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On the Solution of Triharmonic Bessel Heat Equation

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Abstract : In this paper, we study the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \otimes_B^k u(x,t)$$

with the initial condition $u(x,0) = f(x)$ for $x \in R_n^+$. The operator $\otimes_B^k$ is the operator iterated $k-$ times and defined by

$$\otimes_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^3 - \left(\sum_{j=p+1}^{p+q} B_{x_i} \right)^3 \right)^k,$$

where $p+q = n$ is the dimension of the $\mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $i = 1, 2, 3, ..., n$, and k is a nonnegative integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in R_n^+ \times (0,\infty)$, $f(x)$ is a given generalized function and c is a positive constant. By the Fourier transform in sense of Distribution theory we obtain the solution of such equation and related to the triharmonic Besel heat equation.

Keywords : spectrum, tempered distribution, diamond Bessel operator.
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1 Introduction

The operator $\diamondsuit^k$ has been first by A. Kananthai [1] and is named as the Diamond operator iterated k times and defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \ p+q=n. \tag{1.1}$$

n is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. The operator $\diamondsuit^k$ can be expressed in the form

$$\diamondsuit^k = \triangle^k \Box^k = \Box^k \triangle^k, \tag{1.2}$$

where $\triangle^k$ is the Laplacian operator iterated $k-$ times defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right)^k \tag{1.3}$$

and $\Box^k$ is the Ultra-hyperbolic operator iterated $k-$ times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \tag{1.4}$$

A. Kananthai [1] has shown that the solution of the convolution form

$$u(x) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x)$$

is a unique elementary solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x) \right) = \delta. \tag{1.5}$$

The function $R^H_\alpha(\upsilon)$ is called the *Ultra-hyperbolic kernel of Marcel Riesz* was introduced by Y. Nozaki (see [2], p. 72) defined by

$$R^H_\alpha(\upsilon) = \begin{cases} \frac{\upsilon^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{1.6}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}. \tag{1.7}$$

And the function $R^e_\alpha(x)$ denoted the elliptic kernel of Marcel Riesz and defined by

$$R^e_\alpha(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \tag{1.8}$$

where

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^{\alpha} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \tag{1.9}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.
In 2004, Hüseyin Yildirim, M. Zeki Sarikaya and Sermin Öztürk (see [3, 4]) first introduced the Bessel diamond operator $\Diamond_B^k$ iterated $k-$times, defined by

$$\Diamond_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right)^k \tag{1.10}$$

where $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, x_i > 0$. The operator $\Diamond_B^k$ can be expressed by $\Diamond_B^k = \triangle_B^k \Box_B^k = \Box_B^k \triangle_B^k$, where

$$\triangle_B^k = \left(\sum_{i=1}^{p} B_{x_i} \right)^k \quad \text{and} \quad \Box_B^k = \left(\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right)^k. \tag{1.11}$$

And, Hüseyin Yildirim, M. Zeki Sarikaya and Sermin Öztürk (see [3,4]) have shown that the solution of the convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is a unique elementary solution of $\Diamond_B^k$ that is

$$\Diamond_B^k((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta. \tag{1.12}$$

The function $S_\alpha(x)$ define by

$$S_\alpha(x) = \frac{|x|^{\alpha - n - 2|\nu|}}{w_n(\alpha)}, \tag{1.13}$$

where $|x| = x_1^2 + x_2^2 + \ldots + x_n^2, |\nu| = \nu_1 + \nu_2 + \ldots + \nu_n$ and,

$$w_n(\alpha) = \frac{\prod_{i=1}^{n} 2^{\nu_i - \frac{1}{2}} \Gamma(\nu_i + \frac{1}{2})}{2^{n+2|\nu|-2\alpha} \Gamma(\frac{n+2|\nu|-\alpha}{2})}.$$

The function $R_\gamma(x)$ defined by

$$R_\gamma(x) = \begin{cases} \frac{V^{\frac{\gamma-n-2|\nu|}{2}}}{K_n(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{1.14}$$

where

$$K_n(\gamma) = \frac{\pi^{\frac{n+2|\nu|-1}{2}} \Gamma\left(\frac{2+\gamma-n-2|\nu|}{2}\right) \Gamma\left(\frac{1-\gamma}{2}\right) \Gamma(\gamma)}{\Gamma\left(\frac{2+\gamma-p-2|\nu|}{2}\right) \Gamma\left(\frac{p-2|\nu|-\gamma}{2}\right)},$$

and γ is a complex number.

