a_1,b_1;c_1;z), \qquad \textit{if} \quad \operatorname{Re} c_1 > \operatorname{Re} b_1 > 0,$$

$${}_2F_1(a_1,b_1;c_1;z) = {}_2F_1(b_1,a_1;c_1;z),$$

$${}_2F_1(a_1,b_1;c_1;z) = (1-z)^{-a_1}\ {}_2F_1\left(a_1, c_1-b_1; c_1; \frac{z}{z-1}\right).$$

Lemma 1.9. (Royster [14]) *The function* $q(z) = (1-z)^{-2ab}$ *is univalent in* U *if and only if* $|2ab-1| \leq 1$ *or* $|2ab+1| \leq 1$.

2 Main Results

We first prove the following subordination theorem involving the operator $J_{s,b}$.

Theorem 2.1. *Let* $\psi \in \mathcal{P}$ *and* $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$*, such that*

$$\frac{z\psi'(z)}{\psi(z)} \quad \textit{is starlike in } \mathrm{U}, \tag{2.1}$$

and

$$\operatorname{Re}\left[(b+1)\psi(z)\right] > 0,\ z \in \mathrm{U}. \tag{2.2}$$

If $f \in \mathcal{M}(s,b;\tau)$ *such that* $J_{s+2,b}f(z) \neq 0$ *for all* $z \in \dot{\mathrm{U}} = \mathrm{U} \setminus \{0\}$, *and*

$$\tau(z) = \psi(z) + \frac{z\psi'(z)}{(b+1)\psi(z)},$$

then $f \in \mathcal{M}(s+1,b;\psi)$.

Proof. Let $f \in \mathcal{M}(s,b;\tau)$, and define the function p by

$$p(z) = \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}. \tag{2.3}$$

Then, the function p is analytic in U, and differentiating both sides of (2.3) with respect to z and making use of the identity (1.5), we obtain that

$$\frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} = p(z) + \frac{zp'(z)}{(b+1)p(z)}.$$

Since $f \in \mathcal{M}(s,b;\tau)$, the above relation shows that

$$(b+1)p(z) + \frac{zp'(z)}{p(z)} \prec (b+1)\psi(z) + \frac{z\psi'(z)}{\psi(z)}.$$

If we let $\theta(w) = (b+1)w$ and $\phi(w) = \dfrac{1}{w}$, then θ and ϕ are analytic in $\mathbb{C} \setminus \{0\}$. Setting

$$Q(z) = z\psi'(z)\phi(\psi(z)) = \frac{z\psi'(z)}{\psi(z)}$$

and

$$h(z) = \theta(\psi(z)) + Q(z) = (b+1)\psi(z) + \frac{z\psi'(z)}{\psi(z)},$$

from the assumptions it follows that the function Q is starlike, and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\frac{\theta'(\psi(z))}{\phi(\psi(z))} + \frac{zQ'(z)}{Q(z)}\right] = \operatorname{Re}\left[(b+1)\psi(z)\right] + \operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0,\ z \in \mathrm{U}.$$

Therefore, by virtue of Lemma 1.5 we conclude that $p(z) \prec \psi(z)$, that is $f \in \mathcal{M}(s+1,b;\psi)$. □

Corollary 2.2. *Let* $-1 \leq B < A \leq 1$, *and* $b \in \mathbb{R} \setminus \{0\}$ *with* $b \geq -\dfrac{1-A}{1-B}$. *Let denote*

$$C = \frac{A+bB}{1+b} \quad \textit{and} \quad \eta(z) = \frac{1+Cz}{1+Bz} + \frac{(C-B)z}{(1+Bz)(1+Cz)}.$$

If $f \in \mathcal{M}(s,b;\eta(z))$ *such that* $J_{s+2,b}f(z) \neq 0$ *for all* $z \in \dot{\mathrm{U}}$, *then* $f \in \widetilde{\mathcal{M}}(s+1,b;A,B)$.

