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On Some Properties Paranormed Sequence Spaces Generated by Generalized Sequence Means and Core Theorems

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Abstract : Recently, Mursaleen and Noman constructed new sequence spaces by using a matrix domain over a normed space [On generalized means and some related sequence spaces, Comput. Math. Appl. 61 (2011) 988-999]. They also studied some topological properties of these spaces. In this work, we generalize the normed sequence space defined by Mursaleen and Noman to paranormed space. Furthermore, we introduce new sequence space over the paranormed space by using the expansion method. Then, we investigate some topological properties. Finally, we introduce generalized means core of a complex-valued sequence prove some inclusion theorems related to this new type of core.

Keywords : paranormed sequence space; $\alpha-$, $\beta-$ and $\gamma-$duals; core theorems; generalized means.

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1 Introduction

By ω, we shall denote the space of all real or complex valued sequences. Any vector subspace of ω is called a sequence space. We shall write ℓ_∞, c_0 and c for the spaces of all bounded, null and convergent sequences respectively. Also by bs, cs for the spaces of all sequences associated with bounded and convergent series. A sequence space is called an $FK-$space if it is a complete metrizable locally convex space ($F-$space) with the property that convergence implies coordinatewise convergence ($K-$space). A normable $FK-$space is called a $BK-$space. The sequence spaces ℓ_∞, c and c_0 are $BK-$spaces with the usual sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$, where the supremum is taken over all $k \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from μ into γ and we denote it by writing $A : \mu \to \gamma$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = \{(Ax)_n\}$, the $A-$transform of x is in γ; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

The notation $(\mu : \gamma)$ denotes the class of all matrices A such that $A : \mu \to \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \mu$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \gamma$ for all $x \in \mu$. A sequence x said to be $A-$summable to α if Ax converges to α which called as the $A-$limit of x. If $(\mu : \gamma)$ are equipped with limits $\mu - \lim$ and $\gamma - \lim$, respectively, $A \in (\mu : \gamma)$ and $\gamma - \lim_n (Ax)_n = \mu - \lim_k x_k$ for all $x \in \mu$, then we say that A regularly maps μ in to γ and we write $A \in (\mu : \gamma)_{reg}$. The matrix domain μ_A of an infinite matrix A in a sequence space μ is defined by

$$\mu_A = \{x = (x_k) \in \omega : Ax \in \mu\}. \tag{1.2}$$

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there exists subadditive function $g : X \longrightarrow \mathbb{R}$ such that $g(\theta) = 0$, $g(-x) = g(x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \longrightarrow 0$ and $g(x_n - x) \longrightarrow 0$ imply $g(\alpha_n x_n - \alpha x) \longrightarrow 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$ were defined and studied by Maddox, Simons and Nakano [1, 2, 3] as follows:

$$\begin{aligned}
\ell_\infty(p) &= \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\right\}, \\
c_0(p) &= \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}, \\
c(p) &= \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\right\}.
\end{aligned}$$

which are the complete paranormed by

$$g(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/M}.$$

Throughout the article by $\mathcal{F}$ and $\mathbb{N}_k$, respectively, we denote the collection of all subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$. In the literature, by using the matrix domain over the paranormed spaces, many authors have defined new sequence spaces (see[4, 5, 6, 7, 8]).

Let $x = (x_k)$ be a sequence in $\mathbb{C}$, the set of all complex numbers and R_k be the least convex closed region of complex plane containing $x_k, x_{k+1}, x_{k+2}, \ldots$. The Knopp Core (or $K - core$) of x is defined by the intersection of R_k for all $(k = 1, 2, \ldots)$, (see [9, pp.137]). In [10], it is shown that

$$K - core(x) = \bigcap_{z\in\mathbb{C}} B_x(z)$$

for any $x \in \ell_\infty$, where $B_x(z) = \{w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z|\}$.

Let K be a subset of $\mathbb{N}$. The natural density $\delta(K)$ of $K \subseteq \mathbb{N}$ is $\lim_n n^{-1}|\{k \leq n : k \in K\}|$ provided it exists, where $|E|$ denotes the cardinality of a set E. A sequence $x = (x_k)$ is called statistically convergent ($st-$convergent) to the number l, denoted $st - \lim x$, if every $\epsilon > 0$, $\delta(\{k : |x_k - l| \geq \epsilon\}) = 0$, [11]. We write st and st_0 to denote the sets of all statistically convergent sequences and statistically null sequences. In [12], the notion of the statistical core (or st-core) of a complex valued sequence has been introduced by Fridy and Orhan and it is shown for a statistically bounded sequence x that

$$st - core(x) = \bigcap_{t\in\mathbb{C}} C_x(t)$$

for any $x \in \ell_\infty$, where $C_x(t) = \{v \in \mathbb{C} : |v - t| \leq st - \limsup_k |x_k - t|\}$.

