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Weak and Strong Convergence Theorems of Some Iterative Methods for Common Fixed Points of Berinde Nonexpansive Mappings in Banach Spaces

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Abstract : In this paper, we construct new iteration method for approximating a common fixed point of Berinde nonexpansive mappings in a Banach space and give some sufficient conditions for weak and strong convergence of the proposed methods.

Keywords : weak and strong convergence; iterative methods; common fixed points; Berinde nonexpansive mappings; Banach spaces.
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1 Introduction

Let C be a nonempty convex subset of a Banach space X and $T : C \to C$ be a mapping. A point $x \in X$ is a fixed point of T if $Tx = x$. We denote by $F(T)$ the set of all fixed points of T. Let $T_i : C \to C,\ i = 1, 2, 3, \dots, N$ be mappings, a point $x \in C$ is a common fixed point of $\{T_i\}_{i=1}^{n}$ if $T_i x = x$, for all i.

In 2003, Berinde [1] introduced a new type of contraction, called weak contraction, and proved a fixed point theorem for this type of mappings in a complete metric space by showing that the Picard sequence converge strongly to its fixed point.

Recently, there are many iterative methods using to approximate fixed points of nonlinear mappings, such as Mann iteration and Ishikawa iteration and Noor iteration.

The Mann iteration (see [2]) is defined by $u_0 \in C$ and

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n \tag{1.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. For $\alpha_n = 1$, the iteration (1.1) called the Picard iteration.

The Ishikawa iteration (see [3]) is defined by $s_0 \in C$ and

$$\begin{aligned} t_n &= (1-\beta_n)s_n + \beta_n T s_n, \\ s_{n+1} &= (1-\alpha_n)s_n + \alpha_n T t_n \end{aligned} \tag{1.2}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

The Noor iteration (see [4]) is defined by $s_0 \in C$ and

$$\begin{aligned} u_n &= (1-\gamma_n)s_n + \gamma_n T s_n, \\ w_n &= (1-\beta_n)s_n + \beta_n T u_n, \\ s_{n+1} &= (1-\alpha_n)s_n + \alpha_n T w_n \end{aligned} \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. It is easy to see that Mann iteration and Ishikawa iteration are special case of Noor iteration.

The Noor iteration is an iteration method defined as follow :

Let $T_i : C \to C, i = 1,2,3$ be a mapping and let $\{s_n\}$ be a sequence defined by $s_0 \in C$ and

$$\begin{aligned} u_n &= (1-\gamma_n)s_n + \gamma_n T_1 s_n, \\ w_n &= (1-\beta_n)s_n + \beta_n T_2 u_n, \\ s_{n+1} &= (1-\alpha_n)s_n + \alpha_n T_3 w_n \end{aligned} \tag{1.4}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0,1]$. We note that when $T_1 = T_2 = T_3$ the iteration (1.4) reduce to the Noor iteration (1.3) for one mapping.

A mapping T is said to be *weak contraction* if there exists $L \geq 0$ and $\delta \in (0,1)$ such that

$$\|Tx - Ty\| \leq \delta \|x-y\| + L\|y - Tx\| \qquad \text{for all } x, y \in C.$$

Berinde [1] proved that in a complete metric space X, every weak contraction mapping has a fixed point and the Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

In 2013, Phuengrattana and Suantai [5] introduced the following iterative method for weak contraction.

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n-\lambda_n)y_n + \alpha_n T y_n + \lambda_n T z_n \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

where $x_1 \in C, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}$ and $\{\alpha_n + \lambda_n\}$ are sequence in $[0,1]$. They also proved a strong convergence theorem of above iterative method and compared the rate of convergence between Mann, Ishikawa, Noor and SP-iterative methods.

In this paper, we propose a new iteration method as the following :

Let C be a nonempty convex subset of a Banach space X and $T_i : C \to C$, $i = 1, 2, 3$ be a mapping. Our iteration is defined by $x_0 \in C$ and

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T_1 x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n, \\ x_{n+1} &= (1-\alpha_n)T_3 z_n + \alpha_n T_3 y_n \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

Then we prove the weak and strong convergence theorems of the proposed iteration method for approximating a common fixed point of Berinde nonexpansive mappings in a Banach space.

2 Preliminaries

We recall some definitions and useful results that will be used for our main results.

