Thai Journal of Mathematics
Volume 15 (2017) Number 2 : 551–563
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Some Iterative Methods for Coincidence Points of Two Continuous Functions on Closed Interval

Supaporn Nirunruttanakit and Suthep Suantai[1]

Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : supaporn_nirun@cmu.ac.th (S. Nirunruttanakit)
suthep.s@cmu.ac.th (S. Suantai)

Abstract : In this paper, we introduce and study an iterative method, called SN-iteration, for approximating a coincidence point of two continuous functions on closed interval in $\mathbb{R}$. A necessary and sufficient condition for convergence of the proposed method is presented. Moreover, we compare the rate of convergence between SN-iteration and W-iteration. Some numerical examples supporting our main results are also given.

Keywords : coincidence point; fixed point; continuous functions; nondecreasing function.
2010 Mathematics Subject Classification : 54H25; 54C05; 47H10; 26A18.

1 Introduction

Many problems in Science and Applied Science are nonlinear problems. Those problems can be formulate as nonlinear equations. In the mathematical point of view, we want to solve a given equation of the form.

$$\text{Find } x \in X \text{ such that } f(x) = g(x) \tag{1.1}$$

where X is a nonempty set and $f, g : X \longrightarrow X$ are two mappings. A point $x \in X$ which is a solution of above equation is called a coincidence point of f and g.

[1]Corresponding author.

It is well known that the existence of a solution to problem (1.1) is, under appropriate conditions, equivalent to the existence of a fixed point for a certain mapping. In this sense, Machuca [1] proved a coincidence theorem by using Banach's contraction principle.

Several physical problems, expressed as a coincidence point equation $Tx = Sx$, are solved by an approximating sequence $\{x_n\} \subseteq X$ generated by an iterative procedure.

In 2015, Ariza-Ruiz [2] introduced an iterative method by using the concept of Mann iteration to prove the existence of coincidence of two mappings.

Fixed point iteration methods are useful tools for solving nonlinear equations. The following clasical iterative methods were used to approximate fixed point of various nonlinear mappings.

Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. A point $p \in E$ is a fixed point of f if $f(p) = p$. We denote by $F(f)$ the set of fixed point of f. It is known that if E is also bounded, then $F(f)$ is nonempty. The Mann iteration (see [3]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0, 1], and will denote by $M(u_1, \alpha_n, f)$. The Ishikawa iteration (see [4]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = (1 - \beta_n)s_n + \beta_n f(s_n) \\ \\ s_{n+1} = (1 - \alpha_n)s_n + \alpha_n f(t_n) \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0, 1], and will denote by $I(s_1, \alpha_n, \beta_n, f)$. The Noor iteration (see [5]) is defined by $a_1 \in E$ and

$$\begin{cases} c_n = & (1 - \gamma_n)a_n + \gamma_n f(a_n) \\ b_n = & (1 - \beta_n)a_n + \beta_n f(c_n) \\ a_{n+1} = & (1 - \alpha_n)a_n + \alpha_n f(b_n) \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in [0, 1], and will denote by $N(a_1, \alpha_n, \beta_n, \gamma_n, f)$. Clearly the Mann and Ishikawa iterations are special cases of Noor iteration. The SP-iteration (see [6]) is defined by $q_1 \in E$ and

$$\begin{cases} r_n = & (1 - \gamma_n)q_n + \gamma_n f(q_n) \\ t_n = & (1 - \beta_n)r_n + \beta_n f(r_n) \\ q_{n+1} = & (1 - \alpha_n)t_n + \alpha_n f(t_n) \end{cases} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in [0, 1], and will denote by $SP(q_1, \alpha_n, \beta_n, \gamma_n, f)$. The S-iteration (see [7]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = & (1 - \beta_n)s_n + \beta_n f(s_n) \\ \\ s_{n+1} = & (1 - \alpha_n)f(s_n) + \alpha_n f(t_n) \end{cases} \tag{1.6}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0, 1], and will denote by $S(s_1, \alpha_n, \beta_n, f)$. The P-iteration (see [8]) is defined by $k_1 \in E$ and

