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Linesearch Algorithms for Split Generalized Equilibrium Problems and Two Families of Strict Pseudo-Contraction Mappings

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Abstract : In this paper, we study linesearch algorithms finding a common solution of a split generalized equilibrium problems and two families of strict pseudo-contraction mappings in Hilbert spaces. Weak and strong convergence theorems for such algorithms are studied. Our results improve many known recent results in the literature.

Keywords : split generalized equilibrium problem; strict pseudo-contraction; linesearch rule; weak and strong convergence.
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1 Introduction

Throughout this paper, let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{N}$ denote the set of all positive integer numbers. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The split feasible problem (SFP) in the sense of Censor and Elfving [1] is to find $x^* \in C$ such that $Ax^* \in Q$. It turns out that SFP provides a unified framework for study

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of many sigificant real-world problems such as in signal processing, medical image reconstruction, intensity-modulated radiation therapy, *et cetera*; see, for example, [2]. To find a solution of SFP in finite-dimensional Hilbert spaces, a basic scheme proposed by Byrne [3], called the CQ-algorithm, is defined as follows:

$$x^{k+1} = P_C(x^k + \gamma A^T(P_Q - I)Ax^k),$$

where I is the identity, mapping, and P_C is projecttion mapping onto C. Xu [4] investigated the SEP setting in infinite-dimensional Hilbert spaces. In this case, the CQ-algorithm becomes

$$x^{k+1} = P_C(x^k + \gamma A^*(P_Q - I)Ax^k),$$

where A^* is the adjoint operator of A.

The split feasibility problem when C or Q are fixed points of mappings or common fixed points of mappings and solutions of variational inequality problems was considered in some recent research papers; see, for instance, [5].

In 2011, Moudafi [6] introduced the following split equilibrium problem (SEP, for short): Let $g_1 : C \times C \to \mathbb{R}$ and $g_2 : Q \times Q \to \mathbb{R}$ are two bifunctions; $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, then the SEP is to find $x^* \in C$ such that

$$g_1(x^*, x) \geq 0, \forall x \in C, \tag{1.1}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } g_2(y^*, y) \geq 0, \forall y \in Q. \tag{1.2}$$

When looked separately, (1.1) is the classical equilibrium problem EP and we denoted its solution set by $\mathrm{EP}(C, g_1)$. The SEP (1.1) and (1.2) constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of the EP (1.1) in $\mathcal{H}_1$ is the solution of another EP(1.2) in another space $\mathcal{H}_2$, we denote the solution set of EP(1.2) by $\mathrm{EP}(Q, g_2)$.

The solution set of SEP (1.1) and (1.2) is denoted by

$$\Omega = \{p \in \mathrm{EP}(C, g_1) : Ap \in \mathrm{EP}(Q, g_2)\}.$$

See [7] for more detail on equilibrium problems.

In 2013, Kazmi and Rizvi [8] proposed the split generalized equilibrium problem (SGEP, for short): SGEP is the problem of finding $x^* \in C$ such that

$$g_1(x^*, x) + h_1(x^*, x) \geq 0, \forall x \in C, \tag{1.3}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } g_2(y^*, y) + h_2(y^*, y) \geq 0, \forall y \in Q. \tag{1.4}$$

where $g_1, h_1 : C \times C \to \mathbb{R}$ and $g_2, h_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $A : H_1 \to H_2$ is a bounded linear operator. We denote the solution set of SGEP

(1.3) and (1.4) by $S_{\text{GEP}}(C, g_1, h_1)$ and $S_{\text{GEP}}(Q, g_2, h_2)$, respectively. The solution set of SGEP is denoted by

$$S_{\text{GEP}} := \{z \in C : z \in S_{\text{GEP}}(C, g_1, h_1) \text{ such that } Az \in S_{\text{GEP}}(Q, g_2, h_2)\}.$$

If $h_1 = 0$ and $h_2 = 0$, then SGEP reduces to SEP. If $h_1 = h_2 = 0$ and $g_2 = 0$, then SGEP reduces to EP.

On the other hand, many researchers have been proposed numerical algorithms for finding a common element of the set of solutions of monotone equilibrium problems and the set of fixed points of nonexpansive mappings;, for example, [9], [10] and the references therein.

Recently, Dinh, *et al.* [11] studied the split equilibrium problem and nonexpansive mapping involving pseudomonotone and monotone equilibrium bifunctions in real Hilbert spaces, that is, let $f : C \times C \to \mathbb{R}$ be a pseudomonotone bifunction with respect to its solution set, $g : Q \times Q \to \mathbb{R}$ be a monotone bifunction, and $S : C \to C$ and $T : Q \to Q$ be nonexpansive mappings. They stated problem as follows (SEPNM(C, Q, A, f, g, S, T)) or SEPNM for short):

$$\text{Find } x^* \in C \text{ such that } x^* \in S_{\text{EP}}(C, f) \cap \text{Fix}(S) \text{ and } Ax^* \in S_{\text{EP}}(Q, g) \cap \text{Fix}(T),$$

where Fix(S) and Fix(T) are the fixed points of the mappings S and T, respectively. They combined the extragradient method incorporated with the Armijo linesearch rule for solving equilibrium problem and the Mann method for finding a fixed point of an nonexpansive mapping. In addition, they combined the proposed algorithm with hybrid cutting technique to get a strong convergence algorithm for SEPNM.

We recall that a mapping $S : C \to C$ is said to be *L-strict pseudo-contractive* (in the sense of Browder-Petryshyn) if there exists $L \in [0, 1)$ such that

$$\|S(x) - S(y)\|^2 \leq \|x - y\|^2 + L\|(I - S)(x) - (I - S)(y)\|^2, \forall x, y \in C, \tag{1.5}$$

where I is the identity mapping on $\mathcal{H}$. Note that the class of strict pseudo-contractions includes the class of nonexpansive mappings, which are mappings S on C such that

$$\|S(x) - S(y)\| \leq \|x - y\|, \forall x, y \in C.$$

The problem of finding fixed points of nonexpansive mappings via Mann's algorithm [12] has been widely investigated in the literature (see e.g. [13]). Mann's algorithm generates, on initializing with an arbitrary $x_1 \in C$, a sequence according to the recursive formula

$$x_1 \in C, x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n, \forall n \geq 1,$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0, 1)$. Furthermore, iterative algorithms for strict pseudo-contractions are still less developed than those for nonexpansive mappings, despite the pioneering work of Browder and Petryshyn [14] dating from 1967. However, strict pseudo-contractions have many applications, due to their ties with inverse stronglymonotone operators. Indeed, if A is a strongly monotone operator, then $S = I - A$ is a strict pseudo-contraction, and so we can redraft a problem of zeros for A in a fixed point problem for S (see e.g. [15], [16]).

This paper, we propose on a split generalized equilibrium problems and two families of strict pseudo-contraction mappings in Hilbert spaces. In detail,let $f : C \times C \to \mathbb{R}$ be a pseudomonotone bifunction with respect to its solution set, $g, h : Q \times Q \to \mathbb{R}$ be a monotone bifunction, and S_i and T_j are L_i and L'_j-strict pseudo-contractions for some $0 \leq L_i < 1$ and $0 \leq L'_j < 1$, respectively, where for each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, p'\}$. The problem considered in this paper can be stated as follows:

$$\text{Find } \bar{x} \in C \text{ such that } \bar{x} \in S_{\text{EP}}(C, f) \cap \big(\cap_{i=1}^{p} \text{Fix}(S_i, C)\big)$$

and

$$A\bar{x} \in S_{\text{GEP}}(Q, g, h) \cap \big(\cap_{j=1}^{p'} \text{Fix}(T_j, Q)\big),$$

where $\text{Fix}(S_i, C)$ is the set of the fixed points set of the mapping $S_i (i = 1, \ldots, p)$ and $\text{Fix}(T_j, Q)$ is the set of the fixed points set of the mapping $T_j (j = 1, \ldots, p')$.

In this paper, motivated and inspired by the work of Dinh, *et al.* [11] and by research going on this area, we shall introduce a linesearch algorithms for split generalized equilibrium problems and two families of strict pseudo-contraction mappings in Hilbert space. Weak and strong convergence theorems for such algorithms are studied. Our results complement many known recent results in the literature.

2 Preliminaries

Let C be a nonempty convex subset of a Hilbert space $\mathcal{H}$. We write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x as $k \to \infty$, and $x^k \to x$ to indicate that the sequence $\{x^k\}$ converges strongly to x as $k \to \infty$. Since C is closed, convex, for any $x \in \mathcal{H}$, there exists an uniquely point in C, denoted by $P_C(x)$ satisfying

$$\|x - P_C(x)\| \leq \|x - y\|, \forall y \in C.$$

P_C is called the metric projection of $\mathcal{H}$ to C.

Lemma 2.1. *Suppose that C is a nonempty closed convex subset in $\mathcal{H}$. Then P_C has the following properties:*

(a) $z = P_C(x)$ *if and only if* $\langle x - z, y - z\rangle \leq 0, \forall y \in C$;

(b) $\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2, \quad \forall x, y \in \mathcal{H}$;

(c) $\langle x - P_C(x), P_C(x) - y\rangle \geq 0, \quad \forall x \in \mathcal{H}, y \in C$;

(d) $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \quad \forall x \in \mathcal{H}, y \in C.$

Lemma 2.2. *Let $\mathcal{H}$ be a real Hilbert space. Then, for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$, we have*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 2.3 (Opial's condition)**.** *For any sequence $\{x^k\} \subset \mathcal{H}$ with $x^k \rightharpoonup x$, we have the inequality*

$$\liminf_{k\to+\infty} \|x^k - x\| < \liminf_{k\to+\infty} \|x^k - y\|$$

hold for all $y \in \mathcal{H}$ such that $y \neq x$.