Furthermore, W. Satsanit has first introduced the Bessel $\otimes_B^k$ (see [5]) and defined by

$$\begin{aligned}\otimes_B^k &= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^3 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^3\right]^k \\ &= \left(\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right)^k \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{i=1}^{p} B_{x_i}\right)\cdot\right. \\ &\quad \left.\left(\sum_{j=p+1}^{p+q} B_{x_j}\right) + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k \\ &= \Box_B^k\left(\triangle_B^2 - \frac{1}{4}(\triangle_B + \Box_B)(\triangle_B - \Box_B)\right)^k \\ &= \left(\frac{3}{4}\diamondsuit_B\triangle_B + \frac{1}{4}\Box_B^3\right)^k \qquad (1.15)\end{aligned}$$

and $\diamondsuit_B$, $\triangle_B$ and $\Box_B$ are defined by (1.10) and (1.11) with $k = 1$ respectively. It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\triangle u(x,t) \qquad (1.16)$$

with the initial condition $u(x,0) = f(x)$, $\triangle = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\int_{\mathbb{R}^n}\exp\left(-\frac{|x-y|^2}{4c^2t}\right)f(y)dy \qquad (1.17)$$

as the solution of (1.16). The equation, (1.17) can be written

$$u(x,t) = E(x,t) * f(x),$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4c^2t}\right). \qquad (1.18)$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$ (see [6], pp. 208-209).

Next, Hüseyin Yildirim, M. Zeki Sarikaya and A. Saglam (see [7]) have study the following equation,

$$\frac{\partial}{\partial t}u(x,t) = c^2\diamondsuit_B^k u(x,t) \qquad (1.19)$$

with the initial condition

$$u(x,0) = f(x), \quad for \quad x \in R_n^+, \tag{1.20}$$

where the operator $\Diamond_B^k$ is named the Bessel diamond operator iterated $k-$ times, and defined by (1.10), k is a positive integer, $u(x,t)$ i an unknown function, $f(x)$ is the given generalized function and c is a constant, $p+q=n$ is the dimension of the $R_n^+ = \{x : x = (x_1, x_2, \ldots, x_n, t), x_i > 0, i = 1, 2, 3, \ldots, n\}$.

They obtain the solution in the classical convolution form

$$u(x,t) = E(x,t) * f(x), \tag{1.21}$$

where the symbol $*$ is the $B-$ convolution in (2.1), as a solution of (1.19) and satisfies (1.20), where

$$E(x,t) = C_v \int_{\Omega^+} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^2 - \left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^2\right]^k} \prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} dy \tag{1.22}$$

and $\Omega^+ \subset R_n^+$ is the spectrum of $E(x,t)$ for any fixed $t > 0$ and $J_{v_i-\frac{1}{2}}(x_i, y_i)$ is the normalized Bessel function.

Now, the purpose of this work is to study the following equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \otimes_B^k u(x,t) \tag{1.23}$$

with the initial condition

$$u(x,0) = f(x), for \quad x \in R_n^+, \tag{1.24}$$

where the operator $\otimes_B^k$ defined by (1.15) $u(x,t)$ is an unknown unction, $f(x)$ is the given generalized function, k is a positive integer, and c is a positive constant.

Moreover, Bessel heat kernel has interesting properties and also related to the kernel of an extension of the heat equation. We obtain the solution in the classical convolution form

$$u(x,t) = E(x,t) * f(x), \tag{1.25}$$

where the symbol $*$ is the $B-$ convolution in (2.1), as a solution of (1.23)and satisfies (1.24), where

$$E(x,t) = C_v \int_{\Omega^+} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3 - \left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} \prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} dy \tag{1.26}$$

and $\Omega^+ \subset R_n^+$ is the spectrum of $E(x,t)$ for any fixed $t > 0$ and $J_{v_i-\frac{1}{2}}(x_i, y_i)$ is the normalized Bessel function. Before going into details, the following definitions and some important concepts are needed.