$$\begin{cases} s_n = & (1-\gamma_n)k_n + \gamma_n f(k_n) \\ t_n = & (1-\beta_n)s_n + \beta_n f(s_n) \\ k_{n+1} = & (1-\alpha_n)f(s_n) + \alpha_n f(t_n) \end{cases} \tag{1.7}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in [0, 1], and will denote by $P(k_1, \alpha_n, \beta_n, \gamma_n, f)$. The W-iteration (see [9]) is defined by $w_1 \in E$ and

$$\begin{cases} u_n = & (1-\gamma_n)w_n + \gamma_n f(w_n) \\ v_n = & (1-\beta_n)f(w_n) + \beta_n f(u_n) \\ w_{n+1} = & (1-\alpha_n)f(u_n) + \alpha_n f(v_n) \end{cases} \tag{1.8}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in [0, 1], and will denote by $W(w_1, \alpha_n, \beta_n, \gamma_n, f)$.

In 1976, Rhoades [10] proved the convergence of the Mann and Ishikawa iterations for the class of continuous and non-decreasing functions on unit closed interval. After that in 1991 Borwein and Borwein [11] proved the convergence of the Mann iteration of the continuous functions on a bounded closed interval. It was shown in [6] that the SP-iteration converges faster than Noor itteration, Ishikawa iteration and Mann iteration. In 2013, Kosol [7] showed that the S-iteration coverges faster than the Ishikawa iteration on an arbitrary interavl, after that, Sainuan [8] showed that the P-iteration converges faster than S-iteration on an arbitrary interval.

Motivated by those works mentioned above, we aim to introduce a new iteration method and discuss convergence analysis of the proposed method and compare rate of convergence among those methods.

2 Preliminaries

In this section, we recall some definitions and useful results which will be used for our main result.

In order to compare the rate of convergence of two iterative methods, we employ the concept introduced by Rhoades in 1976 [10]

Definition 2.1. Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. Suppose that $\{x_n\}$ and $\{y_n\}$ are two iterations which converge to a fixed point p of f. Then $\{x_n\}$ is said to *converge faster than* $\{y_n\}$ if $|x_n - p| \leq |y_n - p|$ for all $n \geq 1$.

Lemma 2.2 ([9])**.** *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $x_1 \in E$, let $\{x_n\}$ be a sequence defined by W-iteration. Then the followwing hold:*

1. *If $f(x_1) < x_1$ then $f(x_n) \leq x_n$, for all $n \geq 1$ and $\{x_n\}$ is non-increasing.*

2. *If $f(x_1) > x_1$ then $f(x_n) \geq x_n$, for all $n \geq 1$ and $\{x_n\}$ is non-decreasing.*

Theorem 2.3 ([9]). *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ and $lim_{n\to\infty}\alpha_n = 0$, $lim_{n\to\infty}\beta_n = 0$, $lim_{n\to\infty}\gamma_n = 0$. For $x_1 \in E$, let $\{x_n\}$ be a sequence defined by W-iteration. Then $\{x_n\}$ is bounded if and only if $\{x_n\}$ converges to a fixed point of f.*

Lemma 2.4 ([9]). *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence defined by W-iteration. Then we have the following :*

i) If $p \in F(f)$ with $x_1 > p$, then $x_n \geq p$ for all $n \geq 1$.

ii) If $p \in F(f)$ with $x_1 < p$, then $x_n \leq p$ for all $n \geq 1$.

Lemma 2.5 ([9]). *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $x_1 = q_1 \in E$, let $\{q_n\}$ be a sequence defined by P-Iteration and $\{x_n\}$ be a sequence defined by W-iteration. Then we have the followwing results:*

i) If $f(q_1) < q_1$, then $x_n \leq q_n$ for all $n \geq 1$.

ii) If $f(q_1) > q_1$, then $x_n \geq q_n$ for all $n \geq 1$.