Proof. If we choose in Theorem 2.1

$$\psi(z) = 1 + \frac{A-B}{1+b}\frac{z}{1+Bz}, \quad -1 \le B < A \le 1,$$

then it is sufficient to show that the function ψ satisfies the conditions (2.1) and (2.2).

First, we may easily see that the function

$$Q(z) = \frac{z\psi'(z)}{\psi(z)} = \frac{(C-B)z}{(1+Bz)(1+Cz)},$$

is starlike in U. Thus,

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = -1 + \operatorname{Re}\frac{1}{1+Bz} + \operatorname{Re}\frac{1}{1+Cz} > -1 + \frac{1}{1+|B|} + \frac{1}{1+|C|}$$
$$= \frac{1-|BC|}{(1+|B|)(1+|C|)},\ z \in \mathrm{U}.$$

From the assumptions we have $|C| \le 1$, hence $1 - |BC| \ge 0$, and from the above inequality it follows that Q is a starlike function in U.

Also, a simple computation shows that

$$\operatorname{Re}[(b+1)\psi(z)] = (1+b)\operatorname{Re}\frac{1+Cz}{1+Bz} > (1+b)\frac{1-C}{1-B} \ge 0,\ z \in \mathrm{U},$$

which completes the proof of our corollary. □

Remark 2.3. *(i) Setting $A = 1-2\alpha$ with $0 \le \alpha < 1$, and $B = -1$ in Corollary 2.2 we get an improvement of a result by Srivastava and Attiya [3].*

(ii) Setting $s = \delta$ with $\delta > 0$, $b = 1$, $A = 1-2\alpha$ with $0 \le \alpha < 1$, and $B = -1$, Corollary 2.2 would yield the corresponding known result due to Attiya [15].

Theorem 2.4. *Let $\psi \in \mathcal{P}$ and $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, such that*

$$\frac{z\psi'(z)}{\lambda - b + (1+b)\psi(z)} \quad \textit{is starlike in } \mathrm{U},$$

and

$$\operatorname{Re}[\lambda - b + (1+b)\psi(z)] > 0,\ z \in \mathrm{U}.$$

Let define the operator $\mathcal{F}_\lambda : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{F}_\lambda f(z) = \frac{\lambda+1}{z^\lambda}\int_0^z t^{\lambda-1} f(t)\,\mathrm{d}\,t, \quad \operatorname{Re}\lambda > -1,$$

and let

$$\mathcal{X}(z) = \psi(z) + \frac{z\psi'(z)}{\lambda - b + (1+b)\psi(z)}.$$

If $f \in \mathcal{M}(s,b;\mathcal{X})$ such that $J_{s+1,b}\,\mathcal{F}_\lambda f(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$, then $\mathcal{F}_\lambda f \in \mathcal{M}(s,b;\psi)$.

Proof. From the definition of $\mathcal{F}_\lambda$ and (1.5) we obtain that

$$\begin{aligned}&(1+\lambda)J_{s+1,b}f(z) = \lambda J_{s+1,b}\,\mathcal{F}_\lambda f(z) + z\left(J_{s+1,b}\,\mathcal{F}_\lambda f(z)\right)'\\ &= (\lambda-b)J_{s+1,b}\,\mathcal{F}_\lambda f(z) + (1+b)J_{s,b}\,\mathcal{F}_\lambda f(z).\end{aligned} \tag{2.4}$$

Defining the function p by

$$p(z) = \frac{J_{s,b}\,\mathcal{F}_\lambda f(z)}{J_{s+1,b}\,\mathcal{F}_\lambda f(z)},$$

then p is analytic in U, and from (2.4) we deduce that

$$(1+\lambda)\frac{J_{s+1,b}f(z)}{J_{s+1,b}\,\mathcal{F}_\lambda f(z)} = \lambda - b + (1+b)p(z). \tag{2.5}$$

Differentiating both sides of (2.5) with respect to z and using the identity (1.5) and (2.4), we obtain that

$$\frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} = p(z) + \frac{zp'(z)}{\lambda - b + (1+b)p(z)}.$$