2 The Paranormed Sequence Spaces $\overline{\ell}_\infty(p)$, $\overline{c}_0(p)$ and $\overline{c}(p)$

In this section, we define the sequence space $\overline{\mu}(p)$, where $\mu \in \{\ell_\infty, c_0, c\}$ and prove that this sequence space according to its paranorm is complete paranormed linear space. In [13], Mursaleen and Noman defined the matrix $\overline{A}(r, s, t) = (\overline{a}_{nk})$ defined by

$$\overline{a}_{nk} = \begin{cases} s_{n-k}t_k/r_n, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Throughout this paper $r_k \neq 0$, $t_k \neq 0$ and $s_0 \neq 0$ for all $k \in \mathbb{N}$.

Now, we introduce the new sequence spaces $\overline{\ell}_\infty(p)$, $\overline{c}_0(p)$ and $\overline{c}(p)$ as follows:

$$\begin{aligned}
\overline{\ell}_\infty(p) &= \left\{x=(x_k)\in\omega : \sup_{n\in\mathbb{N}}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_kx_k\right|^{p_n}<\infty\right\},\\
\overline{c}_0(p) &= \left\{x=(x_k)\in\omega : \lim_{n\to\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_kx_k\right|^{p_n}=0\right\},\\
\overline{c}(p) &= \left\{x=(x_k)\in\omega : \lim_{n\to\infty}\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_kx_k\right|^{p_n}=l \text{ for some } l\in\mathbb{R}\right\}.
\end{aligned}$$

By the notation of (1.2), we can redefine the space $\overline{\mu}(p)$ as follows:

$$\overline{\mu}(p)=[\mu(p)]_{\overline{A}(r,s,t)}.$$

Then, we have the following special cases.
(i) If $r_n=\sum_{k=0}^{n}s_{n-k}t_k\neq 0$ for all n, then $\overline{A}(r,s,t)$ reduces to the matrix (N,r,s,t) of generalized Nörlund means [14, 15]. If $t=e$, then $\overline{A}(r,s,t)$ reduces of the famous matrix (N,s) of Nörlund means [16, 17].
(ii) If $s=e$, $t_n>0$ and $r_n=\sum_{k=0}^{n}t_k\neq 0$ for all n, then $\overline{\ell}_\infty(p)=r^t_\infty(p)$, $\overline{c}_0(p)=r^t_0(p)$ and $\overline{c}(p)=r^t_c(p)$ [18].
(iii) If $s=e$, $r_n=1/v_n$ and $t_k=v_k$, then $\overline{\ell}_\infty(p)=\ell_\infty(u,v;p)$, $\overline{c}_0(p)=c_0(u,v;p)$ and $\overline{c}(p)=c(u,v;p)$ [19].
(iv) If $s=e$, $r_n=\lambda_n$ and $t_k=\lambda_k-\lambda_{k-1}$, then $\overline{\ell}_\infty(p)=\ell_\infty(\lambda;p)$, $\overline{c}_0(p)=c_0(\lambda;p)$ and $\overline{c}(p)=c(\lambda;p)$ [20].
(v) If $0<\alpha<1$, $s=e$, $t_k=1+\alpha^k$ and $r_n=n+1$, then $\overline{A}(r,s,t)$ reduces to the matrix A^α [21, 22, 23].
(vi) If $r=t=e$ and $s=(r',s',t',0,0,\cdots)$, then $\overline{\ell}_\infty(p)=\ell_\infty(B;p)$, $\overline{c}_0(p)=c_0(B;p)$ and $\overline{c}(p)=c(B;p)$ [24].
(vii) If $0<\alpha<1$ and $r_n=1/k!$, $s_k=(1-\alpha)^k/k!$ and $t_k=\alpha^k/k!$ then, $\overline{A}(r,s,t)$ reduces to the matrix (E,α) [25, 26].

