Common Fixed Point and Coupled Coincidence Point Theorems for θ-ψ Contraction Mappings with Two Metrics Endowed with a Directed Graph

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Abstract : The purpose of this paper is to present some existence and uniqueness results for common fixed point theorems for θ-ψ contraction mappings with two metrics endowed with a directed graph. In addition, by using our main results, we obtain some results about coupled coincidence point endowed with a directed graph. Our results also generalize those presented in [1, 2].

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1 Introduction and Preliminaries

Geraghty [3] introduced an interesting class θ of functions $\theta : [0, \infty) \to [0, 1)$ satisfying that:

$$\theta(t_n) \to 1 \quad \Longrightarrow \quad t_n \to 0,$$

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as well as some result which is a generalization of Banach's contraction principle in 1973. Amini-Harandi and Emami [4] then extended the result in [3] in the context of partially ordered complete metric spaces.

Let d', d be two metrics on X. By $d < d'$ (resp., $d \leq d'$), we mean $d(x, y) < d'(x, y)$ (resp., $d(x, y) \leq d'(x, y)$) for all $x, y \in X$.

When equipping a metric space with two metrics, we can also extend the concept of fixed point theory. There are several research papers in this area, for example see [5, 6]. Recently, some new common fixed point theorems for Geraghty's type contraction mappings using the monotone property with two metrics were shown in [1] by using d-compatibility and g-uniform continuity defined as follows;

Definition 1.1 ([7])**.** Let (X, d) and (Y, d') be two metric spaces. Let $f : X \to Y$, and $g : X \to X$ be two mappings.

(i) The mappings g and f are said to be *d-compatible* if

$$\lim_{n \to \infty} d(gfx_n, fgx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n$.

(ii) A mapping f is said to be *g-uniformly continuous* on X if, for any real number $\epsilon > 0$, there exists $\delta > 0$ such that $d'(fx, fy) < \epsilon$ whenever $x, y \in X$ and $d(gx, gy) < \delta$. If g is the identity mapping, it is obvious that f is uniformly continuous on X.

Let (X, d) be a metric space, and Δ be a diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$.

Throughout the paper we shall say that G with the above mentioned properties satisfies the *standard conditions*.

The fixed point theorem using the context of metric spaces endowed with a graph was initiated by Jachymski [8], which generalizes the Banach contraction principle to mappings on a metric spaces with a graph. Also, the definitions of G-continuous and the property A were given.

Definition 1.2 ([8])**.** Let (X, d) be a metric space, and let $E(G)$ be the set of the edges of the graph.

(i) A mapping $f : X \to X$ is called *G-continuous* if for any $x \in X$ such that there exists a sequence (x_n) in X, $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $f(x_n) \to f(x)$.

(ii) The triple (X, d, G) is said to have the *property A*, if for any sequence (x_n) in X with $x_n \to x$, as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$.

Consequently, several authors have studied the problem of existence of a fixed points for a single-valued mapping and multi-valued mappings in several spaces with a graph; see [9–13].

We now state a collection of notions and definitions from [2].

Definition 1.3 ([2]). Let (X, d) be a metric space, and let $E(G)$ be the set of the edges of the graph. Let $F : X^2 \to X$ and $g : X \to X$, then

(i) A pair of mappings F and g is called *G-edge preserving* if

$$[(gx, gu), (gy, gv) \in E(G)] \Rightarrow [(F(x, y), F(u, v)), (F(y, x), F(v, u)) \in E(G)].$$

(ii) The mapping F is called *G-continuous* if for all $(x^*, y^*) \in X^2$ and for any sequence $(n_i)_i \in \mathbb{N}$ of positive integers, with $F(x_{n_i}, y_{n_i}) \to x^*$, $F(y_{n_i}, x_{n_i}) \to y^*$, as $i \to \infty$, and
$(F(x_{n_i}, y_{n_i}), F(x_{n_i+1}, y_{n_i+1})), (F(y_{n_i}, x_{n_i}), F(y_{n_i+1}, x_{n_i+1})) \in E(G)$, we have that

$$F(F(x_{n_i}, y_{n_i}), F(y_{n_i}, x_{n_i})) \to F(x^*, y^*)$$
$$F(F(y_{n_i}, x_{n_i}), F(x_{n_i}, y_{n_i})) \to F(y^*, x^*),$$
$$\text{as } i \to \infty.$$

(iii) The set $E(G)$ is said to satisfies the *transitivity property* if, for all $x, y, a \in X$, $(x, a), (a, y) \in E(G) \Rightarrow (x, y) \in E(G)$.