Proposition 2.6 ([9]). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in E; f(p) = p\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$. If $f(x_1) < x_1$, then the sequence $\{x_n\}$ defined by W-iteration dose not converges to a fixed point of f.*

Proposition 2.7 ([9]). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in E; f(p) = p\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$. If $f(x_1) > x_1$, then the sequence $\{x_n\}$ defined by W-iteration dose not converges to a fixed point of f.*

Theorem 2.8 ([9]). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded. For $x_1 = q_1 \in E$, let $\{q_n\}$ and $\{x_n\}$ be the sequences defined by P-iteration and W-iteration, respectively. If $\{q_n\}$ converges to a fixed point p of f, then $\{x_n\}$ converges to p. Moreover $\{x_n\}$ converges faster than $\{q_n\}$.*

3 Main Results

We first introduce our iteration method, called N-iteration, as follows : $x_1 \in E$ and

$$\begin{cases} z_n = & (1-\gamma_n)x_n + \gamma_n f(x_n) \\ y_n = & (1-\beta_n)f(z_n) + \beta_n f^2(z_n) \\ x_{n+1} = & (1-\alpha_n)f(z_n) + \alpha_n f^2(y_n) \end{cases} \tag{3.1}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in [0, 1], and will denote by SN$(x_1, \alpha_n, \beta_n, \gamma_n, f)$. In order to prove convergence of our proposed method, the following Lemma is needed.

Lemma 3.1. *Let E be a closed interval on real line and $f : E \to E$ be continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$be sequences in $[0, 1]$. For $x_1 \in E$, let $\{x_n\}$be a sequence defined by SN-iteration. Then the following hold:*

1. *If $f(x_1) < x_1$, then $f(x_n) \leq x_n$, for all $n \geq 1$ and $\{x_n\}$ is non-increasing.*
2. *If $f(x_1) > x_1$, then $f(x_n) \geq x_n$, for all $n \geq 1$ and $\{x_n\}$ is non-decreasing.*

Proof. i) Let $\{x_n\}$ be a sequence defined by N-iteration and $f(x_1) < x_1$, from the definition of z_1, we get $z_1 = (1-\gamma_1)x_1 + \gamma_1 f(x_1)$, so

$$f(x_1) \leq z_1 \leq x_1.$$

Since f is non-decreasing, we have

$$f^3(x_1) \leq f^2(z_1) \leq f^2(x_1) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1.$$

From $y_1 = (1-\beta_1)f(z_1) + \beta_1 f^2(z_1)$ and $f^2(z_1) \leq f(z_1)$, we get

$$f^2(z_1) \leq y_1 \leq f(z_1)$$

Since f is non-decreasing,

$$f^4(z_1) \leq f^2(y_1) \leq f^3(z_1) \leq f(y_1) \leq f^2(z_1) \leq y_1 \leq f(z_1)$$

From $x_2 = (1-\alpha_1)f(z_1) + \alpha_1 f^2(y_1)$, we get

$$f^2(y_1) \leq x_2 \leq f(z_1) \leq x_1.$$

For x_2, we consider three cases:
<u>Case 1</u>: $f^2(y_1) \leq x_2 \leq f(y_1)$. Since f is non-decreasing, we get

$$f(x_2) \leq f^2(y_1) \leq x_2 \leq f(y_1) \leq f(z_1) \leq x_1.$$

$$\text{Thus } f(x_2) \leq x_2 \text{ and } x_2 \leq x_1.$$

Case 2: $f(y_1) \leq x_2 \leq y_1$. Since f is non-decreasing, we have

$$f(x_2) \leq f(y_1) \leq x_2 \leq y_1 \leq f(z_1) \leq x_1.$$

Thus, we have $f(x_2) \leq x_2$ and $x_2 \leq x_1$.
Case 3: $y_1 \leq x_2 \leq f(z_1)$. Then

$$f(y_1) \leq f(x_2) \leq f^2(z_1) \leq y_1 \leq x_2 \leq f(z_1) \leq x_1.$$

Hence, we have $f(x_2) \leq x_2$ and $x_2 \leq x_1$.
By Case 1, 2 and 3, we have $f(x_2) \leq x_2$ and $x_2 \leq x_1$. By continuing in this way, we can show that $f(x_n) \leq x_n$ and $x_{n+1} \leq x_n$ for all $n \geq 1$.