Definition 2.1. Let C be a nonempty subset of Banach space X. A mapping $T : C \to C$ is said to be *contraction* if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\| \le k\,\|x - y\| \quad \text{for all } x, y \in C. \tag{2.1}$$

Definition 2.2. Let C be a nonempty subset of Banach space X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in C. \tag{2.2}$$

Definition 2.3 (Condition $(*)$). Let C be a nonempty subset of Banach space X and a mapping $T : C \to C$ is satisfy *condition $(*)$* if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq \|x - y\| + L\|x - Tx\| \quad \text{for all } x, y \in C. \tag{2.3}$$

Definition 2.4. Let C be a nonempty subset of Banach space X. A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to be *F-contraction* if there exists $k \in [0, 1)$ such that

$$\|Tx - p\| \leq k\|x - p\| \quad \text{for all } x \in C, p \in F(T). \tag{2.4}$$

Definition 2.5. Let C be a nonempty subset of Banach space X. A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\| \quad \text{for all } x \in C, p \in F(T). \tag{2.5}$$

It is clear that a F-contraction is quasi-nonexpansive.

Definition 2.6. Let C be a nonempty subset of Banach space X. A mapping $T : C \to C$ is said to be *Berinde nonexpansive* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq \|x - y\| + L\|y - Tx\| \quad \text{for all } x, y \in C. \tag{2.6}$$

Definition 2.7. A Banach space X is said to satisfy *Opials condition* if any sequence $\{x_n\}$ in $C \subset X$, $x_n \rightharpoonup x$ as $n \to \infty$ implies that $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ for all $y \in C$ with $y \neq x$.

Definition 2.8. Let X be a normed space and $C \subset X$. A mapping $T : C \to X$ is said to be *demicompact* if for any sequence $\{x_n\}$ in X such that

$$\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty,$$

there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converge strongly to $x \in C$.

We will use the notation :

1. $\to$ for strong convergence and $\rightharpoonup$ for weak convergence.
2. $\omega(x_n) = \{x | \exists x_{n_k} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Lemma 2.9. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r, r > 0\}$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|) \tag{2.7}$$

for all $x, y, z \in B_r$ and $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

3 Main Results

We will prove weak and strong convergence theorems of our iteration method. To prove this, the following Lemma are needed.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space X and $T_i : C \to C, i = 1,2,3$ be quasi-nonexpansive mappings. Assume that $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and $\{x_n\}$ be a sequence generated by (1.6) and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. Then for $p \in \bigcap_{i=1}^{3} F(T_i)$,*

(1) $\|x_{n+1} - p\| \leq \|x_n - p\|, \quad \forall n \in \mathbb{N},$

(2) $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. (1) Let $p \in \bigcap_{i=1}^{3} F(T_i)$. By using (1.6) and $T_i : C \to C,\ i = 1,2,3$ are quasi-nonexpansive, we have

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T_1 x_n - p\| \\
&= \|(1-\gamma_n)(x_n - p) + \gamma_n (T_1 x_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n \|T_1 x_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n \|x_n - p\| = \|x_n - p\|
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n T_2 z_n - p\| \\
&= \|(1-\beta_n)(x_n - p) + \beta_n (T_2 z_n - p)\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \|T_2 z_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \|z_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \|x_n - p\| = \|x_n - p\|.
\end{aligned}$$

From above inequalities, we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)T_3 z_n + \alpha_n T_3 y_n - p\| \\
&= \|(1-\alpha_n)(T_3 z_n - p) + \alpha_n (T_3 y_n - p)\| \\
&\leq (1-\alpha_n)\|T_3 z_n - p\| + \alpha_n \|T_3 y_n - p\| \\
&\leq (1-\alpha_n)\|z_n - p\| + \alpha_n \|y_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\| = \|x_n - p\|.
\end{aligned}$$

Hence, we have $\|x_{n+1} - p\| \ \leq\ \|x_n - p\| \ \ \forall n \in \mathbb{N}$.

(2) From (1) and $\{\|x_n - p\|\}$ is non-increasing and bouded below, then we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Theorem 3.2. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T_i : C \to C$, $i = 1, 2, 3$ be Berinde nonexpansive and quasi nonexpansive mappings. Assume that $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and let $\{x_n\}$ be a sequence generated by (1.6) where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ which satisfy the following conditions*

(a) $\limsup_{n\to\infty} \alpha_n < 1$ *and* $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

(b) $\liminf_{n\to\infty} \alpha_n > 0$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(c) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then

(1) $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$, $\lim_{n\to\infty} \|x_n - T_2z_n\| = 0$ *and* $\lim_{n\to\infty} \|T_3z_n - T_3y_n\| = 0$.