Define the sequence $y=(y_k)$, which will be frequently used, as the $\overline{A}-$transform of a sequence $x=(x_k)$, i.e.,

$$y_k=\frac{1}{r_k}\sum_{j=0}^{k}s_{k-j}t_jx_j. \tag{2.1}$$

Theorem 2.1. *We have the following*

(a) *$\overline{\mu}(p)$ is the complete linear metric space paranormed by h, defined by*

$$h(x)=\sup_n\left|\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_kx_k\right|^{\frac{p_n}{M}},$$

where $0<p_n\leq H<\infty$ for all $n\in\mathbb{N}$.

(b) *Then, $\overline{\mu}(p)$ is $BK-$space with the norm $\|x\|_{\overline{\mu}(p)} = \|\overline{A}(r,s,t)x\|_{\ell_\infty}$. That is,*

$$\|x\|_{\overline{\mu}(p)} = \sup_n |\left(\overline{A}(r,s,t)x\right)_n|. \tag{2.2}$$

Proof. (a) Since this can be shown by a routine verification, we omit the detail.

(b) Since the sequence space μ endowed with the norm $\|.\|_\infty$ are BK-spaces (see[27, Example 7.3.2(b),(c)]) and the matrix $\overline{A}(r,s,t)$ is triangle, Theorem 4.3.2 of Wilansky [28, p.61] gives the fact that the spaces $\overline{\mu}(p)$ is BK-spaces with the norm in (2.2). □

Theorem 2.2. *The sequence spaces $\overline{\ell}_\infty(p)$, $\overline{c}_0(p)$ and $\overline{c}(p)$ of none-absolute type is linearly isomorphic to the spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$, respectively, where $0 < p_k \le H < \infty$.*

Proof. To prove the fact that $\overline{c}_0(p) \cong c_0(p)$ we should show the existence of a linear bijection between the spaces $\overline{c}_0(p)$ and $c_0(p)$, where $0 < p_k \le H < \infty$. Consider the transformation T defined with the notation of (1.2) from $\overline{c}_0(p)$ to $c_0(p)$ by $x \mapsto y = Tx = \overline{A}(r,s,t)x$. The linearity of T is trivial. Further, it is clear that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y = (y_k) \in c_0(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \frac{1}{t_k}\sum_{j=0}^{k}(-1)^{k-j}D^s_{k-j}r_jy_j \quad \text{for each } k \in \mathbb{N}, \tag{2.3}$$

where

$$D^s_n = \frac{1}{s_0^{n+1}}\begin{bmatrix} s_1 & s_0 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & s_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 \end{bmatrix}$$

for $n = (1,2,3,\ldots)$ and $D^s_0 = 1/s_0$. Then, we have

$$h(x) = \sup_n \left|[\overline{A}(r,s,t)x]_n\right|^{\frac{p_n}{M}} = \sup_n |y_n|^{\frac{p_n}{M}} = g(y).$$

Thus, we have that $x \in \overline{c}_0(p)$. As a result, T is surjective. Hence, T is linear bijection and this tells us the spaces $\overline{c}_0(p)$ and $c_0(p)$ are linearly isomorphic, for $0 < p \le H < \infty$ as desired. This completes the proof. □

3 The Alpha-, Beta- and Gamma-Duals of the Space $\overline{\mu}(p)$

In this section, we give the theorems determining the $\alpha-, \beta-$ and $\gamma-$duals of the spaces $\overline{\mu}(p)$ for $\mu \in \{\ell_\infty, c_0, c\}$. We start with the definition of the alpha, beta and gamma duals. If x and y are sequences and X and Y are subsets of ω, then we write $x \cdot y = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$ and

$$M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a : a \cdot x \in Y \text{ for all } x \in X\}.$$

The $\alpha-, \beta-$and $\gamma-$duals of a sequence space, which are respectively denoted by X^{α}, X^{β} and X^{γ} are defined by

$$X^{\alpha} = M(X, \ell_1), \quad X^{\beta} = M(X, cs) \quad \text{and} \quad X^{\gamma} = M(X, bs).$$

Theorem 3.1. *Let us define the matrix $\overline{T} = (\overline{t}_{nk})$ by*

$$\overline{t}_{nk} = \begin{cases} \frac{a_n}{t_n}(-1)^{n-k} D^s_{n-k} r_k, & 0 \leq k \leq n, \\ \\ 0, & k > n. \end{cases} \tag{3.1}$$