ii) By using the same argument as i), we obtain the desired result. □

Theorem 3.2. *Let E be a closed interval on real line and $f : E \to E$ be continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ and $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$. For $x_1 \in E$, let $\{x_n\}$ be a sequence defined by SN-iteration. Then $\{x_n\}$ is bounded if and only if $\{x_n\}$ converges to a fixed point of f.*

Proof. It is easy to see that if $\{x_n\}$ converges to a fixed point of f, then $\{x_n\}$ is bounded. Next suppose that $\{x_n\}$ is bounded.

Case 1. $f(x_1) = x_1$: From definition of N-iteration, we obtain that $x_n = x_1$ for all $n \geq 1$. Thus $\{x_n\}$ converge to $x_1 \in F(f)$.

Case 2. $f(x_1) \neq x_1$:

If $f(x_1) < x_1$, by Lemma 3.11, we have $\{x_n\}$ is non-increasing. It follows that $\{x_n\}$ is convergent.

If $f(x_1) > x_1$, by Lemma 3.12, we have $\{x_n\}$ is non-decreasing. It follows that $\{x_n\}$ is convergent.

Next, we will show $\{x_n\}$ converges to a fixed point of f. Let $p = \lim_{n\to\infty} x_n$. By continuity of f, we have that $f(x_n)$ converges to $f(p)$. From $z_n = (1-\gamma_n)x_n + \gamma_n f(x_n)$ and $\lim_{n\to\infty} \gamma_n = 0$, we obtain that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = p.$$

By continuity of f, we have that $f(z_n)$ converges to $f(p)$ and $f^2(z_n)$ converges to $f^2(p)$. From $y_n = (1-\beta_n)f(z_n) + \beta_n f^2(z_n)$ and $\lim_{n\to\infty} \beta_n = 0$, we get

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(z_n) = f(p).$$

By continuity of f, we get $f(y_n)$ converges to $f(p)$ and $f^2(y_n)$ converge to $f^2(p)$. From

$$x_{n+1} = (1-\alpha_n)f(z_n) + \alpha_n f^2(y_n) \text{ and } \lim_{n\to\infty} \alpha_n = 0,$$

we have $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(z_n) = f(p)$. So we get

$$p = f(p).$$

Thus $\{x_n\}$ converges to a fixed point of f. □

Lemma 3.3. *let E be a closed interval on real line and $f : E \to E$ be continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1]$. Let $\{x_n\}$ be a sequence defined by SN-iteration. Then we have the following :*

1. *If $p \in F(f)$ with $x_1 > p$ then $x_n \geq p$, for all $n \geq 1$.*
2. *If $p \in F(f)$ with $x_1 < p$ then $x_n \leq p$, for all $n \geq 1$.*

Proof. i) Let $p \in F(f)$ and $x_1 > p$. Since f is non-decreasing, we get $f(x_1) > f(p)$. From $z_1 = (1-\gamma_1)x_1 + \gamma_1 f(x_1)$ and $x_1 > p$, we have

$$z_1 > (1-\gamma_1)p + \gamma_1 f(p) = p.$$

Since f is non-decreasing, we have $f(z_1) \geq p$ and $f^2(z_1) \geq p$. From $y_1 = (1-\beta_1)f(z_1) + \beta_1 f^2(z_1)$, we get $y_1 \geq p$. It follows that

$$f^2(y_1) \geq p.$$

From $x_2 = (1-\alpha_1)f(z_1) + \alpha_1 f^2(y_1)$, $f(z_1) \geq p$ and $f^2(y_1) \geq p$ we get $x_2 \geq p$. Next, we assume $x_k \geq p$, we will show $x_{k+1} \geq p$. From $z_k = (1-\gamma_k)x_k + \gamma_k f(x_k)$ and $x_k \geq p$, we get