From the above relation, since $f \in \mathcal{M}(s,b;\mathcal{X})$, we deduce

$$p(z) + \frac{zp'(z)}{\lambda - b + (1+b)p(z)} \prec \psi(z) + \frac{z\psi'(z)}{\lambda - b + (1+b)\psi(z)}.$$

Letting

$$\theta(w) = w \quad \text{and} \quad \phi(w) = \frac{1}{\lambda - b + (1+b)w},$$

we have that θ and ϕ are analytic in $\mathbb{C} \setminus \left\{\dfrac{b-\lambda}{b+1}\right\}$. If we set

$$Q(z) = z\psi'(z)\phi(\psi(z)) = \frac{z\psi'(z)}{\lambda - b + (1+b)\psi(z)}$$

and

$$h(z) = \theta(\psi(z)) + Q(z) = \psi(z) + \frac{z\psi'(z)}{\lambda - b + (1+b)\psi(z)},$$

from the first of our assumptions, the function Q is starlike, and combining this fact with the second assumption we deduce that

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left[(\lambda - b) + (1+b)\psi(z)\right] + \operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0,\ z \in \mathrm{U}.$$

Now, according to Lemma 1.5 we conclude that $p(z) \prec \psi(z)$, that is $\mathcal{F}_\lambda f \in \mathcal{M}(s,b;\psi)$. □

Theorem 2.5. *Let $\alpha, \beta \in \mathbb{C}$, and $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Suppose that $\psi \in \mathcal{P}$ is a convex function in* U *satisfying the inequality*

$$\operatorname{Re}\left[\frac{(1+b)\,(\alpha+2\beta\psi(z))}{\beta}+\frac{z\psi''(z)}{\psi'(z)}+1\right]>0,\ z\in \mathrm{U}. \tag{2.6}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$\alpha\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}+\beta\frac{J_{s,b}f(z)}{J_{s+2,b}f(z)}\prec \alpha\psi(z)+\beta\psi^2(z)+\frac{\beta}{1+b}z\psi'(z),$$

then $f \in \mathcal{M}(s+1, b; \psi)$.

Proof. If we define the analytic function p by

$$p(z)=\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)},$$

using the relation (1.5) a simple computation shows that

$$\alpha\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}+\beta\frac{J_{s,b}f(z)}{J_{s+2,b}f(z)}=\alpha p(z)+\beta p^2(z)+\frac{\beta}{1+b}zp'(z).$$

Setting $\theta(w) = \alpha w + \beta w^2$ and $\phi(w) = \dfrac{\beta}{1+b}$, the convexity of the function ψ together with the assumption (2.6) show that the conditions (i) and (ii) of Lemma 1.5 are satisfied, hence $p(z) \prec \psi(z)$, and thus $f \in \mathcal{M}(s+1, b; \psi)$. □

Corollary 2.6. *Let $\alpha, \beta \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $-1 \le B < A \le 1$, such that*

$$\operatorname{Re}\left[2b+\frac{\alpha(1+b)}{\beta}\right]+\frac{2(1-A)}{1-B}+\frac{1-|B|}{1+|B|}\ge 0. \tag{2.7}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$\alpha\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}+\beta\frac{J_{s,b}f(z)}{J_{s+2,b}f(z)}\prec \alpha+\beta+\frac{(\alpha+2\beta)(A-B)z}{(1+b)(1+Bz)}+\frac{\beta(A-B)\left[(A-B)z+1\right]z}{(1+b)^2(1+Bz)^2},$$

then $f \in \widetilde{\mathcal{M}}(s+1, b; A, B)$.