Then,

$$\{\overline{\mu}(p)\}^{\alpha} = \{a = (a_k) \in w : \overline{T} \in (\mu(p) : \ell_1)\}.$$

Proof. Let $a = (a_n) \in w$. Then by using (2.3), we immediately derive for every $n \in \mathbb{N}$ that

$$a_n x_n = \frac{1}{t_n} \sum_{k=0}^{n} (-1)^{n-k} D^s_{n-k} y_k r_k a_n = \overline{T}_n(y). \tag{3.2}$$

Thus, we observe that by (3.2) $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \overline{\mu}(p)$ if and only if $\overline{T} y \in \ell_1$ whenever $y = (y_k) \in \mu(p)$. This means that $a = (a_k) \in \{\overline{\mu}(p)\}^{\alpha}$ if and only if $\overline{T} \in (\mu(p) : \ell_1)$. □

The result of the theorem above corresponds to the special case $q_n = 1$ for all $n \in \mathbb{N}$ in [29, Theorem 5.1 (1-3)].

As direct consequence of Theorem 3.1, we have following.

Corollary 3.2. *Let $K^* = K \cap \{n \in \mathbb{N} : n-1 \leq k \leq n\}$ for $K \subset \mathcal{F}$ and $M \in \mathbb{N}_2$. Define the sets $w_1(p), w_2(p), w_3(p)$ as follows:*

$$\begin{aligned} w_1(p) &:= \bigcap_{M>1} \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K^*} \overline{t}_{nk} M^{1/p_k} \right| < \infty \right\}, \\ w_2(p) &:= \bigcup_{M>1} \left\{ a = (a_k) \in \omega : \sum_n \left| \sum_k \overline{t}_{nk} \right| < \infty \right\}, \\ w_3(p) &:= \bigcup_{M>1} \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K^*} \overline{t}_{nk} M^{-1/p_k} \right| < \infty \right\}. \end{aligned}$$

Then, $\{\overline{\ell}_{\infty}(p)\}^{\alpha} = w_1(p)$, $\{\overline{c}_0(p)\}^{\alpha} = w_3(p)$ and $\{\overline{c}(p)\}^{\alpha} = w_2(p) \cap w_3(p)$.

Theorem 3.3. *The matrix* $\overline{H} = (\overline{h}_{nk})$ *is defined via the sequence* $a = (a_n) \in w$ *by*

$$\overline{h}_{nk} = \begin{cases} \sum_{j=k}^{n} \frac{a_j}{t_j} D_{j-k}^{s} (-1)^{j-k} r_k, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases} \tag{3.3}$$

for all $n, k \in \mathbb{N}$*. Then, we have*

$$\{\overline{\mu}(p)\}^{\beta} = \{a = (a_k) \in w : \overline{H} \in (\mu(p) : c)\},$$

$$\{\overline{\mu}(p)\}^{\gamma} = \{a = (a_k) \in w : \overline{H} \in (\mu(p) : \ell_{\infty})\}.$$

Proof. Let us consider the following equation

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\frac{1}{t_k} \sum_{j=0}^{k} (-1)^{k-j} D_{k-j}^{s} r_j y_j \right] a_k \\ &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \frac{a_j}{t_j} (-1)^{j-k} D_{j-k}^{s} \right] r_k y_k \\ &= \sum_{k=0}^{n} \overline{h}_{nk} y_k \\ &= \overline{H}(y) \quad \text{for all} \quad k \in \mathbb{N}, \end{aligned} \tag{3.4}$$

where $\overline{H} = (\overline{h}_{nk})$ defined by (3.3). We obtain from (3.4) that $ax = (a_n x_n) \in cs$ or bs whenever $x = (x_k) \in \overline{\mu}(p)$ if and only if $\overline{H}y \in c$ or ℓ_{∞} whenever $y = (y_k) \in \mu(p)$. This means that $a = (a_k) \in \{\overline{\mu}(p)\}^{\beta}$ or $a = (a_k) \in \{\overline{\mu}(p)\}^{\gamma}$ if and only if $\overline{H} \in (\mu(p) : c)$ or $\overline{H} \in (\mu(p) : \ell_{\infty})$. This completes the proof. □

As direct consequence of Theorem 3.3, we have following.