In 2015, Suantai et al. [2] used the above definitions to present some existence and uniqueness results for coupled coincidence point and common fixed point of θ-ψ contraction mappings in complete metric spaces endowed with a directed graph. Their results also generalize others in partially ordered metric spaces.

The aim of this paper is to present some existence and uniqueness results for common fixed point theorems for θ-ψ contraction mappings with two metrics endowed with a directed graph. By using our main results, we are able to obtain some results for coupled coincidence point endowed with a directed graph. Our results generalize the realted results given in [1, 2].

2   Main Results

We define the concept of $G(g,f)$-edge preserving which is an effective tool as follows.

Definition 2.1. Let G be a directed graph. We say that $f,g : X \to X$ are $G(g,f)$-*edge preserving* if

$$(gx, gy) \in E(G) \Rightarrow (fx, fy) \in E(G).$$

Let Ψ denote the class of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(ψ_1) ψ is nondecreasing;

(ψ_2) ψ is continuous;

(ψ_3) $\psi(t) = 0 \Leftrightarrow t = 0$.

We now introduce a new class of the Geraghty type contractions in the following definition.

Definition 2.2. Let (X, d) be a metric space endowed with a directed graph G. A pair of mappings $f, g : X \to X$ is called a θ-ψ-*contraction* if

(1) The pair f and g is $G(g,f)$-edge preserving;

(2) there exists $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y \in X$ such that $(gx, gy) \in E(G)$,

$$\psi(d(fx, fy)) \leq \theta(d(gx, gy))\psi(d(gx, gy)). \tag{2.1}$$

Let (X, d) be a metric space endowed with a directed graph G satisfying the standard conditions, and let $f, g : X \to X$. We define some important subsets of X as follows:

1. $X(f, g) = \{u \in X : (gu, fu) \in E(G)\}$.

2. $C(f, g) = \{u \in X : fu = gu\}$, i.e., the set of all coincidence points of mappings f and g by $C(f, g)$.

3. $Cm(f, g) = \{u \in X : fu = gu = u\}$, i.e., the set of all common fixed points of mapping f and g by $Cm(f, g)$.

We now ready to present and prove the main result. We shall begin with a lemma.

Lemma 2.3. *Let (X, d) be a complete metric space endowed with a directed graph G, and let $g, f : X \to X$ be two mappings such that f and g are a θ-ψ-contraction. Assume that $x_0, y_0 \in X$ and $f(X) \subseteq g(X)$. Then*

(i) *There exists sequences $\{x_n\}, \{y_n\}$ in X for which*

$$gx_n = fx_{n-1} \quad and \quad gy_n = fy_{n-1} \quad for \ all \ n \in \mathbb{N}. \qquad (2.2)$$

(ii) *If $(gx_n, gy_n) \in E(G)$ for all $n \in \mathbb{N}$, then*

$$\lim_{n \to \infty} d(gx_n, gy_n) = 0.$$

Proof. (i) Let $x_0, y_0 \in X$. By the assumption that $f(X) \subset g(X)$ and $f(x_0), f(y_0) \in g(X)$, it is easy to construct sequences $\{x_n\}$ and $\{y_n\}$ in X for which

$$gx_n = fx_{n-1} \quad and \quad gy_n = fy_{n-1}$$

for all $n \in \mathbb{N}$.

(ii) Let $(gx_n, gy_n) \in E(G)$ for all $n \in \mathbb{N}$. It follows from the contractive condition (2.1) that

$$\begin{aligned}
\psi(d(gx_{n+1}, gy_{n+1})) &= \psi(d(fx_n, fy_n)) \\
&\leq \theta(d(gx_n, gy_n))\psi(d(gx_n, gy_n)) \\
&< \psi(d(gx_n, gy_n)) \qquad (2.3)
\end{aligned}$$

for all $n \in \mathbb{N}$. By the properties of ψ_1, we have that

$$d(gx_{n+1}, gy_{n+1}) < d(gx_n, gy_n).$$

Thus the sequence $\{d_n\} := \{d(gx_n, gx_n)\}$ is a decreasing sequence. It follows that $d_n \to d$ as $n \to \infty$ for some $d \geq 0$.