$$z_k \geq (1-\gamma_k)p + \gamma_k p = p.$$

Since f is non-decreasing , we have $f(z_k) \geq p$. From $y_k = (1-\beta_k)f(z_k) + \beta_k f^2(z_k), f(x_k) \geq p$ and $f^2(z_k) > p$, we have

$$y_k \geq (1-\beta_k)p + \beta_k p = p.$$

It follows that $f^2(y_k) \geq p$. From $x_{k+1} = (1-\alpha_k)f(z_k) + \alpha_k f^2(y_k)$and $f(z_k) \geq p$, we get

$$x_{k+1} \geq p \text{ for all n } \geq 1.$$

By induction, we can conclude that $x_n \geq p$ for all $n \geq 1$

ii) By using the same argument as i), we obtain the desired result. □

Theorem 3.4. *Let E be closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1]$. For $a_1 = x_1 \in E$, let $\{w_n\}$ be a sequence defined by W-Iteration and $\{x_n\}$ be a sequence defined by SN-Iteration. Then we have the following results :*

1. *If $f(w_1) < w_1$ then $x_n \leq w_n$ for all $n \geq 1$.*
2. *If $f(w_1) > w_1$ then $x_n \geq w_n$ for all $n \geq 1$.*

Proof. i) Suppose that $f(w_1) < w_1$. From definition of u_1. we have

$$f(w_1) \leq u_1 \leq w_1.$$

Since f is non-decreasing, we get

$$f^3(w_1) \leq f^2(u_1) \leq f^2(w_1) \leq f(u_1) \leq f(w_1) \leq u_1 \leq w_1.$$

Since $w_1 = x_1$, we get $f(w_1) = f(x_1)$. We note that

$$z_1 - u_1 = (1-\gamma_1)(x_1 - w_1) + \gamma_1(f(x_1) - f(w_1)) = 0,$$

so $z_1 = u_1$. Since $f(u_1) \leq f(w_1), f^2(u_1) \leq f(u_1)$ and $z_1 = u_1$ we get

$$f(z_1) \leq f(w_1) \text{ and } f^2(z_1) \leq f(u_1).$$

From definition of r_1, we get $f(u_1) \leq v_1 \leq f(w_1)$. We note that

$$y_1 - v_1 = (1-\beta_1)(f(z_1) - f(w_1)) + \beta_1(f^2(z_1) - f(u_1)) \leq 0$$

so $y_1 \leq v_1$. Since f is non-decreasing, we get

$$f^2(y_1) \leq f^2(v_1) \leq f^2(u_1) \leq f(v_1) \text{ and } f(z_1) = f(u_1).$$

It follows that $x_2 - w_2 = (1-\gamma_1)(f(y_1) - f(u_1)) + \alpha_1(f^2(y_1) - f(v_1))$, so $x_2 \leq w_2$.

Next, we assume that $x_k \leq w_k$. Since f is non - decreasing, we get $f(x_k) \leq f(w_k)$. It follows that

$$z_k - u_k = (1-\gamma_k)(x_k - w_k) + \gamma_k(f(x_k) - f(w_k)) \leq 0,$$

so $z_k \leq u_k$. Since f is non decreasing, $f^2(z_k) \leq f^2(u_k)$.
From Lemma 2.2 1, Lemma 3.1 1 we get $f(w_k) \leq w_k$ and $f(x_k) \leq x_k$ respectively. From definition $z_k = (1-\gamma_k)x_k + \gamma_k f(x_k)$, we have $f(x_k) \leq z_k \leq x_k$. Since f is non-decreasing, we get

$$f^3(x_k) \leq f^2(z_k) \leq f^2(x_k) \leq f(x_k) \leq z_k \leq x_k.$$

From definition $u_k = (1-\gamma_k)w_k + \gamma_k f(w_k)$ and $f(w_k) \leq w_k$, we get

$$f(w_k) \leq u_k \leq w_k.$$

Since f is non decreasing, we have

$$f^2(w_k) \leq f(u_k) \leq f(w_k).$$

Since $f^2(z_k) \leq f(z_k)$, we get

$$f^2(z_k) \leq y_k \leq f(z_k) \leq f(u_k) \leq f(w_k) \leq u_k$$

$$\text{and } f^4(z_k) \leq f^2(y_k) \leq f^3(z_k) \leq f^3(u_k) \leq f^3(w_k) \leq f^2(u_k) \leq f^2(w_k) \leq f(u_k)$$

