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A Graphical Proof of the Brouwer Fixed Point Theorem

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Abstract : By simplifying the proof in [1], we give a new proof of the Brouwer fixed point theorem without using the Tietze (continuous) extension theorem.

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1 Introduction

The Brouwer fixed point theorem states that: For the unit cube $[0,1]^d$ of the Euclidean space $\mathbb{R}^d$, any continuous mapping $T:[0,1]^d \to [0,1]^d$ has a fixed point, i.e., a point $x \in [0,1]^d$ with $T(x) = x$.

As in [1], we proof the theorem by induction on the dimension d.

2 Preliminaries

We recall notations introduced in [1]. Put $K = [0,1]^d$ and let $\{e_1, \ldots, e_d\}$ be the standard basis for $\mathbb{R}^d$, that is, based on the Kronocker delta δ_{ji}, $e_j = (\delta_{ji})_{i=1}^d$. For $j = 1, \ldots, d$, write $\square_j = \left\{\sum_{i=1.i\neq j}^d x_i e_i : 0 \leq x_i \leq 1, i = 1, \ldots, d, i \neq j\right\}$ and $\square_{j'} = \square_j + e_j$. Let H_u for $0 \leq u \leq \sqrt{d}$ be the hyperplane passing through $(u, \ldots, u) \in \mathbb{R}^d$ having $\bar{1} = (1,1,...,1) \in \mathbb{R}^d$ as its normal vector and put $\Delta_u = K \cap H_u$.

The mapping

$$\pi_p : (x_1, \ldots, x_d) \mapsto \sum_{i=1, i\neq p}^{d} x_i e_i \ , \quad \text{for } (x_1, \ldots, x_d) \text{ in } K$$

is the projection onto $\square_p$ for $p = j, j'$. Set $\square_{uj}$ to be the component of a subset of $[\square_j \setminus \pi_j(\Delta_u)] \cup [\square_{j'} \setminus \pi_{j'}(\Delta_u)] \cup \Delta_u$ containing Δ_u. Above the face $\square_j$, let S_{uj} be the continuous surface consisting of Δ_u together with $\square_{uj}$.

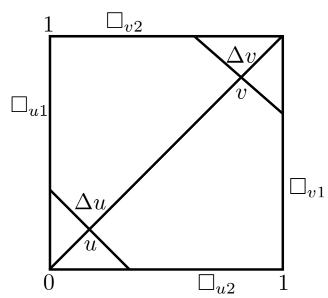

Figure 1

Write the given continuous function $T = (f_1, \ldots, f_d)$ where $f_j : K \to [0,1]$ is continuous for each j. For each u, draw the graph of f_j restricted to S_{uj} via the formula

$$\begin{aligned} g_{uj} : (x_1, \ldots, &x_{j-1}, 0, x_{j+1}, \ldots, x_d) \\ &\mapsto (x_1, \ldots, x_{j-1}, f_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d), x_{j+1}, \ldots, x_d) \end{aligned}$$

for each $(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) \in S_{uj}$.

Observe that $(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d) = \pi_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d)$. Thus the graph of f_j at u means the set of points

$$(x_1, \ldots, x_{j-1}, f_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d), x_{j+1}, \ldots, x_d)$$

for $(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) \in S_{uj}$. Write f_{uj} for $f_j\big|_{S_{uj}}$.

Our proof relies on this result:

Lemma 2.1. [1, Lemma 3.1] *For each u, the graphs of $f_1, \ldots, f_d$ intersect at a point.*

In the sequent, we will refer to "a point of intersection of the graphs of $f_1, \ldots, f_d$" shortly as "a point of intersection of $f_1, \ldots, f_d$". Define $F_u(f_1, ..., f_k)$ to be the set of the points of intersection of $f_1, .., f_k$. For example, if s_1 is the identity mapping on S_{u1}, $F_u(s_1, f_2)$ is the intersection of the graph of f_2 and S_{u1}. The negative part $N^0(f_1, u)$ and the positive part $P^0(f_1, u)$ of f_1 over S_{u1} are defined as

$$\begin{aligned} N^0(f_1, u) &= \{(x_1, \ldots, x_d) \in S_{u1} : f_1(x_1, \ldots, x_d) < x_1\}, \\ P^0(f_1, u) &= \{(x_1, \ldots, x_d) \in S_{u1} : f_1(x_1, \ldots, x_d) > x_1\}. \end{aligned}$$

The nonpositive $N(f_j, u)$ and the nonnegative $P(f_j, u)$ are defibed by replacing $<$ and $>$ by $\leq$ and $\geq$ respectively. Clearly, $N^0(f_1, u)$ and $P^0(f_1, u)$ can be partitioned into relatively open components, say,

$$N^0(f_1, u) = \bigcup_\alpha N^0_\alpha(f_1, u) \ , \quad P^0(f_1, u) = \bigcup_\beta P^0_\beta(f_1, u).$$