We claim that the constant d is 0. Assume on the contrary that $d > 0$. Then by (2.3), we have

$$\frac{\psi(d_{n+1})}{\psi(d_n)} \leq \theta(d_n) < 1.$$

Taking $n \to \infty$ and using properties ψ_3 and ψ_4, we have $\lim_{n\to\infty} \frac{\psi(d_{n+1})}{\psi(d_n)} = 1$. It implies that $\theta(d_n) \to 1$ as $n \to \infty$. Since $\theta \in \Theta$, we have $d_n \to 0$ as $n \to \infty$ which is a contradiction. Finally, we can conclude that $d_n = d(gx_n, gy_n) \to 0$ as $n \to \infty$. $\qquad \square$

Theorem 2.4. *Let (X, d') be a complete metric space endowed with a directed graph G, and let d be another metric on X. Suppose that $f, g : X \to X$ are a θ-ψ-contraction. Suppose that:*

(1) $g : (X, d') \to (X, d')$ is continuous and $g(X)$ is d'-closed;

(2) $f(X) \subseteq g(X)$;

(3) $E(G)$ satisfies the transitivity property;

(4) if $d \not\geq d'$, assume that $f : (X, d) \to (X, d')$ is g-uniformly continuous;

(5) if $d \neq d'$, assume that $f : (X, d') \to (X, d')$ is G-continuous and g and f are d'-compatible;

(6) if $d = d'$, assume that (a) f is G-continuous and g and f are compatible or (b) (X, d, G) has the property A.

Then under these conditions,

$$X(f, g) \neq \emptyset \quad \text{if and only if} \quad C(f, g) \neq \emptyset.$$

Proof. Suppose that $C(f, g) \neq \emptyset$. Let $u \in C(f, g)$. We have $fu = gu$. Then $(gu, fu) = (gu, gu) \in \Delta \subset E(G)$. Hence $(gu, gu) = (gu, fu) \in E(G)$ which means that $u \in X(f, g)$ and thus $X(f, g) \neq \emptyset$.

Suppose now $X(f, g) \neq \emptyset$. Let $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$. By Lemma 2.3, we have a sequence $\{x_n\}$ in X such that

$$gx_n = fx_{n-1}$$

for all $n \in \mathbb{N}$. If $gx_{n_0} = gx_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0-1} is a coincidence point of the mappings g and f. Therefore, we assume that, for each $n \in \mathbb{N}$, $gx_n \neq gx_{n-1}$ holds.

Since $(gx_0, fx_0) = (gx_0, gx_1) \in E(G)$ and the $G(g, f)$-edge preserving property of f and g, we have $(fx_0, fx_1) = (gx_1, gx_2) \in E(G)$. Continue inductively, we obtain that $(gx_{n-1}, gx_n) \in E(G)$ for each $n \in \mathbb{N}$. From Lemma 2.3 (ii), we have

$$\lim_{n \to \infty} d(gx_{n-1}, gx_n) = 0. \tag{2.4}$$

Now, we show that $\{gx_n\}$ is a Cauchy sequence with respect to d. Suppose that $\{gx_n\}$ is not a Cauchy sequence with respect to d. Then there exists $\epsilon > 0$ for which we can find subsequences $\{gx_{n_k}\}$, $\{gx_{m_k}\}$ of $\{gx_n\}$ such that $n_k > m_k \geq k$ satisfying

$$d(gx_{n_k}, gx_{m_k}) \geq \epsilon, \quad d(gx_{n_k-1}, gx_{m_k}) < \epsilon. \tag{2.5}$$

Using (2.5) and the triangle inequality, we have

$$\epsilon \leq r_k := d(gx_{n_k}, gx_{m_k})$$
$$\leq d(gx_{n_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{m_k})$$
$$< d(gx_{n_k}, gx_{n_k-1}) + \epsilon.$$