2 Preliminaries

The shift operator according to the law remark that this shift operator connected to the Bessel differential operator (see [8–10]).

$$T_x^y \varphi(x) = C_v^* \int_0^{\pi} \cdots \int_0^{\pi} \varphi(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, \cdots \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n}) \left(\Pi_{i=1}^n sin^{2v_i - 1}\theta_i\right) d\theta_1 \cdots d\theta_n$$

where $x, y \in R_n^+, C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected to the Bessel differential operator [see2,3,5]

$$\frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}$$

$$U(x,0) = f(x) \ \ , \ U_y(x,0) = 0.$$

The convolution operator determined by the T_x^y is as follows:

$$(f * \varphi)(y) = \int_{R_n^+} f(y) T_x^y \varphi(x) \left(\Pi_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.1}$$

Convolution (2.1) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator:

(1) $T_x^y \cdot 1 = 1$

(2) $T_x^0 f(x) = f(x)$

(3) If $f(x), g(x) \in C(R_n^+)$, $g(x)$ is a bounded function for all $x > 0$ and $\int_0^{\infty} |f(x)| \left(\prod_{i=1}^n x_i^{2v_i}\right) dx < \infty$ then $\int_{R_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy = \int_{R_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy$.

(4) From (3), we have the following equality for $g(x) = 1$:

$$\int_{R_n^+} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy = \int_{R_n^+} f(y) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy.$$

(5) $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows

$$\left(\mathcal{F}_B f\right)(x) = C_v \int_{R_n^+} f(y) \left(\prod_{i=1}^n J_{v_i - \frac{1}{2}}(x_i, y_i) y_i^{2v_i}\right) dy, \tag{2.2}$$

$$\left(\mathcal{F}_B^{-1} f\right)(x) = \left(\mathcal{F}_B f\right)(-x), \quad C_v = \left(\prod_{i=1}^n 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right)\right)^{-1}, \tag{2.3}$$

where $J_{v_i-\frac{1}{2}}(x_i, y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. The following equalities for Fourier-Bessel transformation are true (see [8–10]).

$$\mathcal{F}_B\delta(x) = 1 \tag{2.4}$$

$$\mathcal{F}_B(f * g)(x) = \mathcal{F}_B f(x).\mathcal{F}_B g(x). \tag{2.5}$$

Definition 2.1. The spectrum of the kernel $E(x,t)$ of (1.26) is the bounded support of the Fourier Bessel transform $\mathcal{F}_B E(y,t)$ for any fixed $t > 0$.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}_n^+$ and denote by

$$\Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \ldots - x_{p+q}^2 > 0 \text{ and } \xi_1 > 0\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω^+ be spectrum of $E(x,t)$ defined by (1.26) for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $F_B E(y,t)$ be the Fourier Bessel transform of $E(x,t)$, defined by

$$\mathcal{F}_B E(y,t) = \begin{cases} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3-\left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} & \text{for } \xi \in \Omega_+, \\ 0 & \text{for } \xi \notin \Omega_+. \end{cases} \tag{2.6}$$

Lemma 2.1 (Fourier Bessel Transform of $\Box_B^k$ Operator)**.**

$$\mathcal{F}_B \Box_B^k u(x) = (-1)^k V_1^k(x) \mathcal{F}_B u(x),$$

where

$$V_1^k(x) = \left(\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2\right)^k.$$

Proof. See [7]. □

Lemma 2.2 (Fourier Bessel Transform of $\triangle_B^k$ Operator)**.**

$$\mathcal{F}_B \triangle_B^k u(x) = (-1)^k |x|^{2k} \mathcal{F}_B u(x),$$

where

$$|x|^{2k} = \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^k.$$

Proof. See [7]. □

Lemma 2.3 (Fourier Bessel Transform of $\otimes_B^k$ Operator)**.**

$$\mathcal{F}_B \otimes_B^k u(x) = (-1)^k V^k(x) \mathcal{F}_B u(x),$$

where

$$V^k(x) = \left(\left(\sum_{i=1}^{p} x_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} x_j^2\right)^3\right)^k.$$