For each pair (u, u_0) for which $0 < u < u_0 < \sqrt{d}$ and for a continuous mapping $h_{uj} : S_{uj} \to [0, 1]$, we write $H^{u_0}_{uj}$ for a copy of h_{uj} by translating h_{uj} along the vector $\frac{u_0 - u}{\sqrt{d}} e_j$. We may need to project $H^{u_0}_{uj}$ back to K if necessary. Thus, $H^{u_0}_{uj}$ can be considered as a continuous mapping defined on $S_{u_0 j}$.
In the course of the proof, we need the following construction:

(2.1) For a given nonempty closed subset A of Δ_u formed by a finite union of closed (d-1) - dimensional boxes and for a pair of continuous mappings $g, h : S_{uj} \to [0, 1]$, we draw the segment joining $g(x)$ and $h(x)$ for $x \in A$. By slight shrinking the graph of g over A and call the new mapping as $\hat{g}$, we obtain a continuous surface $\hat{h}$ so that $\hat{h} = h$ over $S_{uj} \setminus A$ and $\hat{h} = \hat{g}$ over A.

We will apply the construction (2.1) to $(g, h) = (f_{u_0 j}, H)$ where u_0 and H are to be specified later.

3 Proof

Assume that $\bar{0}$ is not a fixed point of T and suppose that $f_1(\bar{0}) > 0$. We shall consider $N^0(f_1, u)$ when u moves from 0 toward $\sqrt{d}$. Obviously, under the above assumption, $N^0(f_1, u) = \emptyset$ for all small u. It is also clear that each point in the intersection

$$F_u \coloneqq F_u(s_1, f_2, f_3, \ldots, f_d) \cap \{(x_1, \ldots, x_d) \in \Delta_u : f(x_1, \ldots, x_d) = x_1\}$$

is a fixed point of T.

For each u, we say that f_2 and f_3 are removable from S_{u1} if there are mappings h_2 and h_3 such that $h_j = f_j$ for $j = 2, 3$ on $P(f_1, u)$ and $F_u(s_1, h_2, h_3, f_4, \dots, f_d) \cap N^0(f_1, u) = \emptyset$. The term "removable" describes the removal of points in $F_u(s_1, f_2, f_3, f_4, \dots, f_d) \cap N^0(f_1, u)$. Let

$$\mathcal{U} = \{u > 0 : \text{for each } v \leq u, F_v = \emptyset \text{ and } f_2, f_3 \text{ are removable from } \; S_{v1}\}.$$

Clearly $\mathcal{U} \neq \emptyset$, let $u_0 = \sup \mathcal{U}$. If $u_0 = \sqrt{d}$, then $\bar{1}$ is a fixed point. This follows from Lemma 2.1 and the fact that under new definition of f_2 and f_3, $F_u(s_1, f_2, f_3, \dots, f_d) \cap N^0(f_1, u) = \emptyset$ for all $u < \sqrt{d}$. Now suppose $u_0 < \sqrt{d}$. If $F_{u_0} \neq \emptyset$, we are done. If $F_{u_0} = \emptyset$, we will find a contradiction. First construct a subset A_α of $N^0_\alpha(f_1, u_0)$ formed by a finite union of (d-1) - dimensional boxes lie in each $N^0_\alpha(f_1, u_0)$ for which $F_{u_0}(s_1, f_2, f_3, \dots, f_d) \cap N^0_\alpha(f_1, u_0) \neq \emptyset$. The set A_α can be constructed so that

$$[F_u(s_1, f_2, f_3, \dots, f_d) + \tfrac{u_0 - u}{\sqrt{d}} e_j] \cap N^0_\alpha(f_1, u_0) \subset A_\alpha$$

for all $u < u_0$ with $u_0 - u$ sufficiently small. For some such u, we apply construction (2.1) to $(g, h) = (f_{u_0 j}, H^{u_0}_{u_j})$ for $j = 2, 3$. It is observed by continuity that, for some u with $u_0 - u$ sufficiently small, $F_{u_0}(s_1, \hat{H}^{u_0}_{u_2}, \hat{H}^{u_0}_{u_3}, f_4, \dots, f_d) \cap N^0(f_1, u_0) = \emptyset$. This shows that $u_0 \in \mathcal{U}$. With the similar argument, we can show that $u \in \mathcal{U}$ for some (and actually for all) $u > u_0$ with $u - u_0$ sufficiently small. We do this by letting (u_0, u) take the role of (u, u_0) in the previous case, and this leads to a contradiction as claimed.

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