Letting $k \to \infty$, we have

$$r_k = d(gx_{n_k}, gx_{m_k}) \to \epsilon. \tag{2.6}$$

Again, by the triangle inequality, we have

$$d(gx_{n_k}, gx_{m_k}) \leq d(gx_{n_k}, gx_{n_k+1}) + d(gx_{n_k+1}, gx_{m_k+1}) + d(gx_{m_k+1}, gx_{m_k})$$
$$= d(gx_{n_k}, gx_{n_k+1}) + d(gx_{m_k+1}, gx_{m_k}) + d(fx_{n_k}, fx_{m_k}),$$

which is equivalent to

$$d(gx_{n_k}, gx_{m_k}) - d(gx_{n_k}, gx_{n_k+1}) - d(gx_{m_k+1}, gx_{m_k}) \leq d(fx_{n_k}, fx_{m_k}).$$

By the property of ψ_1, we have

$$\psi(d(gx_{n_k}, gx_{m_k}) - d(gx_{n_k}, gx_{n_k+1}) - d(gx_{m_k+1}, gx_{m_k}))$$
$$\leq \psi(d(fx_{n_k}, fx_{m_k}))$$
$$\leq \theta(d(gx_{n_k}, gx_{m_k}))\psi(d(gx_{n_k}, gx_{m_k})).$$

Letting $k \to \infty$, using the property of ψ_2 and (2.6), we have

$$\lim_{k \to \infty} \theta(d(gx_{n_k}, gx_{m_k})) = 1.$$

It follows that

$$\lim_{k \to \infty} d(gx_{n_k}, gx_{m_k}) = \lim_{k \to \infty} r_k = 0$$

which contradicts. Therefore $\{gx_n\}$ is a Cauchy sequence with respect to d. Applying the similar argument as the proof of Theorem 3.1 in [1], we can see that $\{gx_n\}$ is a Cauchy sequence with respect to d and d'. Since $g(X)$ is a d'-closed subset of the complete metric space (X, d'), there exists $u = gx \in g(X)$ such that

$$\lim_{n \to \infty} gx_n = \lim_{n \to \infty} fx_n = u.$$

Finally, we shall show that u is a common fixed point of f and g. We consider two cases:

Case I: $d \neq d'$. By the triangle inequality, we have

$$d'(gu, f(fx_{n-1})) = d'(gu, fgx_n) \leq d'(gu, gfx_n) + d'(gfx_n, fgx_n).$$

Taking $n \to \infty$, using the d'-compatibility of g and f, the continuity of g in right side and the G-continuity of f in left side, we have that $d'(gu, fu) = 0$, i.e., $gu = fu$.

Case II: $d = d'$. In order to avoid the repetition, we can only consider (b) of the condition (7). In this case, there exists $x \in X$ such that $(gx_n, gx) = (gx_n, u) \in E(G)$ for each $n \in \mathbb{N}$. By triangle inequality, we have

$$d(gx_n, fx) \leq d(gx_n, gx) + d(gx, fx) \tag{2.7}$$

and

$$d(gx, fx) \leq d(gx, gx_n) + d(gx_n, fx). \tag{2.8}$$

Taking $n \to \infty$ in (2.7) and (2.8), then

$$\lim_{n \to \infty} d(gx_n, fx) = d(gx, fx).$$

Since $(gx_n, gx) = (gx_n, u) \in E(G)$ and using (2.1), we get

$$\begin{aligned}
\psi(d(gx_{n+1}, fx)) &= \psi(d(fx_n, fx)) \\
&\leq \theta(d(gx_n, gx))\psi(d(gx_n, gx)) \\
&< \psi(d(gx_n, gx)).
\end{aligned}$$

Taking $n \to \infty$, we have $\lim_{n \to \infty} \psi(d(gx_{n+1}, fx)) = 0$. Therefore, by property ψ_2 and ψ_3, we can conclude that $d(fx, gx) = 0$. Hence $gx = fx$. The proof is now complete. $\square$

Taking $d = d'$ in Theorem 2.4, we have the following.

Theorem 2.5. *Let (X, d) be a complete metric space endowed with a directed graph G and $g, f : X \to X$ be two mappings such that f and g are a θ-ψ contraction. Suppose that the following conditions hold:*

(1) g is continuous and $g(X)$ is closed;

(2) $f(X) \subseteq g(X)$;

(3) (a) f is G-continuous and g and f are compatible or (b) (X, d, G) has the property A;

(4) $E(G)$ satisfies the transitivity property.