The concept of strict pseudo-contraction is considered in [17], which defined as follows.

Definition 2.4. We say that an operator $S : \mathcal{H} \to \mathcal{H}$ is *demiclosed at* 0 if, for any sequence $\{x^k\}$ such that $x^k \rightharpoonup x$ and $Sx^k \to 0$ as $k \to \infty$, we have $Sx = 0$.

The following proposition lists some useful properties of a strict pseudo-contraction mapping.

Proposition 2.5. [17] *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $S : C \to C$ be a L-strict pseudo-contraction and for each $i = 1, \cdots, p, S_i : C \to C$ is a L_i -strict pseudo-contraction for some $0 \le L_i < 1$. Then:*

1. *S satisfies the following Lipschitz condition:*

$$\|S(x) - S(y)\| \le \frac{1+L}{1-L}\|x-y\|, \ \forall x, y \in C;$$

2. *$I - S$ is demiclosed at 0. That is, if the sequence $\{x^k\}$ contains in C such that $x^k \rightharpoonup \bar{x}$ and $(I-S)(x^k) \to 0$ then $(I-S)(\overline{x}) = 0$;*
3. *The set of fixed points $Fix(S)$ is closed and convex;*
4. *If $\eta_i > 0 (i = 1, \cdots, p)$ and $\sum_{i=1}^p \eta_i = 1$ then $\sum_{i=1}^p \eta_i S_i$ is a $\overline{L}$-strict pseudo-contraction with $\overline{L} := \max\{L_i : 1 \le i \le p\}$;*
5. *If η_i is chosen as in* (iv) *and $\{S_i : i = 1, \ldots, p\}$ has a common fixed point then:*

$$\mathrm{Fix}\left(\sum_{i=1}^p \eta_i S_i\right) = \bigcap_{i=1}^p \mathrm{Fix}(S_i, C).$$

Lemma 2.6. [18] *Suppose that $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences of nonnegative real numbers such that*

$$\alpha_{k+1} \le \alpha_k + \beta_k, \quad k \ge 0,$$

where $\sum_{k=0}^\infty \beta_k < \infty$. Then the sequence $\{\alpha_k\}$ is convergent.

Now, we assume that the equilibrium bifunction f, g and h satisfy the following assumptions I, II and III, respectively.

Assumption I : Assume that $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) f is pseudomonotone on C, that is, if $f(x, y) \geq 0$ implies $f(y, x) \leq 0$ for all $x, y \in C$;

(A2) $f(x, \cdot)$ is convex and subdifferentiable on C for all $x \in C$;

(A3) f is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x^k\}$ and $\{y^k\} \subset C$ converge weakly to x and y, respectively, then $f(x^k, y^k) \to f(x, y)$ as $k \to \infty$.

Assumption II : Assume that $g : Q \times Q \to \mathbb{R}$, let us assume that g satisfies the following conditions:

(B1) $g(x, x) = 0$ for all $x \in Q$;

(B2) g is monotone, i.e, $g(x, y) + g(y, x) \leq 0$ for all $x, y \in Q$;

(B3) for each $x, y, z \in Q$, $\lim_{t \to 0} g(tz + (1 - t)x, y) \leq g(x, y)$;

(B4) for each $x \in Q$, $y \mapsto g(x, y)$ is convex and lower semicontinuous.

Assumption III : Let the bifunction $h : Q \times Q \to \mathbb{R}$ be satisfied

(C1) $h(x, x) \geq 0, \forall x \in Q$,

(C2) For each $y \in Q$ fixed, the function $x \longmapsto h(x, y)$ is upper semicontinuous,

(C3) For each $x \in Q$ fixed, the function $y \longmapsto h(x, y)$ is convex and lower semicontinuous,

Assumption IV : For fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H and $x \in C \cap K$ such that

$$f(x, y) + h(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \forall y \in C \backslash K.$$

Let f be an equilibrium bifunction defined on $C \times C$. For $x, y \in C$, we denote by $\partial f(x, y)$ the subgradient of the convex function $f(x, \cdot)$ at y, that is,

$$\partial f(x, y) := \big\{\hat{t} \in \mathcal{H} : f(x, z) \geq f(x, y) + \langle \hat{t}, z - y\rangle, \text{ for all } z \in C\big\}.$$

In particular,

$$\partial f(x, x) := \big\{\hat{t} \in \mathcal{H} : f(x, z) \geq \langle \hat{t}, z - y\rangle, \text{ for all } z \in C\big\}.$$

Let Δ be an open convex set containing C. The next lemma can be considered as infinite-dimentional version of Theorem 24.5 in [19].

Lemma 2.7. [20] *Let $f : \Delta \times \Delta \to \mathbb{R}$ be an equilibrium bifunction satisfiying condition* (B2) *on Δ and* (B4) *on C. Let $\bar{x}, \bar{y} \in \Delta$, and let $\{x^k\}, \{y^k\}$ be two sequences in Δ converging weakly to $\bar{x}, \bar{y}$, respectively. Then, for any $\varepsilon > 0$, there exist $\eta > 0$ and $k_\varepsilon \in \mathbb{N}$ such that*

$$\partial f\big(x^k, y^k\big) \subset \partial f\big(\bar{x}, \bar{y}\big) + \frac{\varepsilon}{\eta}B$$

for every $k > k_\varepsilon$, where B denotes the closed unit ball in $\mathcal{H}$.

Lemma 2.8. [11] *Let the equilibrium bifunction f satisfy assumptions* (B2) *on Δ and* (B4) *on C, and $\{x^k\} \subset C, 0 < \rho' \leq \rho'', \{\rho_k\} \subset [\rho', \rho'']$. Consider the sequence $\{y^k\}$ defined as*

$$y^k = argmin\Big\{ f\big(x^k, y\big) + \frac{1}{2\rho_k}\|y - x^k\|^2 : y \in C \Big\}.$$

If $\{x^k\}$ is bounded, then $\{y^k\}$ is also bounded.

Lemma 2.9. [21] *Let g satisfy Assumtion II. Then, foe all $r > 0$ and $u \in \mathcal{H}$, there exists $w \in Q$ such that*

$$g(w, v) + \frac{1}{\alpha}\langle v - w, w - u\rangle \geq 0, \forall v \in Q.$$

Lemma 2.10. [22] *Assume that the bifunctions $g, h : Q \times Q \to \mathbb{R}$ satisfy Assumption II, Assumption III, respectively. For $\alpha > 0$ and $x \in \mathcal{H}$, define a mapping $T_\alpha^{(g,h)} : \mathcal{H} \to Q$ as follows:*

$$T_\alpha^{(g,h)}(x) = \Big\{ z \in Q : g(z, y) + h(z, y) + \frac{1}{\alpha}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in Q \Big\}.$$

Then, the following hold:

(i) $T_\alpha^{(g,h)}(x) \neq \emptyset$.

(ii) $T_\alpha^{(g,h)}$ *is single-valued.*

(iii) $T_\alpha^{(g,h)}$ *is firmly nonexpansive, i.e., for any $x, y \in \mathcal{H}$,*

$$\|T_\alpha^{(g,h)}x - T_\alpha^{(g,h)}y\|^2 \leq \langle T_\alpha^{(g,h)}x - T_\alpha^{(g,h)}y, x - y\rangle.$$

(iv) $\mathrm{Fix}(T_\alpha^{(g,h)}) = S_{\mathrm{GEP}}(Q, g, h)$.

(v) $S_{\mathrm{GEP}}(Q, g, h)$ *is compact and convex.*

Lemma 2.11. [23] *Let $g : Q \times Q \to \mathbb{R}$ be a bifunction satisfying* Assumption II *hold and let $T_\alpha^{(g,h)}$ be defined as in Lemma 2.10 with $\alpha, \beta > 0$. Then, for any $x, y \in \mathcal{H}$ and*

$$\|T_\alpha^{(g,h)}x - T_\beta^{(g,h)}y\| \leq \|x - y\| + \Big|\frac{\beta - \alpha}{\beta}\Big| \|T_\beta^{(g,h)}x - x\|.$$

Lemma 2.12. [23] *Let $g : Q \times Q \to \mathbb{R}$ be a bifunction satisfying* Assumption II *and $T_\alpha^{(g,h)}, T_\beta^{(g,h)}$ be defined as in Lemma 2.10 with $\alpha, \beta > 0$. Then the following holds:*

$$\|T_\alpha^{(g,h)}x - T_\beta^{(g,h)}x\|^2 \leq \frac{\alpha - \beta}{\alpha}\langle T_\alpha x - T_\beta x, T_\alpha x - x\rangle$$

for all $x \in \mathcal{H}$.

Lemma 2.13. [24] *Let C be a convex subset of a real Hilbert space $\mathcal{H}$ and $g : C \to \mathbb{R}$ be subdifferentiable on C. Then x^* is a solution to the following convex problem:*

$$\min\{g(x) : x \in C\}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the (outward) normal cone of C at $x^* \in C$.

3 Main Results

3.1 A Weak Converegence Algorithm

Algorithm I : Initialization.