Corollary 3.4. *Define the sets* $t_1(p), t_2(p), t_3(p), t_4(p)$ *and* $t_5(p)$ *as follows:*

$$t_1(p) = \left\{ a = (a_k) \in \omega : \left\{ \sum_{j=k}^{\infty} \frac{a_j}{t_j} D_{j-k}^{s} (-1)^{j-k} r_k \right\} \in cs \right\},$$

$$t_2(p) = \left\{ a = (a_k) \in \omega : \left\{ \sum_{j=k}^{\infty} \frac{a_j}{t_j} D_{j-k}^{s} (-1)^{j-k} r_k \right\} \in bs \right\},$$

$$t_3(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sum_{k} \left| \sum_{j=k}^{n} \frac{a_j}{t_j} D_{j-k}^{s} (-1)^{j-k} r_k \right| B^{-1/p_k} < \infty \right\},$$

$$t_4(p)=\bigcap_{B>1}\left\{a=(a_k)\in\omega:\sum_k\left|\sum_{j=k}^{n}\tfrac{a_j}{t_j}D^s_{j-k}(-1)^{j-k}r_k\right|B^{1/p_k}\right.$$
$$\left.\textit{convergent uniformly in } n\right\},$$
$$t_5(p)=\bigcap_{B>1}\left\{a=(a_k)\in\omega:\sum_k\left|\sum_{j=k}^{n}\frac{a_j}{t_j}D^s_{j-k}(-1)^{j-k}r_k\right|B^{1/p_k}<\infty\right\}.$$

Then, we have $\{\overline{\ell}_\infty(p)\}^\beta=t_4(p)$, $\{\overline{c}_0(p)\}^\beta=\{\overline{c}_0(p)\}^\gamma=t_3(p)$, $\{\overline{c}(p)\}^\beta=t_3(p)\cap t_1(p)$, $\{\overline{c}(p)\}^\gamma=t_3(p)\cap t_2(p)$ *and* $\{\overline{\ell}_\infty(p)\}^\gamma=t_5(p)$.

4 $\overline{A}(r,s,t)-Core$

Following Knopp, a core theorem is characterized by a class of matrices for which the core of the transformed sequence is included by the core of the original sequence. In the present section, we introduce a new type core, $\overline{A}(r,s,t)-$core of complex sequence and also determine the necessary and sufficient conditions on matrix A for which $\overline{A}(r,s,t)-core(Ax)\subseteq K-core(x)$ and $\overline{A}(r,s,t)-core(Ax)\subseteq st-core(x)$ for all $x\in\ell_\infty$.

Now, let us write

$$\overline{f}_n(x)=\{\overline{A}(r,s,t)x\}_n=\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_kx_k;\ \ (n,k\in\mathbb{N}).$$

Then, we can define $\overline{A}(r,s,t)-$core of complex sequence as follows:

Definition 4.1. Let G_n be the least closed convex hull containing $\overline{f}_n(x),\overline{f}_{n+1}(x)$, $\overline{f}_{n+2}(x),\ldots$. Then, $\overline{A}(r,s,t)-$core of x is the intersection of all G_n, that is,

$$\overline{A}(r,s,t)-core(x)=\bigcap_{n=1}^{\infty}G_n.$$

Then, we have following special cases.
(*i*) If $s=e$, $t_n>0$ and $r_n=\sum_{k=0}^{n}t_k\neq0$ for all n, then $\overline{A}(r,s,t)-core$ reduces to K_q-core (see [30]).
(*ii*) If $s=e$, $r_n=1/v_n$ and $t_k=v_k$, then $\overline{A}(r,s,t)-core$ reduces to $Z-core$ (see [31]).
(*iii*) If $0<\alpha<1$, $s=e$, $t_k=1+\alpha^k$ and $r_n=n+1$, then $\overline{A}(r,s,t)-core$ reduces to K_r-core (see [32]).

In fact, we define $\overline{A}(r,s,t)-$core of x by the $K-$core of the sequence $(\overline{f}_n(x))$. Therefore, we can establish the following theorem which is an analogue of $K-$core, [10].