Under these conditions,

$$X(f, g) \neq \emptyset \quad \text{if and only if} \quad C(f, g) \neq \emptyset.$$

Theorem 2.6. *In addition to the hypotheses of Theorem 2.4, assume that*

(8) *for any $x, u \in C(f, g)$ such that $gx \neq gy$, we have $(gx, gu) \in E(G)$.*
If f and g are d'-compatible and $X(f, g) \neq \emptyset$, then $Cm(f, g) \neq \emptyset$.

Proof. Theorem 2.4 implies that there exists a coincidence point $x \in X$, that is, $gx = fx$. Suppose that there exists another coincidence point $y \in X$ such that $gy = fy$. Assume that $gx \neq gy$. By assumption (8), we have

$$\psi d(fx, fy) \leq \theta(d(gx, gy))\psi(d(gx, gy))$$
$$< \psi(d(gx, gy)) = \psi(d(fx, fy)),$$

which is a contradiction. Therefore, $gx = gy$. The proof of $Cm(f, g) \neq \emptyset$ can be derived using a similar argument as in Theorem 2.8 in [1] with the d'-compatibility of f and g. $\square$

Remark 2.7. *Let $(X, \preceq)$ be a partially ordered set, d and d' be two metrics on X such that (X, d') is a complete metric space. Let $E(G) = \{(x, y) \in X \times X : x \preceq y\}$ and $\psi(t) = t$. In this case, we simply obtain results from [1].*

Example 2.8. Let $X = [0, \infty) \subseteq \mathbb{R}$ and the metrics $d, d' : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \max\{x, y\}, & \text{if } x \neq y, \end{cases}$$

and

$$d'(x, y) = |x - y|$$

for all $x, y \in X$, respectively. It is easy to see that $d \geq d'$.

Now, we consider $E(G)$ given by

$$E(G) = \{(x, y) : x = y \text{ or } [x, y \in [0, 1/4] \text{ with } x \leq y]\},$$

where $\leq$ is the usual order.

Consider the mappings $f : X \to X$ and $g : X \to X$ defined by

$$gx = x^2 \text{ and } fx = x^4$$

for all $x \in X$, respectively.

Next, we show that the conditions (1)–(2) in Definition 2.2 hold as follows:

(1) Let $(gx, gy) \in E(G)$.
If $gx = gy$, then $fx = fy$ and $(fx, fy) \in E(G)$.

If $gx, gy \in E(G)$ with $gx \leq gy$, then we obtain $gx = x^2$, $gy = y^2 \in [0, 1/4]$ and $x^2 = gx \leq gy = y^2$. We have $fx = x^4 \leq fy = y^4$ and $fx, fy \in [0, 1/4]$. This implies that $(fx, fy) \in E(G)$;

(2) Let $\psi(t) = \frac{t}{2}$, and let $\theta \in \Theta$ be defined by

$$\theta(t) = \begin{cases} \frac{1}{4}, & \text{if } 0 \leq t < 1, \\ t^2 + 2, & \text{if } t \geq 1. \end{cases}$$

Let x, y be arbitrary points in X and $(gx, gy) \in E(G)$. If $gx = gy$, we have $x = y$ and hence the contractive condition (2.1) holds for this case. In another case, we have

$$gx = x^2, \ gy = y^2 \in [0, 1/4] \text{ with } gx \leq gy.$$

Then we obtain $x, y \in [0, 1/2]$ and $x \leq y$. Also, we have

$$\begin{aligned}
\psi(d(fx, fy)) = \frac{d(fx, fy)}{2} &= \frac{\max\{x^4, y^4\}}{2} \\
&= \frac{y^4}{2} \\
&\leq \frac{1}{4}\frac{y^2}{2} \\
&= \theta(y^2)\frac{y^2}{2} \\
&= \theta(\max\{x^2, y^2\})\frac{\max\{x^2, y^2\}}{2} \\
&= \theta(d(gx, gy))\psi(d(gx, gy)).
\end{aligned}$$

Similarly, we can also prove that the condition (2.1) holds for case of $gx \geq gy$. Therefore, f and g are a θ-ψ contraction.