- Pick $x^0 \in C$ and choose the parameters $\beta, \eta, \theta \in (0,1)$, $0 < \rho' \leq \rho''$, $\{\rho_k\} \subset [\rho', \rho'']$, $0 < \gamma' \leq \gamma'' < 2$, $\{\gamma_k\} \subset [\gamma', \gamma'']$, $0 < \alpha$, $\{\alpha_k\} \subset [\alpha, \infty)$, $\mu \in (0, \frac{1}{\|A\|})$.
- For each $i = 1, 2, \ldots, p$, $\{\eta_{k,i}\}$ is a real sequence of nonnegative numbers satisfying $\sum_{i=1}^{p} \eta_{k,i} = 1$ for all $k \geq 1$.
- For each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, p'\}$, $S_i : C \to C$ and $T_j : Q \to Q$ are L_i and L'_j-strict pseudo-contractions for some $0 \leq L_i < 1$ and $0 \leq L'_j < 1$, respectively.
- $\{\beta_k\}$ is a nonnegative real sequence satisfying $0 < \bar{L} < \beta_k < 1$ and $\beta_k \to \frac{1}{2}$ as $k \to \infty$, where $\bar{L} := \max\{L_i : i = 1, 2, \ldots, p\}$ and $L' := \max\{L'_j : j = 1, 2, \ldots, p'\}$.

For each $k, (k = 0, 1, 2, \ldots)$, the sequence $\{x_k\}$ is generated by the following steps:

Step I : Solve the strongly convex program :

$$y^k := \operatorname{argmin}\Big\{ f(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 : y \in C \Big\}$$

to obtain its unique solution y^k. If $y^k = x^k$,then set $u^k = x^k$ and go to Step III. Otherwise, go to Step II.

Step II : (Armijo linesearch rule) Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^m, \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\theta}{2\rho_k}\|x^k - y^k\|^2. \end{cases} \tag{3.1}$$

Set $\eta_k = \eta^{m_k}, z^k = z^{k,m_k}$.

Step III : Take $t^k \in \partial f(z^k, x^k)$, $\sigma_k = \frac{f(z^k, x^k)}{\|t^k\|^2}$, and denote

$$\begin{cases} u^k = P_C(x^k - \gamma_k \sigma_k t^k), \\ v^k = \beta_k u^k + (1 - \beta_k)\sum_{i=1}^{p} \eta_{k,i} S_i(u^k), \\ w^k = T^{(g,h)}_{\alpha_k} A v^k, \\ x^{k+1} := P_C(v^k + \mu A^*(\sum_{j=1}^{p'} \eta'_{k,j} T_j(w^k) - Av^k)) \end{cases}$$

and go to iteration k with k replaced by $k + 1$.

Applying Lemma 4.1, Lemma 4.2 and Lemma 4.3 obtained in [25], we obtain the following Lemma immediately.

Lemma 3.1. *Suppose that $p \in \mathrm{EP}(C, f)$, $f(x, \cdot)$ is convex subdifferentiable on C for all $x \in C$ and that f is pseudomonotone on C. Then, we have:*

(a) *The Armijo linesearch rule* (3.1) is well defined;

(b) $f(z^k, x^k) > 0$;

(c) $0 \notin \partial_2 f(z^k, x^k)$;

(d) $\|u^k - p\|^2 \leq \|x^k - p\|^2 - \gamma_k(2-\gamma_k)\big(\sigma_k\|t^k\|\big)^2$.

Now, we are in a position to state and prove the main weak convergence theorem for the given iterative scheme.

Theorem 3.2. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two real Hilbert spaces and $C \subset \mathcal{H}_1, Q \subset \mathcal{H}_2$ be nonempty closed convex subsets. For each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, p'\}$, $S_i : C \to C$ and $T_j : Q \to Q$ are L_i and L'_j-strict pseudo-contractions for some $0 \leq L_i < 1$ and $0 \leq L'_j < 1$, respectively. Let the bifunctions f, g and h satisfy Assumptions I, II and III, respectively. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint A^*. If*

$$\Omega := \left\{ x^* \in S_{\text{EP}}(C,f) \cap \Big(\bigcap_{i=1}^{p} \text{Fix}(S_i, C) \Big) : Ax^* \in S_{\text{GEP}}(Q,g,h) \cap \Big(\bigcap_{j=1}^{p'} \text{Fix}(T_j, Q) \Big) \right\}$$

is nonempty set, then the sequences $\{x^k\}, \{u^k\}$ and $\{v^k\}$ generated by Algorithm I converge weakly to an element $\bar{x} \in \Omega$, and $\{w^k\}$ converges weakly to $A\bar{x}$.

Proof. Let $x^* \in \Omega$. Then $x^* \in S_{\text{EP}}(C,f) \cap \big(\bigcap_{i=1}^{p} \text{Fix}(S_i, C)\big)$ and $Ax^* \in S_{\text{GEP}}(Q,g,h) \cap \big(\bigcap_{j=1}^{p'} \text{Fix}(T_j, Q)\big)$. From Lemma 3.1(d), we have

$$\|u^k - x^*\|^2 \quad \leq \quad \|x^k - x^*\|^2 - \gamma_k(2-\gamma_k)\big(\sigma_k\|t^k\|\big)^2 \leq \|x^k - x^*\|^2. \qquad (3.2)$$

For each $k \geq 1$, let the mapping $\bar{S_k}$ be given by

$$\bar{S_k} := \sum_{i=1}^{p} \eta_{k,i} S_i.$$

By Proposition 2.5, we see that $\bar{S_k}$ is a $\bar{L}$-strict pseudocontraction on C. Then, for all $k \geq 1$, we have

$$\begin{aligned}
\|v^k - x^*\|^2 &= \|\beta_k u^k + (1-\beta_k)\bar{S_k}(u^k) - x^*\|^2 \\
&= \|\beta_k(u^k - x^*) + (1-\beta_k)(\bar{S_k}(u^k) - x^*)\|^2 \\
&= \beta_k\|u^k - x^*\|^2 + (1-\beta_k)\|\bar{S_k}(u^k) - x^*\|^2 \\
&\quad -\beta_k(1-\beta_k)\|\bar{S_k}(u^k) - u^k\|^2 \\
&= \beta_k\|u^k - x^*\|^2 + (1-\beta_k)\|\bar{S_k}(u^k) - \bar{S_k}(x^*)\|^2 \\
&\quad -\beta_k(1-\beta_k)\|\bar{S_k}(u^k) - u^k\|^2 \\
&\leq \beta_k\|u^k - x^*\| \\
&\quad +(1-\beta_k)\Big(\|u^k - x^*\|^2 + \bar{L}\|(I-\bar{S_k})(u^k) - (I-\bar{S_k})(x^*)\|^2\Big)
\end{aligned}$$

$$\begin{aligned}
&\quad -\beta_k(1-\beta_k)\|\bar{S}_k(u^k)-u^k\|^2 \\
&= \|u^k-x^*\|^2 \\
&\quad +(1-\beta_k)\Big(\bar{L}\|\bar{S}_k(u^k)-u^k\|^2\Big)-\beta_k(1-\beta_k)\|\bar{S}_k(u^k)-u^k\|^2 \\
&= \|u^k-x^*\|^2+(1-\beta_k)(\bar{L}-\beta_k)\big\|\bar{S}_k(u^k)-u^k\big\|^2. \qquad (3.3)
\end{aligned}$$

Since $0<\bar{L}<\beta_k<1$, it follows from (3.3) that

$$\|v^k-x^*\|^2 \le \|u^k-x^*\|^2 \le \|x^k-x^*\|^2. \qquad (3.4)$$

By Lemma 2.10, we have

$$\begin{aligned}
\big\|T^{(g,h)}_{\alpha_k}Av^k-Ax^*\big\|^2 &= \big\|T^{(g,h)}_{\alpha_k}Av^k-T^{(g,h)}_{\alpha_k}Ax^*\big\|^2 \\
&\le \big\langle T^{(g,h)}_{\alpha_k}Av^k-T^{(g,h)}_{\alpha_k}Ax^*, Av^k-Ax^*\big\rangle \\
&= \big\langle T^{(g,h)}_{\alpha_k}Av^k-Ax^*, Av^k-Ax^*\big\rangle \\
&= \frac{1}{2}\Big[\big\|T^{(g,h)}_{\alpha_k}Av^k-Ax^*\big\|^2+\big\|Av^k-Ax^*\big\|^2\Big] \\
&\quad -\frac{1}{2}\big\|T^{(g,h)}_{\alpha_k}Av^k-Av^k\big\|^2.
\end{aligned}$$

Hence,

$$\big\|T^{(g,h)}_{\alpha_k}Av^k-Ax^*\big\|^2 \le \big\|Av^k-Ax^*\big\|^2-\big\|T^{(g,h)}_{\alpha_k}Av^k-Av^k\big\|^2.$$

For each $k\ge 1$, let $\bar{T}_k$ be a mapping defined by

$$\bar{T}_k=\sum_{j=1}^{p'}\eta'_{k,j}T_j.$$

By Proposition 2.5, we see that $\bar{T}_k$ is a $\bar{L}$-strict pseudo-contraction on Q and the sequence $\{x^k\}$ generated by Algorithm 1 can be rewritten as

$$x^{k+1}=P_C(v^k+\mu A^*(\bar{T}_kw^k-Av^k)), \forall k\ge 1.$$

Then, for all $k\ge 1$, we have

$$\begin{aligned}
\|\bar{T}_kw^k-Ax^*\|^2 &= \big\|\bar{T}_kw^k-\bar{T}_kAx^*\big\|^2 \\
&\le \big\|w^k-Ax^*\big\|^2+\bar{L}\|(I-\bar{T}_k)(w^k)-(I-\bar{T}_k)(Ax^*)\|^2 \\
&= \big\|w^k-Ax^*\big\|^2+\bar{L}\|\bar{T}_k(w^k)-w^k\|^2 \\
&< \big\|w^k-Ax^*\big\|^2+\|\bar{T}_k(w^k)-w^k\|^2 \\
&= \big\|T^{(g,h)}_{\alpha_k}Av^k-Ax^*\big\|^2+\|\bar{T}_k(w^k)-w^k\|^2 \\
&\le \big\|Av^k-Ax^*\big\|^2-\big\|T^{(g,h)}_{\alpha_k}Av^k-Ax^*\big\|^2 \\
&\quad +\|\bar{T}_k(w^k)-w^k\|^2. \qquad (3.5)
\end{aligned}$$