Proof. If we choose in Theorem 2.5

$$\psi(z)=1+\frac{A-B}{1+b}\frac{z}{1+Bz},$$

then it is sufficient to show that ψ is a convex function in U which satisfies the inequality (2.6). We may easily check that

$$\operatorname{Re}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)=\operatorname{Re}\frac{1-Bz}{1+Bz}>\frac{1-|B|}{1+|B|}\ge 0,\ z\in \mathrm{U},$$

which shows that ψ is a convex function in U. According to the assumption (2.7), we deduce that

$$\begin{aligned}&\operatorname{Re}\left[\frac{(1+b)(2\beta\psi(z)+\alpha)}{\beta}+\frac{z\psi''(z)}{\psi'(z)}+1\right]\\&=\operatorname{Re}\left[2b+\frac{\alpha(1+b)}{\beta}\right]+2\operatorname{Re}\frac{1+Az}{1+Bz}+\operatorname{Re}\frac{1-Bz}{1+Bz}\\&>\operatorname{Re}\left[2b+\frac{\alpha(1+b)}{\beta}\right]+\frac{2(1-A)}{1-B}+\frac{1-|B|}{1+|B|}\geq 0,\ z\in\mathrm{U},\end{aligned}$$

and the proof is complete. □

Setting $b=0$ and $s=-2$ in Corollary 2.6, we get following special case:

Corollary 2.7. *Let $\alpha,\beta\in\mathbb{C}$ and $-1\leq B<A\leq 1$, such that*

$$\operatorname{Re}\frac{\alpha}{\beta}+\frac{2(1-A)}{1-B}+\frac{1-|B|}{1+|B|}\geq 0.$$

If $f\in\mathcal{A}$ satisfies the subordination

$$\beta\frac{z^2f''(z)}{f(z)}+(\alpha+\beta)\frac{zf'(z)}{f(z)}\prec\alpha+\beta+\frac{(\alpha+2\beta)(A-B)z}{(1+Bz)}+\frac{\beta(A-B)\left[(A-B)z+1\right]z}{(1+Bz)^2},$$

then $f\in\mathcal{S}^*[A,B]$.

Remark 2.8. *(i) Taking $\alpha=\lambda-1$, and $\beta=1$ in Corollary 2.7, we get the result due to Xu and Yang [16, p. 581, Theorem 1].*

(ii) Taking $\alpha=1-\beta$, $0<\beta\leq 1$, $A=1$, and $B=-1$ in Corollary 2.7, we get the result due to Padmanabhan [17].

Theorem 2.9. *Let $-1\leq B<A\leq 1$, $b\in\mathbb{R}\setminus\{0\}$ with $b\geq-\dfrac{1-A}{1-B}$, and let denote*

$$C=\frac{A+bB}{1+b}.$$

(i) If $f\in\widetilde{\mathcal{M}}(s,b;A,B)$ such that $J_{s+2,b}f(z)\neq 0$ for all $z\in\dot{\mathrm{U}}$, then

$$\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}\prec\frac{1}{(b+1)\widetilde{Q}(z)}=q(z)\prec\frac{1+Cz}{1+Bz},\tag{2.8}$$

where

$$\widetilde{Q}(z)=\begin{cases}\displaystyle\int_0^1 t^b\left(\frac{1+Btz}{1+Bz}\right)^{(b+1)(C-B)/B}\mathrm{d}\,t, & \text{if}\quad B\neq 0,\\ \displaystyle\int_0^1 t^b\exp\left[(b+1)(t-1)Cz\right]\mathrm{d}\,t, & \text{if}\quad B=0,\end{cases}\tag{2.9}$$

and q is the best dominant of (2.8).

(ii) Supposing, in addition, that $-1 \leq B < 0$*, and* $C < -\dfrac{B}{b+1}$*, then*

$$\operatorname{Re}\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)} > \left[{}_2F_1\left(1, \frac{(b+1)(B-C)}{B}; b+2; \frac{B}{B-1}\right)\right]^{-1}, \ z \in \mathrm{U}, \quad (2.10)$$

and this result is the best possible.