Proof. We can use the mathematical induction method, for $k=1$, we have

$$\begin{aligned}
&\mathcal{F}_B(\otimes_B u)(x)\\
&\quad = C_v \int_{R_n^+} (\otimes_B u(y)) \left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i,y_i)\, y_i^{2v_i}\right) dy\\
&\quad = C_v \int_{R_n^+} \Box_B \left(\frac{3}{4}\triangle_B^2 + \frac{1}{4}\Box_B^2\right) u(y) \left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i,y_i)\, y_i^{2v_i}\right) dy\\
&\quad = C_v \int_{R_n^+} \left(\frac{3}{4}\triangle_B^2 + \frac{1}{4}\Box_B^2\right) g(y) \left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i,y_i)\, y_i^{2v_i}\right) dy, \quad g(y) = \Box_B u(y)\\
&\quad = \mathcal{F}_B \left(\frac{3}{4}\triangle_B^2 + \frac{1}{4}\Box_B^2\right)(x)\\
&\quad = \frac{3(-1)^2(x_1^2+\ldots+x_n^2)^2 + (-1)^2(x_1^2+\ldots+x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2)^2}{4}\mathcal{F}_B g(x)\\
&\quad = \left(\left(\sum_{i=1}^{p} x_i^2\right)^2 + \left(\sum_{i=1}^{p} x_i^2\right)\left(\sum_{j=p+1}^{p+q} x_j^2\right) + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^2\right)\mathcal{F}_B \Box_B u(x)\\
&\quad = \left(\left(x_1^2+x_2^2+\ldots+x_p^2\right)^3 - \left(x_{p+1}^2+\ldots+x_{p+q}^2\right)^3\right)\mathcal{F}_B u(x)\\
&\quad = V(x)\mathcal{F}_B u(x),
\end{aligned}$$

where $V(x) = \left(x_1^2+x_2^2+\ldots+x_p^2\right)^3 - \left(x_{p+1}^2+\ldots+x_{p+q}^2\right)^3$. By inverse Fourier transform we obtain

$$\otimes_B u(x) = \mathcal{F}_B^{-1} V(x)\mathcal{F}_B u(x).$$

Assume the statement is true for $k-1$, that is

$$\otimes_B^{k-1} u(x) = \mathcal{F}_B^{-1} V^{k-1}(x)\mathcal{F}_B u(x).$$

We must prove that is also true for $k \in N$. So we have

$$\begin{aligned}
\otimes_B^{k} u(x) &= \otimes_B\left(\otimes_B^{k-1} u(x)\right)\\
&= \mathcal{F}_B^{-1} V(x)\mathcal{F}_B \mathcal{F}_B^{-1} V^{k-1}(x)\mathcal{F}_B u(x)\\
&= \mathcal{F}_B^{-1} V^{k}(x)\mathcal{F}_B u(x).
\end{aligned}$$

This completes the proof. □

Lemma 2.4. *For $t,\ v>0$ and $x,y \in \mathbb{R}^n$, we have*

$$\int_0^\infty e^{-c^2x^2t} x^{2v} dx = \frac{\Gamma(v)}{2c^{2v+1}t^{v+\frac{1}{2}}} \tag{2.7}$$

and

$$\int_0^\infty e^{-c^2x^2t} J_{v-\frac{1}{2}}(xy) x^{2v} dx = \frac{\Gamma(v+\frac{1}{2})}{2\left(c^2t\right)^{v+\frac{1}{2}}} e^{-\frac{y^2}{4c^2t}}, \tag{2.8}$$

where c is a positive constant.

Proof. See [7]. □

Lemma 2.5. *Let the operator L be defined by*

$$L = \frac{\partial}{\partial t} - c^2 \otimes_B^k, \tag{2.9}$$

where $\otimes_B^k$ is the operator iterated $k-$ times, and is given by

$$\otimes_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^3 - \left(\sum_{j=p+1}^{p+q} B_{x_i} \right)^3 \right)^k,$$

and

$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i}$$

$p+q=n$ is the dimension $\mathbb{R}_n^+$, k is a positive integer, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$, and c is a positive constant. Then

$$E(x,t) = C_v \int_{\Omega^+} e^{c^2 t \left[\left(y_1^2 + \ldots + y_p^2\right)^3 - \left(y_{p+1}^2 + \ldots + y_{p+q}^2\right)^3 \right]^k} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left(x_i, y_i\right) y_i^{2v_i} dy \tag{2.10}$$

is the elementary solution of (2.9) in the spectrum $\Omega^+ \subset R_n^+$ for $t > 0$.