(2) $\lim_{n\to\infty} \|z_n - x_n\| = 0$, $\lim_{n\to\infty} \|y_n - x_n\| = 0$ *and* $\lim_{n\to\infty} \|x_{n+1} - T_3z_n\| = 0$.

(3) $\lim_{n\to\infty} \|x_n - T_2x_n\| = 0$.

Proof. (1) Let $p \in \bigcap_{i=1}^{3} F(T_i)$. From Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists and hence $\{\|x_n - p\|\}$ is bounded. Then there exixts $M > 0$ such that $\|x_n - p\| \leq M$, $\forall n \in \mathbb{N}$. By quasi-nonexpansiveness of T_i, we have $\{x_n - p\}, \{T_1x_n - p\}, \{T_2z_n - p\}$, $\{T_3z_n - p\}, \{T_3y_n - p\} \subseteq B_M$. By Lemma 2.9, there exists a continuous strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, with $g(0) = 0$ such that

$$\begin{aligned} \|z_n - p\|^2 &= \|(1 - \gamma_n)(x_n - p) + \gamma_n(T_1x_n - p)\|^2 \\ &\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n \|T_1x_n - p\|^2 - (1 - \gamma_n)\gamma_n g(\|x_n - T_1x_n\|) \\ &\leq (1-\gamma_n) \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 - (1 - \gamma_n)\gamma_n g(\|x_n - T_1x_n\|) \\ &= \|x_n - p\|^2 - (1 - \gamma_n)\gamma_n g(\|x_n - T_1x_n\|), \end{aligned} \tag{3.1}$$

$$\begin{aligned} \|y_n - p\|^2 &= \|(1 - \beta_n)(x_n - p) + \beta_n(T_2z_n - p)\|^2 \\ &\leq (1-\beta_n) \|x_n - p\|^2 + \beta_n \|T_2z_n - p\|^2 - (1-\beta_n)\beta_n g(\|x_n - T_2z_n\|) \\ &\leq (1-\beta_n) \|x_n - p\|^2 + \beta_n \|z_n - p\|^2 - (1-\beta_n)\beta_n g(\|x_n - T_2z_n\|) \\ &\leq (1-\beta_n) \|x_n - p\|^2 + \beta_n \|x_n - p\|^2 - (1-\beta_n)\beta_n g(\|x_n - T_2z_n\|) \\ &= \|x_n - p\|^2 - (1 - \beta_n)\beta_n g(\|x_n - T_2z_n\|). \end{aligned} \tag{3.2}$$

From above inequalities, we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)T_3z_n+\alpha_nT_3y_n-p\|^2\\
&= \|(1-\alpha_n)(T_3z_n-p)+\alpha_n(T_3y_n-p)\|^2\\
&\le (1-\alpha_n)\|T_3z_n-p\|^2+\alpha_n\|T_3y_n-p\|^2\\
&\qquad -(1-\alpha_n)\alpha_ng(\|T_3z_n-T_3y_n\|)\\
&\le (1-\alpha_n)\|z_n-p\|^2+\alpha_n\|y_n-p\|^2\\
&\qquad -(1-\alpha_n)\alpha_ng(\|T_3z_n-T_3y_n\|)\\
&\le (1-\alpha_n)(\|x_n-p\|^2-(1-\gamma_n)\gamma_ng(\|x_n-T_1x_n\|)\\
&\qquad +\alpha_n(\|x_n-p\|^2-(1-\beta_n)\beta_ng(\|x_n-T_2z_n\|)\\
&\qquad\quad -(1-\alpha_n)\alpha_ng(\|T_3z_n-T_3y_n\|)\\
&\le \|x_n-p\|^2-(1-\alpha_n)(1-\gamma_n)\gamma_ng(\|x_n-T_1x_n\|)\\
&\qquad -\alpha_n(1-\beta_n)\beta_ng(\|x_n-T_2z_n\|)\\
&\qquad\quad -(1-\alpha_n)\alpha_ng(\|T_3z_n-T_3y_n\|). \qquad (3.3)
\end{aligned}
$$

From (3.3), we have

$$\|x_{n+1}-p\|^2 \;\le\; \|x_n-p\|^2-(1-\alpha_n)(1-\gamma_n)\gamma_ng(\|x_n-T_1x_n\|).$$