Proof. Let $f \in \widetilde{\mathcal{M}}(s, b; A, B)$, and define the function p by

$$p(z) = \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}. \quad (2.11)$$

The function p is analytic in U, and differentiating both sides of (2.11) with respect to z, and using the identity (1.5) we obtain

$$\frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} = p(z) + \frac{zp'(z)}{(b+1)p(z)}.$$

Since $f \in \widetilde{\mathcal{M}}(s, b; A, B)$, the above relation shows that

$$p(z) + \frac{zp'(z)}{(b+1)p(z)} \prec 1 + \frac{A-B}{b+1}\frac{z}{1+Bz} = \frac{1+Cz}{1+Bz}.$$

Therefore, by using Lemma 1.6 for $\beta = b+1$ and $\gamma = 0$ we obtain (2.8), where the function $\widetilde{Q}$ is given by (2.9).

Next, if we set

$$a_1 := \frac{(b+1)(B-C)}{B}, \quad b_1 := b+1, \quad \text{and} \quad c_1 := b+2,$$

then $c_1 > b_1 > 0$, and according to Lemma 1.8 we deduce from (2.9) that

$$\widetilde{Q}(z) = (1+Bz)^{a_1}\int_0^1 t^{b_1-1}(1+Btz)^{-a_1}\,\mathrm{d}\,t = \frac{\Gamma(b_1)}{\Gamma(c_1)}\,{}_2F_1\left(1, a_1; c_1; \frac{Bz}{Bz+1}\right), \quad (2.12)$$

whenever $B \neq 0$.

In order to prove the inequality (2.10), we need to show that

$$\operatorname{Re}\frac{1}{\widetilde{Q}(z)} > \frac{1}{\widetilde{Q}(-1)}, \ z \in \mathrm{U}. \quad (2.13)$$

If $-1 \leq B < 0$, and $C < -\dfrac{B}{b+1}$, then $c_1 > a_1 > 0$, and from (2.12) we obtain that the function $\widetilde{Q}$ may be written as

$$\widetilde{Q}(z) = \int_0^1 g(t, z)\,\mathrm{d}\,\mu(t),$$

where

$$g(t,z)=\frac{1+Bz}{1+(1-t)Bz},\ 0\leq t\leq 1,$$

and

$$\mathrm{d}\,\mu(t)=\frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}t^{a_1-1}(1-t)^{c_1-a_1-1}\,\mathrm{d}\,t$$

is a positive measure on $[0,1]$.

For $-1\leq B<1$ we have that $\operatorname{Re}g(t,z)>0$, $g(t,-r)$ is real for $0\leq r<1$ and $t\in[0,1]$, and

$$\operatorname{Re}\frac{1}{g(t,z)}\geq\frac{1-(1-t)Br}{1-Br}=\frac{1}{g(t,-r)},\quad\text{for}\quad |z|\leq r<1\quad\text{and}\quad t\in[0,1].$$

According to Lemma 1.7 we get

$$\operatorname{Re}\frac{1}{\widetilde{Q}(z)}\geq\frac{1}{\widetilde{Q}(-r)},\quad\text{for}\quad |z|\leq r<1,$$

and letting $r\to 1^-$ we conclude that the inequality (2.13) holds. The result is sharp because the function q is the best dominant of (2.8). □

Setting $A=1-2\delta$ with $0\leq\delta<1$, and $B=-1$ in Theorem 2.9, we have:

Corollary 2.10. *Let $0\leq\delta<1$, and $b\geq-\delta$ with $b\neq 0$. If $f\in\widetilde{\mathbb{M}}(s,b,\delta)$ such that $J_{s+2,b}f(z)\neq 0$ for all $z\in\dot{\mathrm{U}}$, then*

$$\operatorname{Re}\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}>\left[{}_2F_1\left(1,2(1-\delta);b+2;\frac{B}{B-1}\right)\right]^{-1},\ z\in\mathrm{U},$$

and this result is the best possible.

In the next result we will consider the inverse problem of Corollary 2.10.