Using (3.5), we have

$$
\begin{aligned}
&\langle A(v^k - x^*), \bar{T}_k w^k - Av^k\rangle \\
&= \langle A(v^k - x^*) + \bar{T}_k w^k - Av^k - (\bar{T}_k w^k - Av^k), \bar{T}_k w^k - Av^k\rangle \\
&= \langle \bar{T}_k w^k - Ax^*, \bar{T}_k w^k - Av^k\rangle - \|\bar{T}_k w^k - Av^k\|^2 \\
&= \frac{1}{2}\left[\left\|\bar{T}_k w^k - Ax^*\right\|^2 + \left\|\bar{T}_k w^k - Av^k\right\|^2 - \left\|Av^k - Ax^*\right\|^2\right] \\
&\quad -\left\|\bar{T}_k w^k - Av^k\right\|^2 \\
&= \frac{1}{2}\left[\left(\left\|\bar{T}_k w^k - Ax^*\right\|^2 - \left\|Av^k - Ax^*\right\|^2\right) - \left\|\bar{T}_k w^k - Av^k\right\|^2\right] \\
&= \frac{1}{2}\left(\left\|\bar{T}_k w^k - Ax^*\right\|^2 - \left\|Av^k - Ax^*\right\|^2\right) \\
&\quad -\frac{1}{2}\left\|\bar{T}_k w^k - Av^k\right\|^2 \\
&\leq \frac{1}{2}\left(\|\bar{T}_k(w^k) - w^k\|^2 - \left\|T^{(g,h)}_{\alpha_k} Av^k - Ax^*\right\|^2\right) \\
&\quad -\frac{1}{2}\left\|\bar{T}_k w^k - Av^k\right\|^2 \\
&= \frac{1}{2}\|\bar{T}_k(w^k) - w^k\|^2 - \frac{1}{2}\left\|T^{(g,h)}_{\alpha_k} Av^k - Av^k\right\|^2 \\
&\quad -\frac{1}{2}\left\|\bar{T}_k w^k - Av^k\right\|^2. \qquad (3.6)
\end{aligned}
$$

By the definition of x^{k+1} we have

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|P_C(v^k + \mu A^*(\bar{T}_k w^k - Av^k)) - P_C(x^*)\|^2 \\
&\leq \|(v^k - x^*) + \mu A^*(\bar{T}_k w^k - Av^k)\|^2 \\
&= \|v^k - x^*\|^2 + \|\mu A^*(\bar{T}_k w^k - Av^k)\|^2 \\
&\quad +2\mu\langle v^k - x^*, A^*(\bar{T}_k w^k - Av^k)\rangle \\
&\leq \|v^k - x^*\|^2 + \mu^2\|A^*\|^2\|\bar{T}_k w^k - Av^k\|^2 \\
&\quad +2\mu\langle A(v^k - x^*), \bar{T}_k w^k - Av^k\rangle.
\end{aligned}
$$

In combination with (3.6) and (3.4), the last inequality becomes

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|v^k - x^*\|^2 - \mu^2\|A^*\|^2\|\bar{T}_k w^k - Av^k\|^2 \\
&\quad +\mu\|\bar{T}_k(w^k) - w^k\|^2 - \mu\|\bar{T}_k w^k - Av^k\|^2 \\
&\quad -\mu\|T^{(g,h)}_{\alpha_k} Av^k - Av^k\|^2 \\
&= \|v^k - x^*\|^2 - \mu(1 - \mu\|A^*\|^2)\|\bar{T}_k w^k - Av^k\|^2 \\
&\quad +\mu\|\bar{T}_k(w^k) - w^k\|^2 - \mu\|w^k - Av^k\|^2 \\
&\leq \|x^k - x^*\|^2 - \mu(1 - \mu\|A^*\|^2)\|\bar{T}_k w^k - Av^k\|^2 \\
&\quad +\mu\|\bar{T}_k(w^k) - w^k\|^2 - \mu\|w^k - Av^k\|^2. \qquad (3.7)
\end{aligned}
$$

From (3.4), (3.7), and $\mu \in \Big(0, \dfrac{1}{\|A\|^2}\Big)$, we get

$$\|x^{k+1} - x^*\| \leq \|v^k - x^*\| \leq \|u^k - x^*\| \leq \|x^k - x^*\| \tag{3.8}$$

and

$$\begin{aligned}\mu\big(1 - \mu\|A^*\|^2\big)\|\bar{T}_k w^k - Av^k\|^2 + \mu\|w^k - Av^k\|^2 - \mu\|\bar{T}_k(w^k) - w^k\|^2 \\ \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2.\end{aligned} \tag{3.9}$$

Therefore, $\lim_{k\to+\infty} \|x^k - x^*\|$ exists, and we get from (3.8) and (3.9) that

$$\begin{aligned}\lim_{k\to+\infty} \|x^k - x^*\| = \lim_{k\to+\infty} \|v^k - x^*\| = \lim_{k\to+\infty} \|u^k - x^*\| \text{ and} \\ \lim_{k\to+\infty} \|\bar{T}_k w^k - Av^k\| = \lim_{k\to+\infty} \|w^k - Av^k\| = 0.\end{aligned} \tag{3.10}$$

From (3.10) and the inequality

$$\|\bar{T}_k w^k - w^k\| \leq \|\bar{T}_k w^k - Av^k\| + \|w^k - Av^k\|,$$

we get

$$\lim_{k\to+\infty} \|\bar{T}_k w^k - w^k\| = 0. \tag{3.11}$$

Besides, Lemma 3.1(d) implies

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma_k(2 - \gamma_k)\big(\sigma_k\|t^k\|\big)^2.$$

Hence,

$$\begin{aligned}\gamma_k(2 - \gamma_k)(\sigma_k\|t^k\|)^2 &\leq \|x^k - x^*\|^2 - \|u^k - x^*\|^2 \\ &= (\|x^k - x^*\| - \|u^k - x^*\|)(\|x^k - x^*\| + \|u^k - x^*\|).\end{aligned}$$

In view of (3.10), we get

$$\lim_{k\to+\infty} \sigma_k\|t^k\| = 0. \tag{3.12}$$

Moreover, by the definition of u^k,$u^k = P_C(x^k - \gamma_k\sigma_k t^k)$. We have

$$\|u^k - x^k\| \leq \gamma_k\sigma_k\|t^k\|.$$

So, we get from (3.12) that

$$\lim_{k\to+\infty} \|u^k - x^k\| = 0. \tag{3.13}$$

From (3.3), we get

$$\|v^k - x^*\|^2 \leq \|u^k - x^*\|^2 + (1 - \beta_k)\Big(\bar{L} - \beta_k)\big\|\bar{S}_k(u^k) - u^k\big\|^2. \tag{3.14}$$

Therefore,

$$(1-\beta_k)(\beta_k-\bar{L})\|\bar{S}_k u^k - u^k\|^2 \leq \|u^k - x^*\|^2 - \|v^k - x^*\|^2.$$

Combining the last inequality with (3.10) , we obtain that

$$\lim_{k\to+\infty} \|\bar{S}_k u^k - u^k\| = 0. \tag{3.15}$$

Moreover,

$$\begin{aligned} \|v^k - x^k\| &\leq \|v^k - u^k\| + \|u^k - x^k\| \\ &= \|\beta u^k + (1-\beta)\bar{S}_k u^k - u^k\| + \|u^k - x^k\| \\ &= (1-\beta)\|\bar{S}_k u^k - u^k\| + \|u^k - x^k\| \end{aligned}$$

Thus, we get from (3.13) and (3.15) that

$$\lim_{k\to+\infty} \|v^k - x^k\| = 0. \tag{3.16}$$

Since $\lim_{k\to+\infty}\|x^k - x^*\|$ exists, $\{x^k\}$ is bounded. By Lemma 2.8 , $\{y^k\}$ is bounded, and consequently $\{z^k\}$ is bounded. By Lemma 2.7 $\{t^k\}$ is bounded. Step III and (3.12) yield

$$\lim_{k\to\infty} f(z^k, x^k) = \lim_{k\to\infty} \left[\sigma_k\|t^k\|\right]\|t^k\| = 0. \tag{3.17}$$

We have

$$0 = f(z^k, z^k) = f(z^k, (1-\eta_k)x^k + \eta_k y^k) \leq (1-\eta_k) f(z^k, x^k + \eta_k f(z^k, y^k),$$

so,we obtain

$$\eta_k\left[f(z^k, x^k) - f(z^k, y^k)\right] \leq f(z^k, x^k).$$

Thus, we get from (3.34) that

$$\frac{\theta}{2\rho_k}\eta_k\|x^k - y^k\|^2 \leq \eta_k\left[f(z^k, x^k) - f(z^k, y^k)\right] \leq f(z^k, x^k).$$

Combining this with (3.17), we have

$$\lim_{k\to\infty} \eta_k\|x^k - y^k\|^2 = 0. \tag{3.18}$$

Suppose that $\bar{x}$ is a weak accumulation point of $\{x^k\}$, that is, there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $\{x^{k_j}\}$ converges weakly to $\bar{x} \in C$ as $j \to +\infty$. Then, it follows from (3.13) and (3.16) that $u^{k_j} \rightharpoonup \bar{x}$, $v^{k_j} \rightharpoonup \bar{x}$, and $Av^{k_j} \rightharpoonup A\bar{x}$. Since $\lim_{k\to+\infty}\|w^k - Av^k\| = 0$, we deduce that $w^{k_j} \rightharpoonup A\bar{x}$. Because $\{w^k\} \subset Q$ and Q is closed and convex, we have that $A\bar{x} \in Q$. From (3.18), we get

$$\lim_{i\to\infty} \eta_{k_i}\|x^{k_i} - y^{k_i}\|^2 = 0. \tag{3.19}$$

We now consider two distinct cases.