Theorem 4.2. *For any* $z \in \mathbb{C}$*, let*

$$H_x(z) = \{w \in \mathbb{C} : |w - z| \leq \limsup_n |\overline{f}_n(x) - z|\}.$$

Then for any $x \in \ell_\infty$*,*

$$\overline{A}(r,s,t) - core(x) = \bigcap_{z \in \mathbb{C}} H_x(z).$$

Now, we prove some lemmas which will be need to the main results of this section. We define the matrix $C = (c_{nk})$ by

$$c_{nk} = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k a_{nk}; \quad (n, k \in \mathbb{N}). \tag{4.1}$$

Lemma 4.3. $C \in (\ell_\infty : \overline{c})$ *if and only if*

$$\|C\| = \sup_n \sum_k |c_{nk}| < \infty, \tag{4.2}$$

$$\lim_{n \to \infty} c_{nk} = \alpha_k, \tag{4.3}$$

$$\lim_n \sum_k |c_{nk} - \alpha_k| = 0, \tag{4.4}$$

where $\overline{c}$ is defined in [13].

Lemma 4.4. $C \in (c : \overline{c})_{reg}$ *if and only if the conditions (4.2) and (4.3) of the Lemma 4.3 hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and*

$$\lim_n \sum_k c_{nk} = 1. \tag{4.5}$$

Proof. This may be obtained in the similar way as mentioned in the proof of Lemma 4.3. So we omit details. ☐

Theorem 4.5. $C \in (st \cap \ell_\infty : \overline{c})_{reg}$ *if and only if* $C \in (c : \overline{c})_{reg}$ *hold and*

$$\lim_n \sum_{k \in E} c_{nk} = 0 \tag{4.6}$$

for every $E \subset \mathbb{N}$ *with* $\delta(E) = 0$*.*

Proof. Because of $c \subset st \cap \ell_\infty$, $C \in (c : \overline{c})_{reg}$. Now, for every $x \in \ell_\infty$ and set $E \subset \mathbb{N}$ with $\delta(E) = 0$. Now define sequence $s = (s_k)$ for all $x \in \ell_\infty$ as

$$s_k = \begin{cases} x_k, & k \in E, \\ 0, & k \notin E. \end{cases}$$

Then, since $s \in st_0$, $Cs \in \overline{c}_0$, where $\overline{c}_0$ is the space of sequences which defined [13]. Then, since

$$\sum_k c_{nk} s_k = \sum_{k \in E} c_{nk} x_k,$$

the matrix $E = (e_{nk})$ defined by

$$e_{nk} = \begin{cases} c_{nk}, & k \in E, \\ 0, & k \notin E \end{cases}$$

for all n, must belong to the class $(\ell_\infty : \overline{c})$. Hence, the necessity of (4.6) follows from Lemma 4.3.

Conversely, let $x = (x_k) \in st \cap \ell_\infty$ with $st - \lim x = p$. Then, the set E is defined by $E = \{k : |x_k - p| \geq \varepsilon\}$ has density zero and $|x_k - p| \leq \varepsilon$ if $k \notin E$. Now, we can write

$$\sum_k c_{nk} x_k = \sum_k c_{nk}(x_k - p) + p \sum_k c_{nk}. \tag{4.7}$$

Since

$$\left| \sum_k c_{nk} x_k \right| = \|x\| \sum_{k \in E} c_{nk} x_k + \varepsilon \|C\|, \tag{4.8}$$

letting $n \longrightarrow \infty$ in (4.7) and using (4.5), (4.6) we have

$$\lim_n \sum_k c_{nk} x_k = p. \tag{4.9}$$

This implies that $C \in (st \cap \ell_\infty : \overline{c})_{reg}$ and this step completes the proof. □

Now, we may begin with quoting the following lemma (see[33]) which is needed for proving our main theorem.

Lemma 4.6. *Let $A = (a_{nk})$ be an infinite matrix satisfying $\sum_k |a_{nk}| < \infty$ and $\lim_n a_{nk} = 0$. Then, there exists $y \in \ell_\infty$ with $\|y\| \leq 1$ such that*

$$\limsup_n \sum_k a_{nk} y_k = \limsup_n \sum_k |a_{nk}|.$$

Theorem 4.7. *Let $C = (c_{nk}) \in (c : \overline{c})_{reg}$. Then, $\overline{A}(r,s,t) - core(Cx) \subseteq K - core(x)$ for all $x \in \ell_\infty$ if and only if*

$$\lim_n \sum_k |c_{nk}| = 1. \tag{4.10}$$

Proof. Let $C \in (c : \overline{c})_{reg}$. The matrix $C = (c_{nk})$ satisfies the condition of Lemma 4.6, since by the condition of regularity, there exists a $y \in \ell_\infty$ with $\|y\| \leq 1$ and $\limsup_n (Cy)_n = \limsup_n \sum_k |c_{nk}|$ (see [34]). Hence,

$$\left\{ v \in \mathbb{C} : |v| \leq \limsup_n \sum_k c_{nk} y_k \right\} = \left\{ v \in \mathbb{C} : |v| \leq \limsup_n \sum_k |c_{nk}| \right\}.$$