Theorem 2.11. *Let $0\leq\delta<1$, and $b\geq-\delta$ with $b\neq 0$. If $f\in\widetilde{\mathbb{M}}(s+1,b,\delta)$, then $f\in\widetilde{\mathbb{M}}(s,b,\delta)$ in $|z|<R(b,\delta)$, where*

$$R(b,\delta)=\begin{cases}\dfrac{2-\delta-\sqrt{(2-\delta)^2-(b+1)(1-b-2\delta)}}{1-b-2\delta}, & \text{if}\quad b<1-2\delta,\\[2mm] \dfrac{2-\delta+\sqrt{(2-\delta)^2-(b+1)(1-b-2\delta)}}{1-b-2\delta}, & \text{if}\quad b>1-2\delta,\\[2mm] \dfrac{1+b}{3+b}, & \text{if}\quad b=1-2\delta.\end{cases}\tag{2.14}$$

The result is the best possible.

Proof. For $f \in \mathcal{A}$, according to the differentiation relation (1.5), it is easy to show that

$$\frac{z\left(J_{s+2,b}f(z)\right)'}{J_{s+2,b}f(z)} = (b+1)\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)} - b.$$

It follows that $f \in \widetilde{\mathbb{M}}(s+1,b,\delta)$ if and only if

$$p(z) := \frac{z\left(J_{s+2,b}f(z)\right)'}{J_{s+2,b}f(z)} = \delta + (1-\delta)u(z), \tag{2.15}$$

where u is an analytic function in U with $u(0) = 1$, and $\operatorname{Re} u(z) > 0$, for $z \in$ U. Using (1.5), form (2.15) we may easily deduce that

$$\begin{aligned}
&\operatorname{Re}\left[\frac{z\left(J_{s+1,b}f(z)\right)'}{J_{s+1,b}f(z)} - \delta\right] = (1-\delta)\operatorname{Re}\left[u(z) + \frac{zu'(z)}{\delta+b+(1-\delta)u(z)}\right] \\
&\geq (1-\delta)\operatorname{Re}\left[u(z) - \frac{|zu'(z)|}{|\delta+b+(1-\delta)u(z)|}\right] \\
&\geq (1-\delta)\operatorname{Re}\left[u(z) - \frac{|zu'(z)|}{\delta+b+(1-\delta)\operatorname{Re} u(z)}\right],\ z \in \mathrm{U},
\end{aligned} \tag{2.16}$$

whenever $0 \leq \delta < 1$, and $b \geq -\delta$. Using in (2.16) the well known estimates (see [18])

$$|zu'(z)| \leq \frac{2r}{1-r^2}\operatorname{Re} u(z) \quad \text{and} \quad \operatorname{Re} u(z) \geq \frac{1-r}{1+r}, \quad \text{if} \quad |z| = r < 1,$$

we obtain that

$$\operatorname{Re}\left[\frac{z\left(J_{s+1,b}f(z)\right)'}{J_{s+1,b}f(z)} - \delta\right] \geq (1-\delta)\operatorname{Re} u(z)\left[1 - \frac{2r}{(\delta+b)(1-r^2)+(1-\delta)(1-r)^2}\right],$$

for $|z| = r < 1$, and the right-hand side of this inequality is positive if $r < R(b,\delta)$, where $R(b,\delta)$ is given by (2.14).

To show that the bound $R(b,\delta)$ is best possible, we consider the function $f \in \mathcal{A}$ defined by

$$\frac{z\left(J_{s+2,b}f(z)\right)'}{J_{s+2,b}f(z)} = \delta + (1-\delta)\frac{1+z}{1-z},\ z \in \mathrm{U}.$$

Noting that

$$\frac{z\left(J_{s+1,b}f(z)\right)'}{J_{s+1,b}f(z)} - \delta = (1-\delta)\left[\frac{1+z}{1-z} + \frac{2z}{(\delta+b)(1-z)^2+(1-\delta)(1-z^2)}\right] = 0,$$

for $z = -R(b,\delta)$, we conclude that the bound (2.14) is the best possible. □