Next, we show that the conditions (1)–(6) in Theorem 2.4 hold as follows:

(1) We can easily check that $g : (X, d') \to (X, d')$ is continuous. Also, we can see that $g(X) = [0, \infty)$ is d'-closed;

(2) By the definition of f and g, we can see that $f(X) = g(X)$;

(3) It is easy to see that $E(G)$ satisfies the transitivity property;

(4) It follows from $d \geq d'$ that we have nothing to show this condition;

(5) Since $d \neq d'$, we will prove that $f : (X, d') \to (X, d')$ is continuous and g and f are d'-compatible. It is easy to see that $f : (X, d') \to (X, d')$ is

continuous. So we will only show that g and f are d'-compatible. Suppose that $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = a.$$

for some $a \in X$. Now, we have

$$d'(gfx_n, fgx_n) = |x_n^8 - x_n^8| = 0$$

for all $n \in \mathbb{N}$. This implies that $d'(gfx_n, fgx_n) \to 0$ as $n \to \infty$;

(6) Since $d \neq d'$, we have nothing to show this condition.

Consequently, all the conditions of Theorem 2.4 hold. Therefore, g and f have a coincidence point and, further, the points 0 and 1 are coincidence points of the mappings g and f.

3   Some Particular Cases

We begin with some useful background. Throughout this section, let X be a nonempty set and $F : X^2 \to X$, $g : X \to X$ be two mappings.

We define two mappings $T_F^2, G^2 : X \times X \to X \times X$ by

$$T_F^2(x, y) = (F(x, y), F(y, x)) \tag{3.1}$$

and

$$G^2(x, y) = (gx, gy) \tag{3.2}$$

for all $x, y \in X$.

The concept of the cross product of the graphs $G_i = (V_i, E_i), i = 1, 2,$ is defined by

$$G_1 \times G_2 := (V_1 \times V_2, \{((x, y), (x', y'))|(x, x') \in E_1 \text{ and } (y, y') \in E_2\}).$$

Now, the following lemma show that the edge preserving and the transitivity property in the 2-dimensional can be interpreted in terms of two mapping T_F^2 and G^2.

Lemma 3.1. *Let $F : X^2 \to X$, $g : X \to X$ be two mappings, and let (X, d) be a metric space endowed with a directed graph G_1 and G_2. Then*

    (i) If F is G_1-edge preserving and g is G_2-edge preserving, then $T_F^2, G^2 : X \times X \to X \times X$ is $G^*(G^2, T_F^2)$-edge preserving.

(ii) If $E(G_1)$ and $E(G_2)$ satisfies the transitivity property, then so does $E(G^*)$.

Proof. (i) Let F and g is G_1 and G_2-edge preserving and $x, y, u, v \in X$. Then we have

$$(G^2(x, y), G^2(u, v)) \in E(G^*)$$
$$\implies ((gx, gy), (gu, gv)) \in E(G^*)$$
$$\implies (gx, gu) \in E(G_1) \quad \text{and} \quad (gy, gv) \in E(G_2)$$
$$\implies (F(x, y), F(u, v)) \in E(G_1) \quad \text{and} \quad (F(y, x), F(v, u)) \in E(G_2)$$
$$\implies ((F(x, y), F(y, x)), (F(u, v), F(v, u))) \in E(G^*)$$
$$\implies (T_F^2(x, y), T_F^2(F(u, v)) \in E(G^*).$$

(ii) Let $E(G_1)$ and $E(G_2)$ satisfies the transitivity property and $x, y, u, v, a, b \in X$. We have

$$((x, y), (u, v)), ((u, v), (a, b)) \in E(G^*)$$
$$\implies ((x, u), (u, a)) \in E(G_1) \quad \text{and} \quad ((y, v), (v, b)) \in E(G_2)$$
$$\implies (x, a) \in E(G_1) \quad \text{and} \quad (y, b) \in E(G_2)$$
$$\implies ((x, y), (a, b)) \in E(G^*). \qquad \square$$

Suantai et al. [2] gave the notion of a θ-ψ-contraction as follows.

Definition 3.2 ([2])**.** Let (X, d) be a complete metric space endowed with a directed graph G. A pair of mappings $F : X^2 \to X$ and $g : X \to X$ is called a *θ-ψ-contraction* if

(1) F and g are G-edge preserving;

(2) there exists $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y, u, v \in X$ satisfying $(gx, gu), (gy, gv) \in E(G)$,

$$\psi(d(F(x, y), F(u, v))) \leq \theta(M(gx, gu, gy, gv))\psi(M(gx, gu, gy, gv))$$

where $M(gx, gu, gy, gv) = \max\{d(gx, gu), d(gy, gv)\}$.