$\leq f(w_k) \leq u_k$. This implies $f(u_k) \leq v_k \leq f(w_k)$. Since f is non-decreasing, we have

$$f^2(u_k) \leq f(v_k) \leq f^2(w_k).$$

From $f^2(y_k) \leq f^2(u_k)$ and $f^2(u_k) \leq f(v_k)$, we get $f^2(y_k) \leq f(v_k)$. It follows that

$$x_{k+1} - w_{x+1} = (1-\alpha_k)(f(z_k) - f(u_k)) + \alpha_k(f^2(y_k) - f(v_k) \leq 0$$

so $x_{k+1} \leq w_{k+1}$. By induction, we obtain $x_n \leq w_n$ for all $n \geq 1$.

ii) By using same argument as (i), we obtain $x_n \geq w_n$ for all $n \geq 1$. □

Proposition 3.5. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in E; f(p) = p\}$. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. If $f(x_1) < x_1$, then the sequence $\{x_n\}$ defined by SN-iteration does not converges to a fixed point of f*

Proof. Suppose $f(x_1) < x_1$. By Lemma 3.11, $\{x_n\}$ is non - increasing. Since the initial point

$$x_1 < \inf\{p \in E; f(p) = p\},$$

it follows that $\{x_n\}$ does not converges to a fixed point of f. □

Proposition 3.6. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in E; f(p) = p\}$. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$be sequence in $[0,1]$. If $f(x_1) > x_1$, then the sequence $\{x_n\}$ defined by SN-iteration does not converges to a fixed point of f.*

Proof. Suppose $f(x_1) > x_1$. By Lemma 3.12, $\{x_n\}$ is non - decreasing. Since the initial point

$$x_1 > \sup\{p \in E; f(p) = p\},$$

it follows that $\{x_n\}$ does not converges to a fixed of f. □

Theorem 3.7. *Let E be a closed interval on the real line and $f : E \to E$ be continuous and non-decreasing function such that $F(f)$ is non empty set and bounded . For $w_1 = x_1 \in E$, let $\{x_n\}$ and $\{w_n\}$ be the sequences defined by SN-iteration and W - iteration, respectively. If $\{w_n\}$ converges to fixed point p of f then $\{x_n\}$converges to p. Moreover $\{x_n\}$ converges faster than $\{w_n\}$.*

Proof. Suppose that $\{w_n\}$ converges to a fixed point p of f. Let $l = \inf F(f)$ and $u = \sup F(f)$. We divide our proof into three cases.

Case 1 : $w_1 = x_1 > u$.

Since $\{w_n\}$ converge to p, by Proposition 3.6, we get $f(w_1) < w_1$. By Lemma 3.41, we get $x_n \leq w_n$ for all $n \geq 1$. Since f is non-decreasing and $w_1 = x_1$, we have

$$u = f(u) \leq f(x_1) < x_1.$$

It follows that

$$f(x_1) \leq z_1 \leq x_1.$$

Since f is non-decreasing, we get

$$u = f(u) \leq f^3(x_1)$$

$$\text{and } f^3(x_1) \leq f^2(z_1) \leq f^2(x_1) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1.$$

From $y_1 = (1-\beta_1)f(z_1) + \beta_1 f^2(z_1)$, we get $f^2(z_1) \leq y_1 \leq f(z_1)$. Since f is non-decreasing, we get

$$u = f(u) \leq f^2(y_1) \leq f^3(z_1) \leq f(y_1) \leq f^2(z_1) \leq f(z_1).$$

From definition of x_2, we have

$$f^2(y_1) \leq x_2 \leq f(y_1)$$

so we get $u = f(u) \leq f^2(y_1) \leq x_2$. Hence $u \leq x_2$.