Theorem 2.12. *Let $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, and let $\psi \in \mathcal{P}$ be a univalent function in* U*, such that*

$$\frac{z\psi'(z)}{\psi(z)} \quad \textit{is starlike in } \mathrm{U}, \tag{2.17}$$

and

$$\operatorname{Re}\left[1+\frac{\beta}{\gamma}\psi(z)-\frac{z\psi'(z)}{\psi(z)}+\frac{z\psi''(z)}{\psi'(z)}\right]>0,\ z\in\mathrm{U}. \tag{2.18}$$

If $f\in\mathcal{A}$, *and* $b\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $J_{s+1,b}f(z)\neq 0$ *for all* $z\in\dot{\mathrm{U}}$, *then*

$$\alpha+\beta\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\delta}+\gamma\delta(b+1)\left(\frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}-1\right)\prec\alpha+\beta\psi(z)+\gamma\frac{z\psi'(z)}{\psi(z)}, \tag{2.19}$$

then

$$\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\delta}\prec\psi(z),$$

and ψ *is the best dominant of* (2.19). *The power of the function is the principal one, i.e.* $\left.\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\delta}\right|_{z=0}=1.$

Proof. If we define the function p by

$$p(z)=\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\delta},$$

then p is analytic in U. Differentiating the above definition formula with respect to z, and using identity (1.5), we obtain that

$$\alpha+\beta\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\delta}+\gamma\delta(b+1)\left(\frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}-1\right)=\alpha+\beta p(z)+\gamma\frac{zp'(z)}{p(z)}.$$

Now, by setting $\theta(w)=\alpha+\beta w$ and $\phi(w)=\dfrac{\gamma}{w}$, our assertion follows easily by applying Lemma 1.5. ☐

Setting in Theorem 2.12 the parameters $\alpha=\beta=0$, $\gamma=\dfrac{1}{\delta(b+1)}$, and taking the function $\psi(z)=(1-z)^{-2\delta(1-a)}$, we have the next result:

Corollary 2.13. *Let* $0\leq a<1$, $0<\delta\leq 1$, *and satisfies either*

$$|2\delta(1-a)+1|\leq 1\quad\text{or}\quad|2\delta(1-a)-1|\leq 1.$$

If $f\in\widetilde{\mathbb{M}}(s,b,a)$, *and* $b\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $J_{s+1,b}f(z)\neq 0$ *for all* $z\in\dot{\mathrm{U}}$, *then*

$$\operatorname{Re}\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\frac{\delta}{2(1-a)}}>2^{-\delta},\ z\in\mathrm{U}, \tag{2.20}$$

and the result is the best possible. (The power of the function is the principal one.)

Proof. First, we remark that if $\beta = 0$, then the conditions (2.17) and (2.18) are identical. Thus, we need to show that $\psi(z) = (1-z)^{-2\delta(1-a)}$ is univalent in U, and the function $\dfrac{z\psi'(z)}{\psi(z)}$ is starlike in U.

According the Lemma 1.9 and the given hypothesis, the function ψ is univalent in U. Also, since

$$h(z) := \frac{z\psi'(z)}{\psi(z)} = \frac{2\delta(1-a)z}{1-z},$$

we have

$$\operatorname{Re}\frac{zh'(z)}{h(z)} = \operatorname{Re}\frac{1}{1-z} > \frac{1}{2},\ z \in \mathrm{U},$$

which shows that h is starlike in U.

If $f \in \widetilde{\mathbb{M}}(s,b,a)$, according to Theorem 2.12 we have

$$\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\frac{\delta}{2(1-a)}} \prec (1-z)^{-\delta}.$$

Thus, there exists a Schwarz function w, that is analytic in U with $w(0) = 0$, and $|w(z)| < 1$ in U, such that

$$\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\frac{\delta}{2(1-a)}} = (1-w(z))^{-\delta},\ z \in \mathrm{U}.$$