Case I. $\limsup_{i\to\infty} \eta_{k_i} > 0$. In this case, there exist $\overline{\eta} > 0$ and a subsequence of $\{\eta_{k_i}\}$, denoted again by $\{\eta_{k_i}\}$, such that, for some $i_0 > 0$, $\eta_{k_i} > \overline{\eta}$ for all $i > i_0$. Using this fact and (3.19), we have

$$\lim_{i\to\infty} \|x^{k_i} - y^{k_i}\| = 0. \tag{3.20}$$

Recall that $x^k \rightharpoonup \bar{x}$, together with (3.20), implies that $y^{k_i} \rightharpoonup \bar{x}$ as $i \longrightarrow \infty$. By the definition of y^{k_i},

$$y^{k_i} := \text{argmin}\{f(x^{k_i}, y) + \frac{1}{2\rho_{k_i}}\|y - x^{k_i}\|^2 : y \in C\},$$

so, we have

$$0 \in \partial f(x^{k_i}, y^{k_i}) + \frac{1}{\rho_{k_i}}\big(y^{k_i} - x^{k_i}\big) + N_C\big(y^{k_i}\big).$$

Thus, there exists $\hat{t}^{k_i} \in \partial f(x^{k_i}, y^{k_i})$ such that

$$\langle \hat{t}^{k_i}, y - y^{k_i}\rangle + \frac{1}{\rho_{k_i}}\langle y^{k_i} - x^{k_i}, y - y^{k_i}\rangle \geq 0, \forall y \in C.$$

Combining this with

$$f(x^{k_i}, y) - f(x^{k_i}, y^{k_i}) \geq \langle \hat{t}^{k_i}, y - y^{k_i}\rangle, \forall y \in C,$$

yields

$$f(x^{k_i}, y) - f(x^{k_i}, y^{k_i}) + \frac{1}{\rho_{k_i}}\langle y^{k_i} - x^{k_i}, y - y^{k_i}\rangle \geq 0, \forall y \in C. \tag{3.21}$$

Since

$$\langle y^{k_i} - x^{k_i}, y - y^{k_i}\rangle \leq \|y^{k_i} - x^{k_i}\|\|y - y^{k_i}\|,$$

from (3.21) we get that

$$f(x^{k_i}, y) - f(x^{k_i}, y^{k_i}) + \frac{1}{\rho_{k_i}}\|y^{k_i} - x^{k_i}\|\|y - y^{k_i}\| \geq 0. \tag{3.22}$$

Letting $i \to \infty$, by the weak continuity of f and (3.20), from (3.22) we obtain in the limit that

$$f(\bar{x}, y) - f(\bar{x}, \bar{x}) \geq 0.$$

Thus,

$$f(\bar{x}, y) \geq 0, \forall y \in C.$$

Hence, $\bar{x}$ is a solution of $\text{EP}(C, f)$.

Case II. $\lim_{i\to\infty} \eta_{k_i} = 0$. From the boundedness of $\{y^{k_i}\}$, without loss of generality, we may assume that $y^{k_i} \rightharpoonup \overline{y}$ as $i \to \infty$. Replacing y by x^{k_i} in (3.22), we get

$$f(x^{k_i}, y^{k_i}) \leq -\frac{1}{\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2. \tag{3.23}$$

On the other hand, by the Armijo linesearch rule (3.34), for $m_{k_i}-1$, we have

$$f(z^{k_i,m_{k_i}-1},x^{k_i})-f(z^{k_i,m_{k_i}-1},y^{k_i})<\frac{\theta}{2\rho_{k_i}}\|y^{k_i}-x^{k_i}\|^2.$$

Combining this with (3.23), we get

$$f(x^{k_i},y^{k_i})\le-\frac{1}{\rho_{k_i}}\|y^{k_i}-x^{k_i}\|^2<\frac{2}{\theta}\big[f(z^{k_i,m_{k_i}-1},y^{k_i})-f(z^{k_i,m_{k_i}-1},x^{k_i})\big].\quad(3.24)$$

According to the algorithm, we have $z^{k_i,m_{k_i}-1}=(1-\eta^{m_{k_i}-1})x^{k_i}+\eta^{m_{k_i}-1}y^{k_i}$. Since $\eta^{m_{k_i}-1}\to 0,\{x^{k_i}\}$ converges weakly to $\bar{x}$, and $\{y^{k_i}\}$ converges weakly to $\bar{y}$, this implies that $z^{k_i,m_{k_i}-1}\rightharpoonup\bar{x}$ as $i\to\infty$. Beside that, $\left\{\frac{1}{\rho_{k_i}}\|y^{k_i}-x^{k_i}\|^2\right\}$ is bounded, so without loss of generality we may assume that $\lim_{i\to+\infty}\frac{1}{\rho_{k_i}}\|y^{k_i}-x^{k_i}\|^2$ exists. Hence, in the limit, from (3.24) we get that

$$f(\bar{x},\bar{y})\le-\lim_{i\to+\infty}\frac{1}{\rho_{k_i}}\|y^{k_i}-x^{k_i}\|^2\le\frac{2}{\theta}f(\bar{x},\bar{y}).$$

Therefore, $f(\bar{x},\bar{y})=0$ and $\lim_{i\to+\infty}\frac{1}{\rho_{k_i}}\|y^{k_i}-x^{k_i}\|^2=0$. By Case I we get $\bar{x}\in S_{\rm EP}(f)$. Next, we prove that any weakly cluster point of the sequence $\{x^k\}$ is a common fixed point of L_i-strict pseudocontraction, for each $i=1,2,\dots,p$. Inparticular, $\bar{x}\in\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)$. Let $\bar{y}$ be any weakly cluster point of $\{x^k\}$ and let $\{x^{k_m}\}$ be a subsequence of $\{x^k\}\subset C$ weakly converging to $\bar{y}$. By convexity and the closedness of C, C is weakly closed. Thus, $\overline{y}\in C$. We first show that

$$\lim_{m\to\infty}\|x^{k_m}-S(x^{k_m})\|=0.\quad(3.25)$$

Since,

$$\|\bar{S}_k(u^k)-x^k\|\le\|\bar{S}_k(u^k)-u^k\|+\|u^k-x^k\|.$$

Then, by (3.15) and (3.13) we obtain

$$\lim_{k\to+\infty}\|\bar{S}_k u^k-x^k\|=0.\quad(3.26)$$

Since,

$$\|\bar{S}_k(x^k)-x^k\|\le\|\bar{S}_k(x^k)-\bar{S}_k(u^k)\|+\|\bar{S}_k(u^k)-x^k\|.$$

Then, by Proprositon 2.5(i) , we obtain

$$\|\bar{S}_k(x^k)-x^k\|\le\frac{1+\bar{L}}{1-\bar{L}}\|x^k-u^k\|+\|\bar{S}_k(u^k)-x^k\|.$$

So, from (3.13) and (3.26), we obtain

$$\lim_{k\to\infty}\|\bar{S}_k(x^k)-x^k\|=0.\quad(3.27)$$

For each $i = 1, 2, \dots, p$, we suppose that $\{\eta_{k_m,i}\}$ converges to η_i as $m \to \infty$ such that $\sum_{i=1}^{p} \eta_i = 1$. Then, for each $1, 2, \dots, p$ and $x \in C$, we have

$$\bar{S}_{k_m}(x) := \sum_{m=1}^{p} \eta_{k_m,i} S_i(x) \to \sum_{i=1}^{p} \eta_i S_i(x) := S(x) \quad \text{as} \quad m \to \infty. \tag{3.28}$$

It follows from (3.27) that

$$\begin{aligned}
\|x^{k_m} - S(x^{k_m})\| &\le \|x^{k_m} - \bar{S}_{k_m}(x^{k_m})\| + \|\bar{S}_{k_m}(x^{k_m}) - S(x^{k_m})\| \\
&= \|x^{k_m} - \bar{S}_{k_m}(x^{k_m})\| + \left\| \sum_{i=1}^{p} \eta_{k_m,i} S_i(x^{k_m}) - \sum_{i=1}^{p} \eta_i S_i(x^{k_m}) \right\| \\
&= \|x^{k_m} - \bar{S}_{k_m}(x^{k_m})\| + \left\| \sum_{i=1}^{p} (\eta_{k_m,i} - \eta_i) S_i(x^{k_m}) \right\| \\
&\le \|x^{k_m} - \bar{S}_{k_m}(x^{k_m})\| + \sum_{i=1}^{p} |\eta_{k_m,i} - \eta_i| \|S_i(x^{k_m})\|.
\end{aligned} \tag{3.29}$$

So, we get

$$\lim_{m \to \infty} \|x^{k_m} - S(x^{k_m})\| = 0. \tag{3.30}$$

By Proprosition 2.5(ii), we have

$$\bar{y} \in \mathrm{Fix}(S) = \mathrm{Fix}\left(\sum_{i=1}^{p} n_i S_i \right) x.$$