Next, we assume $u \leq x_k$. Then $u = f(u) \leq f(x_k)$. By Lemma 3.11 and definition of z_k, we get

$$u = f(u) \leq f(x_k) \leq z_k \leq x_k,$$

so $u \leq z_k \leq x_k$. Since f is non-decreasing, we get

$$u = f(u) \leq f^2(z_k) \leq f^2(x_k) \leq f(z_k) \leq f(x_k) \leq z_k \leq x_k.$$

From difinition of y_k, we get $f^2(z_k) \leq y_k \leq f(z_k)$. Then

$$u = f(u) = f^2(y_k) \leq f^3(z_k) \leq f(z_k).$$

From definition of x_{k+1}, we have $f^2(y_k) \leq x_{k+1} \leq f(z_k)$.
So we get

$$u = f(u) \leq f^2(y_k) \leq x_{k+1}.$$

By induction, we can conclude that $u \leq x_n$ for all $n \geq 1$.
By Lemma 3.31 we have $p \leq x_n \leq w_n$ for all $n \geq 1$. It implies that

$$|x_n - p| \leq |w_n - p| \text{ for all n } \geq 1.$$

It follows that $\{x_n\}$ converges to p faster than $\{w_n\}$ converges to p.
Case 2: $w_1 = x_1 < l$.
Since $\{w_n\}$ converges to p, by Proposition3.5, we get $f(w_1) > w_1$. By Lemma 3.42, we get

$$x_n \geq w_n \text{ for all n } \geq 1.$$

By using the same proof as above, we can show that $x_n \leq l$ for all $n \geq 1$. It implies that

$$|x_n - p| \leq |w_n - p|\,, \text{ for all n } \geq 1.$$

It follows that $\{x_n\}$ converges to p faster than $\{w_n\}$ converges to p.
Case 3: $l < w_1 = x_1 < u$.
If $f(x_1) = x_1$, it follows by definitions of $\{x_n\}$ and $\{w_n\}$ that $x_n = w_n = x_1$ for all $n \geq 1$. So

$$p = x_1 \text{ and } |x_n - p| = |w_n - p| \text{ for all n } \geq 1.$$

If $f(x_1) < x_1$, by Lemma 3.11, we get $\{w_n\}$ is non-increasing. It follows that $p \leq w_n$ for all n ≥ 1. By Lemma 3.41, we get

$$p \leq x_n \leq w_n \text{ for all n } \geq 1.$$

It implies that

$$|x_n - p| \leq |w_n - p| \text{ for all n } \geq 1.$$

It follows that $\{x_n\}$ converges to p faster than $\{w_n\}$ converges to p.
If $f(x_1) > x_1$, by Lemma 3.12, we get $\{w_n\}$ is non-increasing. It follows that

$$p \leq w_n \text{ for all n } \geq 1.$$

By Lemma 3.42, we get $p \leq x_n \leq w_n$ for all $n \geq 1$. It implies that

$$|x_n - p| \leq |w_n - p| \text{ for all n } \geq 1.$$

It follows that $\{x_n\}$ converges to p faster than $\{w_n\}$ converges to p. □

Example 3.8. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x^2-3x}{7}$ and $g(x) = \frac{4x-12}{7}$. We would like to find a coincidence point of f and g.

Consider

$$f(x) - g(x) = \frac{x^2 - 7x + 12}{7}$$

$$\text{so } f(x) - g(x) + x = \frac{x^2 + 12}{7}.$$

To do this, let

$$h(x) = f(x) - g(x) + x.$$

We see that a coincidence point of f and g is a fixed point of h. It is clear that h is continuous and non-decreasing function.

The numerical results of P, W and SN-iterations are given in Table 1 when $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$ for all $n \geq 1$ and initial point $x_1 = 1$. It is observed that $\{k_n\}$, $\{w_n\}$ and $\{x_n\}$ converge to 3 which is a fixed point h (a coincidence point of f and g) and note that the sequence $\{x_n\}$ generated by SN-iteration converges faster than the sequence generated by P and W - iterations.