Hence, we have

$$(1-\alpha_n)(1-\gamma_n)\gamma_ng(\|x_n-T_1x_n\|) \;\le\; \|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Since $\lim_{n\to\infty}\|x_n-p\|$ exists and by assumption (a), we get

$$\lim_{n\to\infty} g(\|x_n-T_1x_n\|)=0.$$

Since g is continuous and $g(0)=0$, we have

$$\lim_{n\to\infty}\|x_n-T_1x_n\| \;=\; 0. \qquad (3.4)$$

From (3.3), we have

$$
\begin{aligned}
&\alpha_n(1-\beta_n)\beta_ng(\|x_n-T_2z_n\|) \;\le\; \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
\text{and}\quad &(1-\alpha_n)\alpha_ng(\|T_3z_n-T_3y_n\|) \;\le\; \|x_n-p\|^2-\|x_{n+1}-p\|^2.
\end{aligned}
$$

By $\lim_{n\to\infty}\|x_n-p\|$ exists and assumption (b) and (c), we obtain

$$\lim_{n\to\infty} g(\|x_n-T_2z_n\|)=0 \quad\text{and}\quad \lim_{n\to\infty} g(\|T_3z_n-T_3y_n\|)=0.$$

Since g is continuous and $g(0) = 0$, we have

$$\lim_{n\to\infty} \|x_n - T_2 z_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|T_3 z_n - T_3 y_n\| = 0. \tag{3.5}$$

(2) From (1.6) and results from (1), we have

$$\begin{aligned} \|z_n - x_n\| &\leq \gamma_n \|x_n - T_1 x_n\| \leq \|x_n - T_1 x_n\| \to 0, \\ \|y_n - x_n\| &\leq \beta_n \|x_n - T_2 z_n\| \leq \|x_n - T_2 z_n\| \to 0, \\ \|x_{n+1} - T_3 z_n\| &\leq \alpha_n \|T_3 z_n - T_3 y_n\| \leq \|T_3 z_n - T_3 y_n\| \to 0. \end{aligned}$$

Hence, we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0, \lim_{n\to\infty} \|y_n - x_n\| = 0 \ \text{ and } \ \lim_{n\to\infty} \|x_{n+1} - T_3 z_n\| = 0.$$

(3) For $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_n - T_2 x_n\| &\leq \|x_n - T_2 z_n\| + \|T_2 z_n - T_2 x_n\| \\ &\leq \|x_n - T_2 z_n\| + \|z_n - x_n\| + L\|x_n - T_2 z_n\|. \end{aligned}$$

It follows that $\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$. □

Theorem 3.3. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T_i : C \to C, i = 1, 2, 3$ be Berinde nonexpansive and quasi nonexpansive mappings. Assume that $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and let $\{x_n\}$ be a sequence generated by* (1.6) *where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$, and satisfly the condition* (a), (b), (c) *of Theorem 3.2. If T_1 or T_2 is demicompact, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{3}$.*

Proof. By Theorem 3.2, we have $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$. Without loss of generality, we assume that T_1 is demicompact. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q, \exists q \in C$. For each $k \in \mathbb{N}$, we have

$$\begin{aligned} \|T_1 x_{n_k} - T_1 q\| &\leq \|x_{n_k} - q\| + L\|q - T_1 x_{n_k}\| \\ &\leq \|x_{n_k} - q\| + L\|q - x_{n_k}\| + \|x_{n_k} - T_1 x_{n_k}\| \to 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \|T_1 x_{n_k} - T_1 q\| = 0. \tag{3.6}$$

By triangle inequality, we have

$$\|q - T_1 q\| \leq \|q - x_{n_k}\| + \|x_{n_k} - T_1 x_{n_k}\| + \|T_1 x_{n_k} - T_1 q\|.$$

It follows from (3.6) that $\|q - T_1q\| = 0$. Hence $T_1q = q$ and

$$\begin{aligned} \|q - T_2q\| &\leq \|q - x_{n_k}\| + \|x_{n_k} - T_2z_{n_k}\| + \|T_2z_{n_k} - T_2q\| \\ &\leq \|q - x_{n_k}\| + \|x_{n_k} - T_2z_{n_k}\| + \|z_{n_k} - q\| + L\,\|q - T_2z_{n_k}\| \\ &\leq \|q - x_{n_k}\| + \|x_{n_k} - T_2z_{n_k}\| + \|z_{n_k} - x_{n_k}\| + \|x_{n_k} - q\| \\ &\quad + L\,\|q - x_{n_k}\| + \|x_{n_k} - T_2z_{n_k}\| \ \to\ 0. \end{aligned}$$