It follows from 2.5(v), we have

$$\bar{y} \in \bigcap_{i=1}^{p} \mathrm{Fix}(S_i, C).$$

In particular, we conclude that $\bar{x} \in \cap_{i=1}^{p} \mathrm{Fix}(S_i, C)$. Hence,

$$\bar{x} \in S_{\mathrm{EP}}(C, f) \cap \big(\bigcap_{i=1}^{p} \mathrm{Fix}(S_i, C) \big). \tag{3.31}$$

Next, we need to show that $A\bar{x} \in S_{\mathrm{GEP}}(Q, g, h) \cap \big(\bigcap_{j=1}^{p'} \mathrm{Fix}(T_j, Q) \big)$. Indeed, we have $S_{\mathrm{GEP}}(Q, g, h) = \mathrm{Fix}(T_\beta^{(g,h)})$. So, if $T_\beta^{(g,h)} A\bar{x} \neq A\bar{x}$, then, using Opial's condition, we have

$$\begin{aligned}
\liminf_{j \to +\infty} \|Av^{k_j} - A\bar{x}\| &< \liminf_{j \to +\infty} \|Av^{k_j} - T_\beta^{(g,h)} A\bar{x}\| \\
&= \liminf_{j \to +\infty} \|Av^{k_j} - w^{k_j} + w^{k_j} - T_\beta^{(g,h)} A\bar{x}\| \\
&\le \liminf_{j \to +\infty} (\|Av^{k_j} - w^{k_j}\| + \|w^{k_j} - T_\beta^{(g,h)} A\bar{x}\|).
\end{aligned}$$

So it follows from (3.12) and Lamma 3.1 that

$$\begin{aligned}\liminf_{j\to+\infty}\|Av^{k_j}-A\bar{x}\| &< \liminf_{j\to+\infty}\|T_{\beta}^{(g,h)}A\bar{x}-w^{k_j}\| \\ &= \liminf_{j\to+\infty}\|T_{\beta}^{(g,h)}A\bar{x}-T_{\alpha_{k_j}}^{(g,h)}Av^{k_j}\| \\ &\le \liminf_{j\to+\infty}\Big\{\|Av^{k_j}-A\bar{x}\|+\frac{|\alpha_{k_j}-\beta|}{\alpha_{k_j}}\|T_{\alpha_{k_j}}^{(g,h)}Av^{k_j}-Av^{k_j}\|\Big\} \\ &= \liminf_{j\to+\infty}\Big\{\|Av^{k_j}-A\bar{x}\|+\frac{|\alpha_{k_j}-\beta|}{\alpha_{k_j}}\|w^{k_j}-Av^{k_j}\|\Big\} \\ &= \liminf_{j\to+\infty}\ \|Av^{k_j}-A\bar{x}\|,\end{aligned}$$

a contradiction. Thus, $A\bar{x}\in$Fix$(T_{\alpha}^{(g,h)})=S_{\text{GEP}}(Q,g,h)$. Moreover, (3.11) imply that $(I-\bar{T}_k)(w^{k_j})\to 0$ and we have $w^{k_j}\rightharpoonup Ap$, then by proposition 2.5(ii) we get $(I-\bar{T}_k)(A\bar{x})\to 0$ is demiclosed at 0, so we obtain that $A\bar{x}\in\bigcap_{j=1}^{p'}\text{Fix}(T_j,Q)$. Therefore,

$$A\bar{x}\in S_{\text{GEP}}(Q,g,h)\cap(\bigcap_{j=1}^{p'}\text{Fix}(T_j,Q)). \tag{3.32}$$

From (3.31) and (3.32) we obtain that $\bar{x}\in\Omega$. To complete the proof, we must show that the whole sequence $\{x^k\}$ converges weakly to $\bar{x}$. Indeed, if there exists a subsequence $\{x^{l_i}\}$ of $\{x^k\}$ such that $x^{l_i}\rightharpoonup q$ with $q\neq\bar{x}$, then we have $q\in\Omega$. By Opial's condition this yields

$$\begin{aligned}\liminf_{i\to+\infty}\|x^{l_i}-q\| &< \liminf_{i\to+\infty}\|x^{l_i}-\bar{x}\| \\ &= \liminf_{j\to+\infty}\|x^{k}-\bar{x}\| \\ &= \liminf_{j\to+\infty}\|x^{k_j}-\bar{x}\| \\ &< \liminf_{j\to+\infty}\|x^{k_j}-q\| \\ &= \liminf_{i\to+\infty}\|x^{l_i}-q\|.\end{aligned} \tag{3.33}$$

This is a contradiction. Hence, $\{x^k\}$ converges weakly to $\bar{x}$. Combining this with (3.13), it is immediate that $\{u^k\},\{v^k\}$ also converge weakly to $\bar{x}$ and $w^{k_j}\rightharpoonup A\bar{x}\in S_{\text{GEP}}(Q,g,h)\cap\big(\bigcap_{j=1}^{p'}\text{Fix}(T_j,Q)\big)$. □

Corollary 3.3. *Let $\mathcal{H}_1,\mathcal{H}_2$ be two real Hilbert spaces and $C\subset\mathcal{H}_1, Q\subset\mathcal{H}_2$ be nonempty closed convex subsets. Let $S:C\to C$ and $T:Q\to Q$ are L and L'-strict pseudo-contractions, respectively, . Let the bifunctions f,g and h satisfy Assumptions I, II and III, respectively. Let $A:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded linear operator with its adjoint A^*. If*

$$\Omega_1:=\{x^*\in S_{\text{EP}}(C,f)\cap\ \mathit{Fix}\ (S,C): Ax^*\in S_{\text{GEP}}(Q,g,h)\cap\ \mathit{Fix}\ (T,Q)\}\neq\emptyset.$$

Let the sequences $\{x^k\}, \{u^k\}$ and $\{v^k\}$ be generated by the following :

Step I : *Solve the strongly convex program :*

$$y^k := \text{argmin}\Big\{f(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 : y \in C\Big\}$$

to obtain its unique solution y^k. If $y^k = x^k$,then set $u^k = x^k$ and go to Step III. Otherwise, go to Step II.

Step II : *(Armijo linesearch rule) Find m_k as the smallest positive integer number m such that*

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^m, \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\theta}{2\rho_k}\|x^k - y^k\|^2. \end{cases} \tag{3.34}$$

Set $\eta_k = \eta^{m_k}, z^k = z^{k,m_k}$.

Step III : *Take $t^k \in \partial f(z^k, x^k)$, $\sigma_k = \frac{f(z^k,x^k)}{\|t^k\|^2}$, and denote*

$$\begin{cases} u^k = P_C(x^k - \gamma_k \sigma_k t^k), \\ v^k = \beta_k u^k + (1 - \beta_k)S(u^k), \\ w^k = T^{(g,h)}_{\alpha_k} Av^k, \\ x^{k+1} := P_C(v^k + \mu A^*(T(w^k) - Av^k)) \end{cases}$$

and go to iteration k with k replaced by $k + 1$.

Then, $\{x^k\}, \{u^k\}$ and $\{v^k\}$ converge weakly to an element $\bar{x} \in \Omega_1$, and $\{w^k\}$ converges weakly to $A\bar{x}$.

3.2 A Strong Converegence Algorithm

Algorithm II : Initialization.

- Pick $x^g \in C_0 = C$ and choose the parameters $\beta, \eta, \theta \in (0, 1)$, $0 < \rho' \leq \rho'', \{\rho_k\} \subset [\rho', \rho'']$, $0 < \gamma' \leq \gamma'' < 2$, $\{\gamma_k\} \subset [\gamma', \gamma'']$, $0 < \alpha$, $\{\alpha_k\} \subset [\alpha, \infty)$, $\mu \in (0, \frac{1}{\|A\|})$.
- For each $i = 1, 2, \ldots, p$, $\{\eta_{k,i}\}$ is a real sequence of nonnegative numbers satisfying $\sum_{i=1}^{p} \eta_{k,i} = 1$ for all $k \geq 1$.
- For each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, p'\}$, $S_i : C \to C$ and $T_j : Q \to Q$ are L_i and L'_j-strict pseudo-contractions for some $0 \leq L_i < 1$ and $0 \leq L'_j < 1$, respectively.
- $\{\beta_k\}$ is a nonnegative real sequence satisfying $0 < \bar{L} < \beta_k < 1$ and $\beta_k \to \frac{1}{2}$ as $k \to \infty$, where $\bar{L} := \max\{L_i : i = 1, 2, \ldots, p\}$.

Iteration : $k, (k = 0, 1, 2, \ldots)$. Having x^k, do the following steps:

Step I : Solve the strongly convex program :

$$y^k := \text{argmin}\Big\{f(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 : y \in C\Big\}$$

to obtain its unique solution y^k.

If $y^k = x^k$, then set $u^k = x^k$ and go to Step III. Otherwise, go to Step II.

Step II : (Armijo linesearch rule) Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1-\eta^m)x^k + \eta^m y^m, \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\theta}{2\rho_k}\|x^k - y^k\|^2. \end{cases} \tag{3.35}$$

Set $\eta_k = \eta^{m_k}, z^k = z^{k,m_k}$.