Table 1: The comparision of the convergence P, W and SN-iterations to the fixed point $p = 3$ of $h(x)$.

	P - iteration	W-iteration	SN - iteration	
n	k_n	w_n	x_n	$\|f(x_n) - x_n\|$
2	2.13553997787607	2.24723450991630	2.51878141774077	0.10182712945297
3	2.41789297649809	2.51672231726532	2.70607402202513	0.05433120835762
4	2.56953075488760	2.65289355428222	2.79338979802626	0.03561399679048
5	2.66755218616547	2.73741728668975	2.84579661620831	0.02542600962349
6	2.73660735743429	2.79527662577310	2.88100739212889	0.01902169265700
7	2.78775338213773	2.83721114345012	2.90620906853544	0.01465538146993
8	2.82689683304958	2.86875447664451	2.92498290256889	0.01152066604829
9	2.85755244535065	2.89310369446013	2.93935526260808	0.00918893165217
10	2.88197164453972	2.91225809600929	2.95057609443455	0.00740951828668

The following graphs show the convergence of the sequence generated by P, W and SN-iterations.

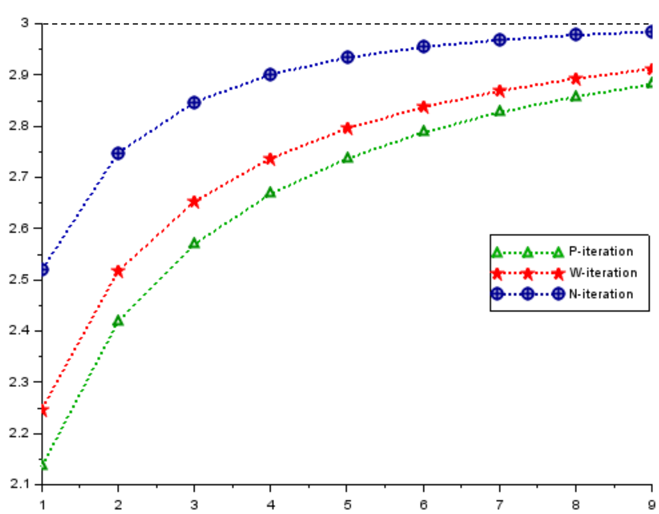

Figure 1: Graph of the convergence of P, W and SN-iterations

Acknowledgements : The authors would like to thank the referees for many comments and suggestions to improve the exposition of this paper and the Thailand Research Fund under the project RTA5780007 and Chiang Mai University, Chiang Mai, Thailand for the financial support.

References

[1] R. Machuca, A coincidence problem, Amer. Math. Month. 74 (1967) 469.

[2] D. Ariza-Ruiz, J. Garcia-Falset, Existence and uniqueness of solution to several kinds of differential equations using the coincidence theory, Taiwanese J. Math. (2015) 1661-1692.

[3] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. (1953) 506-510.

[4] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 4 (1974) 147-150.

[5] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 215 (2000) 217-229.

[6] S. Suantai, W. Phuengrattana, On the rate of convergence of Mann, Ishikawa, Noor and SP-iteration for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 235 (2011) 3006-3014.

[7] S. Kosol, Rate of convergence of S-iteration and Ishikawa iteration for continuous functions on closed itervals, Thai J. Math. 11 (2013) 703-709.

[8] P. Sainuan, Rate of convergence of P-iteration and S-iteration for continuous functions on closed intervals, Thai J. Math. 13 (2015) 1-9.

[9] S. Wongyai, S. suantai, Convergence Theorem and Rate of Convergence of a New Iterative Method for Continuous Functions on Closed Interval, AMM and APAM Conference Proceedings, 2016, Chulalongkorn University, May 23-25, (2016), 111-118.

[10] B.E. Rhoades, Comments on two fixed point iteration methods, J. Math. Anal. Appl. 56 (1976) 741-750.

[11] D. Borwein, J. Borwein. Fixed point iterations for real functions. J. Math. Anal. Appl. 157 (1991) 112-126.

(Received 14 May 2017)
(Accepted 31 August 2017)