Let (X, d) be a metric space endowed with a directed graph G satisfying the standard conditions. We consider the set denoted by $(X^2)_g^F$ which is defined by

$$(X^2)_g^F = \{(x, y) \in X^2 : (gx, F(x, y)), (gy, F(y, x)) \in E(G)\}.$$

We denote the set of all coupled coincidence points of mappings $F : X^2 \to X$ and $g : X \to X$ by $CcFix(F)$. In other words,

$$CcFix(F) = \{(x, y) \in X^2 : F(x, y) = gx \text{ and } F(y, x) = gy\}.$$

We ready to present an application of Theorem 2.4 in order to deduce coupled fixed point results.

Theorem 3.3. *Let (X, d') be a complete metric space endowed with a directed graph G, d be another metric on X, and $F : X \times X \to X$ and $g : X \to X$ be a pair of mappings which is a θ-ψ-contraction. Suppose that the following hold:*

(1) $g : (X, d') \to (X, d')$ is continuous and $g(X)$ is d'-closed;

(2) $F(X \times X) \subseteq g(X)$;

(3) if $d \not\geq d'$, assume that $F : (X, d) \times (X, d) \to (X, d')$ is g-uniformly continuous;

(4) if $d \neq d'$, assume that $F : (X, d') \times (X, d') \to (X, d')$ is continuous and g and F are d'-compatible;

(5) if $d = d'$, assume that (a) F is G-continuous and g and F are compatible or (b) (X, d, G) has the property A;

(6) $E(G)$ satisfies the transitivity property.

Under these conditions, $CcFix(F) \neq \emptyset$ if and only if $(X^2)_g^F \neq \emptyset$.

Proof. It is only necessary to apply Theorem 2.4 to the mappings T_F^2 and G^2 in a complete metric space $(X \times X, D')$ and a metric space $(X \times X, D)$, where

$$D'((x, y), (u, v)) = \max\{d'(x, u), d'(y, v)\},$$
$$D((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}$$

and put $G^* = G \times G$, we have

$$((x, y), (u, v)) \in E(G^*) \Leftrightarrow (x, u), (y, v) \in E(G)$$

for all $(x, y), (u, v) \in X \times X$. By Lemma 3.1, we have T_F^2, $G^2 : X \times X \to X \times X$ is $G^*(G^2, T_F^2)$-edge preserving and $E(G^*)$ satisfies the transitivity property. Since $F(X \times X) \subseteq g(X)$, let $x_0, y_0 \in X$, we can construct sequences $\{gx_n\}, \{gy_n\}$ for which

$$gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}) \quad \text{for all} \quad n \in \mathbb{N}.$$

From this, it easy to see that if F is G-continuous, then T_F^2 is G-continuous. Applying the similar argument as the proof of Theorem 3.5 in [1], we have T_F^2 and G^2 are D'-compatible. This completes the proof. $\square$

Remark 3.4. Taking $d = d'$ in Theorem 3.3, we simply obtain the result in [2].

Let us denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}.$$

Now, we defined the notion of $F : X^2 \to X$ and $g : X \to X$ is edge preserving as follow:

Definition 3.5. We say that a pair of mappings $F : X^2 \to X$ and $g : X \to X$ is edge preserving if

$$[(gx, gu) \in E(G), (gy, gv) \in E(G^{-1})]$$
$$\Rightarrow [(F(x,y), F(u,v)) \in E(G), (F(y,x), F(v,u)) \in E(G^{-1})].$$

Remark 3.6. If $F : X^2 \to X$ is edge preserving, then the pair $F : X^2 \to X$ and $I : X \to X$ is edge preserving, where $I(x)$ is the identity map.

Definition 3.7. Let (X, d) be a complete metric space endowed with a directed graph G. A pair of mappings $F : X^2 \to X$ and $g : X \to X$ is called a θ-G-contraction if :

(1) The pair F and g is edge preserving;

(2) there exists $\theta \in \Theta$ such that for all $(gx, gu) \in E(G), (gy, gv) \in E(G^{-1})$ satisfying,

$$d(F(x,y), F(u,v)) \leq \theta \left(\frac{(d(gx,gu) + d(gy,gv))}{2} \right) \frac{(d(gx,gu) + d(gy,gv))}{2}.$$

The next theorem is also obtained by applying our result.