Step III : Take $t^k \in \partial f(z^k, x^k)$ and compute $\sigma_k = \frac{f(z^k,x^k)}{\|t^k\|^2}$, and denote

$$\begin{cases} u^k = P_C(x^k - \gamma_k\sigma_k t^k), \\ v^k = \beta_k u^k + (1-\beta_k)\sum_{i=1}^p \eta_{k,i} S_i(u^k), \\ w^k = T_{\alpha_k}^{(g,h)} A v^k, \\ d^k := P_C(v^k + \mu A^*(\sum_{i=1}^p \eta'_{k,i} T_i(w^k) - Av^k)), \\ C_{k+1} = \{x \in C_k : \|x - d^k\| \leq \|x - v^k\| \leq \|x - x^k\|\}, \\ x^{k+1} := P_{C_{k+1}}(x^g) \end{cases}$$

and go to iteration k with k replaced by $k+1$.

Now, we are in a position to state and prove the main strong convergence theorem for the given iterative scheme. Throughout this section, we suppose the following :

Theorem 3.4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two real Hilbert spaces and $C \subset \mathcal{H}_1, Q \subset \mathcal{H}_2$ be nonempty closed convex subsets. For each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, p'\}$, S_i and T_j are L_i and L'_j-strict pseudo-contractions for some $0 \leq L_i < 1$ and $0 \leq L'_j < 1$, respectively. Let the bifunctions f, g and h satisfy Assumptions I, II and III, respectively. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint A^*. If*

$$\Omega := \left\{ x^* \in S_{\mathrm{EP}}(C, f) \cap \Big(\bigcap_{i=1}^p \mathrm{Fix}(S_i, C) \Big) : Ax^* \in S_{\mathrm{GEP}}(Q, g, h) \cap \Big(\bigcap_{j=1}^{p'} \mathrm{Fix}(T_j, Q) \Big) \right\}$$

is nonempty set, then the sequences $\{x^k\}, \{u^k\}$ and $\{v^k\}$ generated by Algorithm II converge strongly to an element $\bar{x} \in \Omega$, and $\{w^k\}$ converges strongly to $A\bar{x} \in S_{\mathrm{GEP}}(Q, g, h) \cap (\bigcap_{j=1}^{p'} \mathrm{Fix}(T_j, Q))$.

Proof. First, we observe that the linesearch rule (3.35) is well defined. Let $x^* \in \Omega$. From (3.7) ,(3.14), and (3.2) we have

$$\begin{aligned} \|d^k - x^*\|^2 &\leq \|v^k - x^*\| - \mu(1-\mu\|A\|^2)\|\bar{T}_k w^k - Av^k\|^2 - \mu\|w^k - Av^k\|^2 \\ &\leq \|u^k - x^*\|^2 + (1-\beta_k)(\bar{L} - \beta_k)\big\|\bar{S}_k(u^k) - u^k\big\|^2 \\ &\quad -\mu(1-\mu\|A\|^2)\|\bar{T}_k w^k - Av^k\|^2 - \mu\|w^k - Av^k\|^2 \\ &\leq \|x^k - x^*\| + (1-\beta_k)(\bar{L} - \beta_k)\big\|\bar{S}_k(u^k) - u^k\big\|^2 \\ &\quad -\mu(1-\mu\|A\|^2)\|\bar{T}_k w^k - Av^k\|^2 - \mu\|w^k - Av^k\|^2. \end{aligned} \tag{3.36}$$

Since $0 < \bar{L} < \beta_k < 1$ and $\mu \in (0, \frac{1}{\|A\|^2})$, (3.36) implies that

$$\|d^k - x^*\| \leq \|v^k - x^*\| \leq \|u^k - x^*\| \leq \|x^k - x^*\|, \forall k. \tag{3.37}$$

Since $x^* \in C_0$, from (3.37) we get by induction that $x^* \in C_k$ for all $k \in \mathbb{N}$ and, consequently, $\Omega \subset C_k$ for all k. By setting

$$D_k = \{x \in \mathcal{H}_1 : \|x - d^k\| \leq \|x - v^k\| \leq \|x - x^k\|\}, k \in \mathbb{N},$$

it is clear that D_k is closed and convex for all k. In addition, $C_0 = C$ is also closed and convex, and $C_{k+1} = C_k \cap D_k$. Hence, C_k is closed and convex for all k. From the definition of x^{k+1} we have $x^{k+1} \in C_{k+1} \subset C_k$ and $x^k = P_{C_k}(x^g)$, so

$$\|x^k - x^g\| \leq \|x^{k+1} - x^g\|, \forall k.$$

Since $x^* \in C^{k+1}$, this implies that

$$\|x^{k+1} - x^g\| \leq \|x^* - x^g\|.$$

Thus,

$$\|x^k - x^g\| \leq \|x^{k+1} - x^g\| \leq \|x^* - x^g\|, \forall k.$$

Consequently, $\{\|x^k - x^g\|\}$ is nondecreasing and bounded, so $\lim_{k\to+\infty} \|x^k - x^g\|$ does exist. Combining this with (3.37), we obtain that $\{d^k\}$ and $\{v^k\}$ are also bounded. For all $m > n$, we have that $x^m \in C_m \subset C_n$ and $x^n = P_{C_n}(x^g)$. Combining this fact with Lemma 2.1, we get

$$\begin{aligned}
\|x^m - x^n\|^2 &\leq \|x^m - x^g\|^2 - \|x^n - x^g\|^2 \\
&= (\|x^m - x^g\| - \|x^n - x^g\|)(\|x^m - x^g\| + \|x^n - x^g\|).
\end{aligned}$$

Since $\lim_{k\to+\infty} \|x^k - x^g\|$ exists, this implies that $\lim_{m,n\to+\infty} \|x^m - x^n\| = 0$. Therefore, $\{x^k\}$ is a Cauchy sequence, so

$$\lim_{k\to\infty} x^k = \bar{x}. \tag{3.38}$$

By Step III we get

$$\|d^k - x^{k+1}\| \leq \|v^k - x^{k+1}\| \leq \|x^k - x^{k+1}\|.$$

Therefore,

$$\begin{aligned}
\|d^k - x^k\| &\leq \|d^k - x^{k+1}\| + \|x^{k+1} - x^k\| \\
&\leq \|x^k - x^{k+1}\| + \|x^k - x^{k+1}\| \\
&= 2\|x^k - x^{k+1}\|
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}
\|v^k - x^k\| &\leq \|v^k - x^{k+1}\| + \|x^{k+1} - x^k\| \\
&\leq \|x^k - x^{k+1}\| + \|x^k - x^{k+1}\| \\
&= 2\|x^k - x^{k+1}\|.
\end{aligned} \tag{3.40}$$

So, from (3.39), (3.40), and (3.38) we get that

$$\lim_{k\to\infty}\|d^k - x^k\| = \lim_{k\to\infty}\|v^k - x^k\| = 0. \tag{3.41}$$

In view of (3.36) and (3.41), we have

$$\begin{aligned}(1-\beta_k)(\beta_k - \bar{L})\|\bar{S}_k u^k - u^k\|^2 + \mu(1-\mu\|A\|^2)\|\bar{T}_k w^k - Av^k\|^2 + \mu\|w^k - Av^k\|^2 \\ \leq \quad & \|x^k - x^*\|^2 - \|d^k - x^*\|^2 \\ = \quad & \big(\|x^k - x^*\| + \|d^k - x^*\|\big)\big(\|x^k - x^*\| - \|d^k - x^*\|\big) \\ \leq \quad & \|x^k - d^k\|\big(\|x^k - x^*\| + \|d^k - x^*\|\big) \to 0 \text{ as } k\to\infty. \end{aligned} \tag{3.42}$$

Since $0 < \bar{L} < \beta_k < 1$ and $\mu \in \left(0, \frac{1}{\|A\|}\right)$, we deduce from (3.42) that

$$\lim_{k\to+\infty}\|\bar{S}_k u^k - u^k\| = 0, \lim_{k\to+\infty}\|\bar{T}_k w^k - Av^k\| = 0, \text{ and } \lim_{k\to+\infty}\|w^k - Av^k\| = 0. \tag{3.43}$$

In addition, from the inequality

$$\|\bar{T}_k w^k - w^k\| \leq \|\bar{T}_k w^k - Av^k\|\|w^k - Av^k\|,$$

combined with (3.43), we get

$$\lim_{k\to+\infty}\|\bar{T}_k w^k - w^k\| = 0. \tag{3.44}$$

Besides, (3.16), (3.40), and $\lim_{k\to+\infty} x^k = \bar{x}$ it imply

$$\lim_{k\to+\infty} u^k = \bar{x}, \lim_{k\to+\infty} v^k = \bar{x}. \tag{3.45}$$

Since

$$\begin{aligned}\|\bar{S}_k\bar{x} - \bar{x}\|^2 \quad & \leq \quad \|\bar{S}_k\bar{x} - \bar{S}_k u^k\|^2 + \|\bar{S}_k u^k - u^k\|^2 + \|u^k - \bar{x}\|^2 \\ & \leq \quad \big(\|u^k - x^*\|^2 + \bar{L}\|(I-\bar{S}_k)(u^k) - (I-\bar{S}_k)(x^*)\|^2\big) \\ & \qquad + \|\bar{S}_k u^k - u^k\| + \|u^k - \bar{x}\| \\ & = \quad 2\|u^k - \bar{x}\|^2 + (\bar{L}+1)\|\bar{S}_k u^k - u^k\|^2,\end{aligned}$$

from (3.43) and (3.45) we get that $\|\bar{S}_k\bar{x} - \bar{x}\| = 0$, that is, $\bar{x} \in \mathrm{Fix}(\bar{S}_k)$. From (3.18) we have

$$\lim_{k\to+\infty}\eta_k\|x^k - y^k\|^2 = 0. \tag{3.46}$$

We now consider two distinct cases.