However, since $K-core(x) \subseteq B_1(0)$, by the definition of $K-core$,

$$\left\{v \in \mathbb{C} : |v| \leq \limsup_n \sum_k |c_{nk}|\right\} \subseteq B_1(0) = \{v \in \mathbb{C} : |v| \leq 1\}.$$

This implies (4.10).

Conversely, let $v \in \overline{A}(r,s,t) - core(Cx)$. Then, for any given $t \in \mathbb{C}$, we can write

$$\begin{aligned} |v-t| &= \limsup_n |\overline{f}_n(Ax) - t| \\ &= \limsup_n \left|t - \sum_k c_{nk}x_k\right| \\ &\leq \limsup_n \left|\sum_k c_{nk}(t-x_k)\right| + \limsup_n |t| \left|1 - \sum_k c_{nk}\right| \\ &= \limsup_n \left|\sum_k c_{nk}(t-x_k)\right|. \end{aligned} \tag{4.11}$$

Now, let $\limsup_k |x_k - t| = s$. Then, for any $\varepsilon > 0$, $|x_k - t| \leq s + \varepsilon$ whenever $k \geq k_0$. Hence,

$$\begin{aligned} \left|\sum_k c_{nk}(t-x_k)\right| &= \left|\sum_{k<k_0} c_{nk}(t-x_k) + \sum_{k\geq k_0} c_{nk}(t-x_k)\right| \\ &\leq \sup_k |x_k - t| \sum_{k<k_0} |c_{nk}| + (s+\varepsilon) \sum_{k\geq k_0} |c_{nk}| \\ &\leq \sup_k |x_k - t| \sum_{k<k_0} |c_{nk}| + (s+\varepsilon) \sum_k |c_{nk}|. \end{aligned} \tag{4.12}$$

Whence, applying $\limsup_n$ to the (4.12) with combining (4.11) and using the hypothesis, we have

$$|v-t| \leq \limsup_n \left|\sum_k c_{nk}(t-x_k)\right| \leq s + \varepsilon$$

which means that $v \in K - core(x)$. This completes the proof. □

Theorem 4.8. *Let $C \in (st \cap \ell_\infty : \overline{c})_{reg}$. Then, $\overline{A}(r,s,t) - core(Cx) \subseteq st - core(x)$ for all $x \in \ell_\infty$ if and only if (4.10) holds.*

Proof. Because of $st - core(x) \subseteq K - core(x)$ for every sequence $x \in \ell_\infty$, the necessity of the condition (4.10) holds by Theorem 4.7.

Conversely, let $v \in \overline{A}(r,s,t) - core(Cx)$. If $st - \limsup |x_k - t| = l$, then for any ε, the set E is defined by $E = \{k : |x_k - t| > l + \varepsilon\}$ has density zero.

Now, we can again write (4.11)

$$\begin{aligned}\left|\sum_k c_{nk}(t - x_k)\right| &= \left|\sum_{k\in E} c_{nk}(t - x_k) + \sum_{k\notin E} c_{nk}(t - x_k)\right| \\ &\leq \sup_k |x_k - t| \sum_{k\in E} |c_{nk}| + (l+\varepsilon)\sum_{k\notin E} |c_{nk}| \\ &\leq \sup_k |x_k - t| \sum_{k\in E} |c_{nk}| + (l+\varepsilon)\sum_{k} |c_{nk}|.\end{aligned}$$

Whence, applying $\limsup_n$ to the above inequality and using conditions (4.5) and (4.6), we have

$$\limsup_n \left|\sum_k c_{nk}(t - x_k)\right| \leq l + \varepsilon. \tag{4.13}$$

If we combine (4.11) and (4.13), we have

$$|v - t| \leq st - \limsup_k |x_k - t|$$

which means that $v \in st - core(x)$. This completes the proof. □

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