Theorem 3.8. *Let (X, d') be a complete metric space endowed with a directed graph G, and let d be another metric on X. Suppose that $g : X \to X$ and $F : X \times X \to X$ are a θ-G-contraction. Also suppose that the following hold:*

   (1) *$g : (X, d') \to (X, d')$ is continuous and $g(X)$ is d'-closed;*
   (2) *$F(X \times X) \subseteq g(X)$;*
   (3) *if $d \not\geq d'$, assume that $F : (X, d) \times (X, d) \to (X, d')$ is g-uniformly continuous;*

(4) *if $d \neq d'$, assume that $F : (X, d') \times (X, d') \to (X, d')$ is continuous and g and F are d'-compatible;*

(5) *if $d = d'$, assume that* (a) *F is G-continuous and g and f are compatible or* (b) *(X, d, G) has the property A.*

Then, $CcFix(F) \neq \emptyset$ if and only if $(X^2)_g^F \neq \emptyset$.

Proof. As in Theorem 3.3. By assume

$$D'((x,y),(u,v)) = \frac{d'(x,u) + d'(y,v)}{2},$$

$$D((x,y),(u,v)) = \frac{d(x,u) + d(y,v)}{2}$$

and put $G^* = G \times G^{-1}$, we have

$$(x,y),(u,v)) \in E(G^*) \Leftrightarrow (x,u) \in E(G) \text{ and } (y,v) \in E(G^{-1})$$

for all $(x,y),(u,v) \in X \times X$. It is easy to see that T_F^2 and G^2 are D'-compatible. By Lemma 3.1, we have T_F^2, $G^2 : X \times X \to X \times X$ is $G^*(G^2, T_F^2)$-edge preserving, $E(G^*)$ satisfies the transitivity property and T_F^2 is G-continuous. Let $\psi(t) = t$. This completes the proof. $\qquad\square$

References

[1] J. Martínez-Moreno, W. Sintunavarat, Y.J. Cho, Common fixed point theorems for Geraghty's type contraction mappings using the monotone property with two metrics, Fixed Point Theory and Applications 2015:174 (2015) doi:10.1186/s13663-015-0426-y.

[2] S. Suantai, P. Charoensawan, T.A. Lampert, Common coupled fixed point theorems for θ-ψ contraction mappings endowed with a directed graph, Fixed Point Theory and Applications 2015:224 (2015) doi:10.1186/s13663-015-0473-4.

[3] M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc. 40 (1973) 604–608.

[4] A. Amini-Harandi, H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, Nonlinear Anal. 72 (2010) 2238–2242.

[5] R. Precup, Discrete continuation method for boundary value problems on bounded sets in Banach spaces, J. Comput. Appl. Math. 113 (2000) 267–281.

[6] R.P. Agarwal, D. O'Regan, Fixed point theory for generalized contractions on spaces with two metrics, J. Math. Anal. Appl. 248 (2000) 402–414.

[7] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986) 771–779.

[8] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Am. Math. Soc. 136 (2008) 1359-1373

[9] M.R. Alfuraidan, Remarks on monotone multivalued mappings on a metric space with a graph, J. Ineq. Appl. 2015:202 (2015) doi:10.1186/s13660-015-0712-6.

[10] M.R. Alfuraidan, M.A. Khamsi, Caristi, Fixed point theorem in metric spaces with a graph, Abstr. Appl. Anal. 2014 (2014) doi:10.1155/2014/303484.

[11] I. Beg, A.R. Butt, S. Radojević, The contraction principle for set valued mappings on a metric space with a graph, Comput. Math. Appl. 60 (2010) 1214-1219.

[12] F. Bojor, Fixed point theorems for Reich type contractions on metric spaces with a graph, Nonlinear Anal. 75 (9) (2012) 3895-3901.

[13] M.R. Alfuraidan, Remarks on Caristi's fixed point theorem in metric spaces with a graph. Fixed Point Theory Appl. 2014:240 (2014) doi:10.1186/1687-1812-2014-240.