Case I. $\limsup_{k\to\infty}\eta_k > 0$. Then there exist $\overline{\eta} > 0$ and a subsequence $\{\eta_{k_i}\} \subset \{\eta_k\}$ such that $\eta_{k_i} > \bar{\eta}$ for all i. So we get from (3.46) that

$$\lim_{k\to+\infty}\|x^{k_i} - y^{k_i}\| = 0. \tag{3.47}$$

Since $x^k \to \bar{x}$, (3.47) implies that $y^{k_i} \to \bar{x}$ as $i \to \infty$. For each $y \in C$, we get from (3.22) that

$$f(x^{k_i}, y) - f(x^{k_i}, y^{k_i}) + \frac{1}{\rho_{k_i}} \|y^{k_i} - x^{k_i}\| \|y - y^{k_i}\| \geq 0. \tag{3.48}$$

Letting $i \to \infty$, by the continuity of f, since $x^{k_i} \to \bar{x}$ and $y^{k_i} \to \bar{x}$, in the limit, from (3.48) we obtain that $f(\bar{x}, y) - f(\bar{x}, \bar{x}) \geq 0$. Hence, $f(\bar{x}, y) \geq 0$, $\forall y \in C$, so $\bar{x}$ is a solution of
EP(C, f).

Case II. $\lim_{k\to\infty} \eta_k = 0$. From the boundedness of $\{y^k\}$, we deduce that there exists $\{y^{k_i}\} \subset \{y^k\}$ such that $y^{k_i} \rightharpoonup \overline{y}$ as $i \to \infty$. Replacing y by y^{k_i} in (3.21), we get

$$f(x^{k_i}, y^{k_i}) + \frac{1}{\rho_{k_i}} \|y^{k_i} - x^{k_i}\|^2 \leq 0. \tag{3.49}$$

On the other hand, by the Armijo linesearch rule (3.35), for $m_{k_i} - 1$, there exists $z^{k_i, m_{k_i}-1}$ such that

$$f(z^{k_i, m_{k_i}-1}, x^{k_i}) - f(z^{k_i, m_{k_i}-1}, y^{k_i}) < -\frac{\theta}{2\rho_{k_i}} \|y^{k_i} - x^{k_i}\|^2.$$

Combining this with (3.49), we get

$$f(z^{k_i, m_{k_i}-1}, y^{k_i}) - f(z^{k_i, m_{k_i}-1}, x^{k_i}) > -\frac{\theta}{2\rho_{k_i}} \|y^{k_i} - x^{k_i}\|^2 \geq \frac{2}{\theta} f\big(x^{k_i}, y^{k_i}\big). \tag{3.50}$$

According to the algorithm, we have $z^{k_i, m_{k_i}-1} = (1 - \eta^{m_{k_i}-1}) x^{k_i} + \eta^{m_{k_i}-1} y^{k_i}$. Since $\eta^{m_{k_i}-1} \to 0, x^{k_i}$ converges weakly to $\bar{x}$, and y^{k_i} converges weakly to $\overline{y}$, this implies that $z^{k_i, m_{k_i}-1} \rightharpoonup \bar{x}$ as $i \to \infty$. Beside that, $\left\{\frac{1}{\rho_{k_i}} \|y^{k_i} - x^{k_i}\|^2\right\}$ is bounded, so without loss of generality, we may assume that $\lim_{i\to+\infty} \frac{1}{\rho_{k_i}} \|y^{k_i} - x^{k_i}\|^2$ exists. Hence, we obtain in the limit (3.50) that

$$f(\bar{x}, \bar{y}) \geq -2 \lim_{i\to+\infty} \frac{1}{\rho_{k_i}} \|y^{k_i} - x^{k_i}\|^2 \geq \theta f(\bar{x}, \bar{y}).$$

Therefore, $f(\bar{x}, \bar{y}) = 0$ and $\lim_{i\to+\infty} \frac{1}{\rho_{k_i}} \|y^{k_i} - x^{k_i}\|^2 = 0$. By Case I we get $\bar{x} \in \bigcap_{i=1}^p \mathrm{Fix}(S_i, C)$. So

$$\bar{x} \in S_{\mathrm{EP}}(f) \cap \Big(\bigcap_{i=1}^{p} \mathrm{Fix}(S_i, C) \Big). \tag{3.51}$$

We get from (3.45) that $\lim_{k\to+\infty} Av^k = A\bar{x}$. Combining this with (3.43) yields

$$\lim_{k\to+\infty} w^k = A\bar{x}. \tag{3.52}$$

Since,

$$\|\bar{T}_k A\bar{x} - A\bar{x}\| \leq \|\bar{T}_k A\bar{x} - \bar{T}_k w^k\| + \|\bar{T}_k w^k - w^k\| + \|w^k - A\bar{x}\|.$$

Then, by Proprositon 2.5(i), we obtain

$$\begin{aligned}\|\bar{T}_k A\bar{x} - A\bar{x}\| &\leq \frac{1+\bar{L}}{1-\bar{L}}\|A\bar{x} - w^k\| + \|\bar{T}_k w^k - w^k\| + \|w^k - A\bar{x}\| \\ &= \Big(1 + \frac{1+\bar{L}}{1-\bar{L}}\Big)\|w^k - A\bar{x}\| + \|\bar{T}_k w^k - w^k\|.\end{aligned}$$

So, from (3.52) and (3.44) we get that

$$\|\bar{T}_k A\bar{x} - A\bar{x}\| = 0, \text{ for each } j \in \{1, 2, \ldots, p'\}.$$

Hence for each $j \in \{1, 2, \ldots, p'\}$, we get $A\bar{x} \in \bigcap_{j=1}^{p'} \mathrm{Fix}(T_j, Q)$. Moreover,

$$\begin{aligned}\|T_\beta^{(g,h)} A\bar{x} - A\bar{x}\| &\leq \|T_\beta^{(g,h)} A\bar{x} - T_{\alpha_k}^{(g,h)} Av^k - Av^k\| + \big\|T_{\alpha_k}^{(g,h)} Av^k - Av^k\big\| \\ &\quad + \|Av^k - A\bar{x}\| \\ &= \big\|T_\beta^{(g,h)} A\bar{x} - T_{\alpha_k}^{(g,h)} Av^k - Av^k\big\| + \big\|w^k - Av^k\big\| \\ &\quad + \|Av^k - A\bar{x}\| \\ &\leq \|Av^k - A\bar{x}\| + \frac{|\alpha_k - \beta|}{\alpha_k}\big\|T_{\alpha_k}^{(g,h)} Av^k - Av^k\big\| + \|w^k - Av^k\| \\ &\quad + \|Av^k - A\bar{x}\| \\ &= 2\|Av^k - A\bar{x}\| + \frac{|\alpha_k - \beta|}{\alpha_k}\big\|w^k - Av^k\big\| + \|w^k - Av^k\|,\end{aligned}$$

where the last inequality comes from Lemma 2.12. Letting $k \to \infty$ and recalling that $\lim_{k\to+\infty} Av^k = A\bar{x}$. Then, we get

$$\|T_\alpha^{(g,h)} A\bar{x} - A\bar{x}\| = 0$$

Thus, $A\bar{x} \in \mathrm{Fix}(T_\alpha^{(g,h)}) = S_{\mathrm{GEP}}(Q, g, h)$. Hence,

$$A\bar{x} \in S_{\mathrm{GEP}}(g, h) \cap \Big(\bigcap_{j=1}^{p'} \mathrm{Fix}(T_j, Q)\Big).$$

So, we conclude that $\bar{x} \in \Omega$, the proof of the theorem is complete. □

For each $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, p'\}$ putting $S_i = S$ and $T_j = S$, where S and T are nonexpansive mappings, and $h = 0$ in Theorem 3.4, we have the following result.

Corollary 3.5. [11] *Let $\mathcal{H}_1, \mathcal{H}_2$ be two real Hilbert spaces and $C \subset \mathcal{H}_1, Q \subset \mathcal{H}_2$ be nonempty closed convex subsets. Let $S : C \to C$; $T : Q \to Q$ be a nonexpansive mapping, and let bifunctions f and g satisfy Assumptions I and II , respectively. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint A^*. If*

$$\Omega := \{x^* \in S_{\mathrm{EP}}(C, f) \cap \ Fix\ (S) : Ax^* \in \ Fix\ (Q, g) \cap \ Fix\ (T)\} \neq \emptyset.$$

Then the sequences $\{x^k\}, \{u^k\}$ and $\{v^k\}$ be generated by the following :

Step I : *Solve the strongly convex program :*

$$y^k := \mathrm{argmin}\Big\{ f(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 : y \in C \Big\}$$

to obtain its unique solution y^k. If $y^k = x^k$,then set $u^k = x^k$ and go to Step III. Otherwise, go to Step II.

Step II : (Armijo linesearch rule) *Find m_k as the smallest positive integer number m such that*

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^m, \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\theta}{2\rho_k}\|x^k - y^k\|^2. \end{cases} \tag{3.53}$$

Set $\eta_k = \eta^{m_k}, z^k = z^{k,m_k}$.

Step III : *Take $t^k \in \partial f(z^k, x^k)$, $\sigma_k = \frac{f(z^k, x^k)}{\|t^k\|^2}$, and denote*

$$\begin{cases} u^k = P_C(x^k - \gamma_k \sigma_k t^k), \\ v^k = (1 - \beta)u^k + \beta S(u^k), \\ w^k = T^g_{\alpha_k} A v^k, \\ x^{k+1} := P_C(v^k + \mu A^*(T(w^k) - Av^k)) \end{cases}$$

and go to iteration k with k replaced by $k + 1$,
converge strongly to an element $\bar{x} \in \Omega$, and $\{w^k\}$ converges strongly to $A\bar{x}$.

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