Proof. Let $LE(x,t) = \delta(x,t)$, where $E(x,t)$ is the elementary solution of L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t} E(x,t) - c^2 \otimes_B^k E(x,t) = \delta(x)\delta(t).$$

Applying the Fourier Bessel transform, which is defined by (2.2) to the both sides of the above equation and using Lemma 2.3 by considering $\mathcal{F}_B \delta(x) = 1$, we obtain

$$\frac{\partial}{\partial t} \mathcal{F}_B E(x,t) - c^2 \left[\left(x_1^2 + x_2^2 + \ldots + x_p^2\right)^3 - \left(x_{p+1}^2 + \ldots + x_{p+q}^2\right)^3 \right]^k \mathcal{F}_B E(x,t) = \delta(t).$$

Thus, we get

$$\mathcal{F}_B E(x,t) = H(t) e^{c^2 t \left[\left(x_1^2 + x_2^2 + \ldots + x_p^2\right)^3 - \left(x_{p+1}^2 + \ldots + x_{p+q}^2\right)^3 \right]^k},$$

where $H(t)$ is the Heaviside function, because $H(t) = 1$ holds for $t \geq 0$.

Therefore,

$$\mathcal{F}_B E(x,t) = e^{c^2 t \left[\left(x_1^2 + x_2^2 + \ldots + x_p^2\right)^3 - \left(x_{p+1}^2 + \ldots + x_{p+q}^2\right)^3 \right]^k},$$

which has been already by (2.5). Thus from (2.3), we have

$$E(x,t) = C_v \int_{\mathbb{R}_n^+} e^{c^2 t \left[\left(y_1^2 + \ldots + y_p^2\right)^3 - \left(y_{p+1}^2 + \ldots + y_{p+q}^2\right)^3 \right]^k} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left(x_i, y_i\right) y_i^{2v_i} dy$$

where Ω^+ is the spectrum of $E(x,t)$. Thus, we obtain

$$E(x,t) = C_v \int_{\Omega^+} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3-\left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} \prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i, y_i\right) y_i^{2v_i} dy$$

as an elementary solution of (2.9) in the spectrum $\Omega^+ \subset R_n^+$ for $t > 0$. □

3 Main Results

Theorem 3.1. *Let us consider the equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2 \otimes_B^k u(x,t) = 0 \tag{3.1}$$

with the initial condition

$$u(x,0) = f(x) \tag{3.2}$$

where $\otimes_B^k$ is the operator iterated $k-$ times, and is defined by

$$\otimes_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i}\right)^3 - \left(\sum_{j=p+1}^{p+q} B_{x_i}\right)^3\right)^k,$$

and

$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i},$$

$p+q=n$ is the dimension $\mathbb{R}_n^+$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}_n^+ \times (0,\infty)$, $f(x)$ is the given generalized function, and c is a positive constant. Then

$$u(x,t) = E(x,t) * f(x) \tag{3.3}$$

is a solution of (3.1) and satisfies (3.2), where $E(x,t)$ is given by (2.10). In particular, if we put $k=1$ and $q=0$ in (3.1), then (3.1) reduces to the equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle_B^3 u(x,t) = 0,$$

which is related the Triharmonic Bessel heat equation.

Proof. Taking the Fourier Bessel transform, which is defined by (2.2), of the both sides of (3.1) for $x \in \mathbb{R}_n^+$ and using Lemma 2.3, we obtain

$$\frac{\partial}{\partial t}\mathcal{F}_B u(x,t) = c^2\left(\left(x_1^2+\ldots+x_p^2\right)^3 - \left(x_{p+1}^2+\ldots+x_{p+q}^2\right)^3\right)^k \mathcal{F}_B u(x,t). \tag{3.4}$$

We consider the initial condition (3.2), then we have the following equality for (3.4)

$$u(x,t) = f(x) * \mathcal{F}_B^{-1} e^{c^2 t\left[\left(x_1^2+\ldots+x_p^2\right)^3-\left(x_{p+1}^2+\ldots+x_{p+q}^2\right)^3\right]^k}. \tag{3.5}$$