Using the elementary inequality

$$\operatorname{Re}\zeta^{\frac{1}{m}} \geq (\operatorname{Re}\zeta)^{\frac{1}{m}},\ \text{for}\ \operatorname{Re}\zeta > 0,\ \text{and}\ m \geq 1,$$

we obtain that

$$\operatorname{Re}\left(\frac{J_{s+1,b}f(z)}{z}\right)^{\frac{\delta}{2(1-a)}} = \operatorname{Re}\left(\frac{1}{1-w(z)}\right)^{\delta} \geq \left(\operatorname{Re}\frac{1}{1-w(z)}\right)^{\delta}$$
$$\geq \left(\frac{1}{1+|w(z)|}\right)^{\delta} > \frac{1}{2^{\delta}},\ z \in \mathrm{U},$$

for $0 < \delta \leq 1$, and this proves our corollary. □

Remark 2.14. *By setting* $\psi(z) = \dfrac{1}{(1-z)^{2b}}$ *with* $b \in \mathbb{C} \setminus \{0\}$, $b = 0$, $s = -1$, $\alpha = \delta = 1$, $\beta = 0$, *and* $\gamma = \dfrac{1}{\eta}$, *Theorem 2.12 reduces to the result obtained by Srivastava and Lashin [19].*

Theorem 2.15. *Let* $g \in \mathcal{A}$ *that satisfies the inequality*

$$\operatorname{Re}\frac{J_{s+1,b}g(z)}{z} > 0,\ z \in \mathrm{U}.$$

If $f \in \mathcal{A}$ such that

$$\left|\frac{J_{s+1,b}f(z)}{J_{s+1,b}g(z)} - 1\right| < 1, \; z \in \mathrm{U}, \tag{2.21}$$

then

$$f \in \widetilde{\mathbb{M}}(s,b,\alpha) \quad in \quad |z| < \frac{\sqrt{17}-3}{4},$$

and the bound $\dfrac{\sqrt{17}-3}{4}$ *is best possible.*

Proof. Letting

$$w(z) = \frac{J_{s+1,b}f(z)}{J_{s+1,b}g(z)} - 1, \tag{2.22}$$

then w is analytic in U with $w(0) = 0$, and $|w(z)| < 1$ for $z \in \mathrm{U}$. By applying the familiar *Schwarz Lemma*, we get

$$w(z) = z\psi(z),$$

where ψ is analytic in U, and $|\psi(z)| \leq 1$ for all $z \in \mathrm{U}$. Therefore (2.22) leads us

$$J_{s+1,b}f(z) = J_{s+1,b}g(z)\left(1 + z\psi(z)\right),$$

which gives that

$$\frac{z\left(J_{s+1,b}f(z)\right)'}{J_{s+1,b}f(z)} = \frac{z\left(J_{s+1,b}g(z)\right)'}{J_{s+1,b}g(z)} + \frac{z\left[\psi(z) + z\psi'(z)\right]}{1 + z\psi(z)}. \tag{2.23}$$

Setting

$$\phi(z) = \frac{J_{s+1,b}g(z)}{z},$$

then ϕ is analytic in U, $\phi(0) = 1$, and $\operatorname{Re}\phi(z) > 0$ for $z \in \mathrm{U}$, and

$$\frac{z\left(J_{s+1,b}g(z)\right)'}{J_{s+1,b}g(z)} = \frac{z\phi'(z)}{\phi(z)} + 1. \tag{2.24}$$

Using (2.24) and the assumption (2.21), together with the well-known estimates [18]

$$\operatorname{Re}\frac{z\phi'(z)}{\phi(z)} \geq -\frac{2r}{1-r^2} \quad \text{and} \quad \operatorname{Re}\frac{z\left[\psi(z) + z\psi'(z)\right]}{1 + z\psi(z)} \geq -\frac{r}{1-r}, \quad \text{for} \quad |z| = r < 1,$$

from (2.23) we deduce that

$$\operatorname{Re}\frac{z\left(J_{s+1,b}f(z)\right)'}{J_{s+1,b}f(z)} \geq 1 - \frac{2r}{1-r^2} - \frac{r}{1-r} = \frac{1-3r-2r^2}{1-r^2}, \quad |z| = r < 1,$$

and the right-hand side of the above inequality is positive, provided that $|z| < \dfrac{\sqrt{17}-3}{4}$. This completes the proof. □

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