Therefore, $\|q - T_2q\| = 0$. Hence $T_1q = q$. Since T_3 is Berinde nonexpansive, we have

$$\begin{aligned} \|T_3z_{n_k} - T_3q\| &\leq \|z_{n_k} - q\| + L\,\|q - T_3x_{n_k}\| \\ &\leq \|z_{n_k} - q\| + L\,\|q - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T_3z_{n_k}\| \ \to\ 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \|T_3z_{n_k} - T_3q\| \ =\ 0. \tag{3.7}$$

By triangle inequality, we have

$$\|q - T_3q\| \leq \|q - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T_3z_{n_k}\| + \|T_3z_{n_k} - T_3q\| \to 0.$$

Hence $\|q - T_3q\| = 0$, so $T_3q = q$. Thus $q \in \bigcap_{i=1}^{3} F(T_i)$. By Lemma (3.1), we have that $\lim_{n\to\infty} \|x_n - q\|$ exists. Since $\lim_{n\to\infty} \|x_{n_k} - q\| = 0$, it follow that $x_n \to q$. □

A mapping T is said to be *demiclosed at* 0 if $\{x_n\} \subset C, x_n \rightharpoonup x$, for some $x \in C$ and $\|x_n - Tx_n\| \to 0$, then $x \in F(T)$.

Theorem 3.4. *Let C be a nonempty closed convex subset of Banach space X that satisfies the Opials condition and $T : C \to C$ be a mapping satisfying the condition (∗). Then T is demiclosed at* 0.

Proof. Let $\{x_n\}$ be the sequence in C such that $x_n \rightharpoonup x$, for some $x \in C$ and $\|x_n - Tx_n\| \to 0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_n - Tx\| &\leq \|x_n - Tx_n\| + \|Tx_n - Tx\| \\ &\leq \|x_n - Tx_n\| + \|x_n - x\| + L\,\|x_nTx_n\|, \end{aligned}$$

which implies

$$\liminf_{n\to\infty} \|x_n - Tx\| \ \leq\ \liminf_{n\to\infty} \|x_n - x\|.$$

Suppose that $Tx \neq x$, by Opials condition, we have

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - Tx\| &\leq \liminf_{n\to\infty} \|x_n - x\| \\ &< \liminf_{n\to\infty} \|x_n - Tx\|,\end{aligned}$$

which is a contradiction. Therefore, $Tx = x$. □

We next prove weak convergence of the iteration (1.6) to a common fixed point of Berinde nonexpansive mappings.

Theorem 3.5. *Let X be a uniformly convex Banach space having Opials condition and C be a nonempty closed convex subset of X and let $T_i : C \to C,\ i = 1, 2, 3$ be Berinde nonexpansive mappings such that T_1 and T_2 satisfy condition (∗) and T_3 is weakly continuous. Suppose $\{x_n\}$ be a sequence generated by (1.6) where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ which satisfy the following conditions:*

(a) $\limsup_{n\to\infty} \alpha_n < 1$ *and* $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

(b) $\liminf_{n\to\infty} \alpha_n > 0$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(c) $\lim_{n\to\infty} \alpha_n = \alpha$, *for some* $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges weakly to $x \in \bigcap_{i=1}^{3} F(T_i)$.

Proof. By Theorem 3.2, we obtain

(1) $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$, $\lim_{n\to\infty} \|x_n - T_2z_n\| = 0$ and $\lim_{n\to\infty} \|T_3z_n - T_3y_n\| = 0$,

(2) $\lim_{n\to\infty} \|z_n - x_n\| = 0$, $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and $\lim_{n\to\infty} \|x_{n+1} - T_3z_n\| = 0$,

(3) $\lim_{n\to\infty} \|x_n - T_2x_n\| = 0$.

By Lemma 3.1, we know that a sequence $\{x_n\}$ is bounded, so it has a weakly convergent subsequence. We may assume that $x_n \rightharpoonup x$, for some $x \in C$. By (2), we obtain $z_n \rightharpoonup x$ and $y_n \rightharpoonup x$. It follows by Theorem 3.4 that $x \in F(T_1)$ and $x \in F(T_2)$. From

$$x_{n+1} = (1 - \alpha_n)T_3z_n + \alpha_nT_3y_n$$

and T_3 is weakly continuous, we obtain

$$x_{n+1} = (1-\alpha_n)T_3 z_n + \alpha_n T_3 y_n \rightharpoonup (1-\alpha)T_3 x + \alpha T_3 x = T_3 x,$$

which implies $x = T_3 x$, this is $x \in F(T_3)$. Therefore, $x_n \rightharpoonup x$ and $x \in \bigcap_{i=1}^{3} F(T_i)$. □

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