Here, if we use (2.2) and (2.3), then we have

$$
\begin{aligned}
u(x,t) &= f(x) * \mathcal{F}_B^{-1} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3-\left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} \\
&= \int_{\mathbb{R}_n^+} \mathcal{F}_B^{-1} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3-\left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy \\
&= \int_{\mathbb{R}_n^+} \left(C_v \int_{\mathbb{R}_n^+} e^{c^2 t V^k(z)} \prod_{i=1}^n J_{v_i-\frac{1}{2}}(y_i, z_i) z_i^{2v_i} dz \right) T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy,
\end{aligned}
\tag{3.6}
$$

where $V(z) = \left(z_1^2 + z_2^2 + \ldots + z_p^2\right)^3 - \left(z_{p+1}^2 + z_{p+2}^2 + \ldots + z_{p+q}^2\right)^3$. Set

$$
E(x,t) = C_v \int_{\mathbb{R}_n^+} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3-\left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} \prod_{i=1}^n J_{v_i-\frac{1}{2}}\left(x_i, y_i\right) y_i^{2v_i} dy.
\tag{3.7}
$$

Since the integral in (3.7) is divergent, therefore we choose $\Omega^+ \subset \mathbb{R}_n^+$ be the spectrum of $E(x,t)$ and by (2.9), we have

$$
\begin{aligned}
E(x,t) = &\quad C_v \int_{\mathbb{R}_n^+} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3-\left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} \prod_{i=1}^n J_{v_i-\frac{1}{2}}\left(x_i, y_i\right) y_i^{2v_i} dy \\
= &\quad C_v \int_{\Omega^+} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3-\left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} \prod_{i=1}^n J_{v_i-\frac{1}{2}}\left(x_i, y_i\right) y_i^{2v_i} dy.
\end{aligned}
\tag{3.8}
$$

Thus (3.6) can be written in the convolution form

$$
u(x,t) = E(x,t) * f(x)
$$

Moreover, since $E(x,t)$ exists, we can see that

$$
\begin{aligned}
\lim_{t\to 0} E(x,t) &= C_v \int_{\Omega^+} \prod_{i=1}^n J_{v_i-\frac{1}{2}}(x_i, y_i) y_i^{2v_i} dy \\
&= C_v \int_{\mathbb{R}_n^+} \prod_{i=1}^n J_{v_i-\frac{1}{2}}(x_i, y_i) y_i^{2v_i} dy \\
&= \delta(x), \quad \text{for} \quad x \in \mathbb{R}_n^+.
\end{aligned}
\tag{3.9}
$$

hold (see [7]). Thus for the solution $u(x,t) = E(x,t) * f(x)$ of (3.1), then we have

$$
\lim_{t\to 0} u(x,t) = u(x,0) = \delta * f(x) = f(x)
$$

which satisfies (3.2). This completes the proof. □

Theorem 3.2. *The kernel $E(x,t)$ defined by (3.8) has the following properties:*

(1) $E(x,t) \in \mathcal{C}^{\infty}$*-the space of continuous function for* $x \in \mathbb{R}^n$*,* $t > 0$ *with infinitely differentiable.*

(2) $\left(\dfrac{\partial}{\partial t} - c^2 \otimes_B^k\right) E(x,t) = 0$ *for all* $x \in \mathbb{R}_n^+, t > 0.$

(3) $\lim_{t \to 0} E(x,t) = \delta$ *for all* $x \in \mathbb{R}_n^+$.

Proof. (1) From (3.8) and

$$\frac{\partial^n}{\partial t^n} E(x,t) = C_v \int_{\mathbb{R}_n^+} \frac{\partial^n}{\partial t^n} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^3-\left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^3\right]^k} \prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i, y_i\right) y_i^{2v_i} dy,$$

we have $E(x,t) \in C^{\infty}$ for $x \in \mathbb{R}_n^+, t > 0$.
(2) We have $u(x,t) = E(x,t)$ since $u(x,t) = E(x,t) * f(x)$ holds. Note here that, we use the fact $f(x) = \delta(x)$ by the Fourier Bessel transformation. Then, we obtain

$$\left(\frac{\partial}{\partial t} - c^2 \otimes_B^k\right) E(x,t) = 0$$

by direct computation.
(3) This case is obvious by (3.9).

In particular, if we put $k = 1$ and $q = 0$ in (3.1) then (3.1) reduces to the equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle_B^3 u(x,t) = 0$$

which has solution

$$u(x,t) = E(x,t) * f(x)$$

where $E(x,t)$ is defined by (2.10) with $k = 1$ which is related to triharmonic Bessel heat equation. This complete the proof. □

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