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Dynamic Analysis of a Fractional Order Phytoplankton-Zooplankton System with a Crowley-Martin Functional Response

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Abstract : In this work, we investigate the dynamical behaviour of a fractional order phytoplankton–zooplankton system(PZS) with a Crowley–Martin functional response. Local stability analysis of biologically feasible equilibrium points is worked out with help of ecological as well as disease basic reproduction numbers. We proved that the equilibrium $E_0 = (0, 0, 0)$ of the PZS is a saddle point. We proved that the equilibrium $E_1 = (\frac{1}{\gamma}, 0, 0)$ of the system is asymptotically stabile if $R_0 < 1$ and $R_0^* < 1$. Also we proved that the equilibrium $E_2 = (S_2, I_2, 0)$ of the system if $R_0(1) > 1$. Numerical simulations are carried out for a hypothetical set of parameter values to substantiate our analytical findings.

Keywords : phytoplankton–zooplankton model; stability conditions; numerical simulation; fractional order.
2010 Mathematics Subject Classification : 26A33, 65R10, 37N25.

1 Introduction

Phytoplankton are microscopic algae found in marine and fresh water and represent a major source of food and oxygen for wildlife inhabiting lakes, rivers, estuaries and oceans. Plankton are microscopic organisms that float freely with oceanic currents and in other bodies of water. They are made up of tiny plants (called phytoplankton) and tiny animals (called zooplankton). Plankton is the basis of all aquatic food chains, and phytoplankton in particular lies on the first trophic level of the food chain. The animals in the plankton community are known as zooplankton. The phytoplankton are consumed by zooplankton, their animal counterparts, which considered to be most favorable food sources for fish and other aquatic animals. The importance of plankton for the wealth of the ocean ecosystem and ultimately for the planet itself is nowadays widely recognized. Mathematical models to study algae blooms are now classical [1], but research to investigate these phenomena and the mechanisms behind them is still ongoing [2]. The manifestation in new geographic areas and the increase in the number of blooms of both toxic as well as non toxic algae species are reported in [3].

Many authors have studied phytoplankton-zooplankton models. A phytoplankton-zooplankton model with harvesting is proposed and investigated in [4]. Rehim and Imran [5] investigated the interaction of toxic–phytoplankton–zooplankton systems and their dynamical behavior based upon nonlinear ordinary differential equation model system. Das and Ray [6] investigated the effect of delay on nutrient cycling in phytoplankton–zooplankton interactions in the estuarine system. Saha and Bandyopadhyay [7] considered a toxin producing phytoplankton–zooplankton model in which the toxin liberation by phytoplankton species follows a discrete time variation. In [8], authors have dealt with a nutrient–plankton model in an aquatic environment in the context of phytoplankton bloom. In [9], models of nutrient–plankton interaction with a toxic substance that inhibits either the growth rate of phytoplankton, zooplankton or both trophic levels are proposed and studied.

In [10] the authors proposed a prey–predator model for the phytoplankton–zooplankton system with the assumption that the viral disease is spreading only among the prey species, and, though the predator feeds on both the susceptible and infected prey, the infected prey is more vulnerable to predation as is seen in nature (see references quoted earlier). The dynamical behaviour of the system is investigated from the point of view of stability and persistence. The model shows that infection can be sustained only above a threshold of force of infection. Gakkhar and Negi [11] investigate the dynamical behaviour of toxin producing phytoplankton (TPP) and zooplankton. The phytoplanktons are divided into two groups, namely susceptible phytoplankton and infected phytoplankton. The conditions for coexistence for the populations are presented. Chattopadhyay et al. [12], deals with the problem of a nutrient–phytoplankton (N–P) populations where phytoplankton population is divided into two groups, namely susceptible phytoplankton and infected phytoplankton. Conditions for coexistence or extinction of populations are derived taking into account general nutrient uptake functions and Holling type-II functional response as an example. In 2010, Dhar and Sharma [13], proposed the role of viral infection in phytoplankton dynamics without and with incubation population class is studied. It is observed that phytoplankton species in the absence of incubated class are unstable around an endemic equilibrium but the presence of delay in the form of incubated class has made it conditionally stable around an endemic equilibrium.

Fractional order differentiation consist in the generalization of classical integer differentiation to real or com- plex orders. We observe that fractional order can be complex in viewpoint of pure mathematics and there is much interest in developing the theoretical analysis and numerical methods to fractional equations, because they have recently proved to be valuable in various fields of science and engineering. Indeed, we can find numerous applications in polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, etc. [14, 15, 16, 17, 18, 19, 20]. For some recent work on fractional differential equations and inclusions, see [21, 22, 23, 24, 25, 27, 28, 29] and the references therein.

In this paper, we consider the following equations describing the time evolution of the fractional order phytoplankton–zooplankton system [30]:

$$\begin{aligned}
\frac{d^\alpha S}{dt^\alpha} &= aS(1-\gamma(S+I)) - \frac{cPS}{(1+\alpha_1 S)(1+\alpha_2 P)} - \zeta SI,\\
\frac{d^\alpha I}{dt^\alpha} &= I(\zeta S - kP - h),\\
\frac{d^\alpha P}{dt^\alpha} &= P(-d + \frac{eS}{(1+\alpha_1 S)(1+\alpha_2 P)} + k_1 I),\\
S(\delta) &= S_0 > 0, \quad I(\delta) = I_0 > 0, \quad P(\delta) = P_0 > 0.
\end{aligned} \tag{1.1}$$

in Caputo fractional derivative sense. Here S and I are the concentrations of the susceptible and the infected prey phytoplanktons, respectively; and P is the concentration of the predator zooplankton, at time t and $\gamma = \frac{b}{a}$. The parameters as it appears in equation (1.1) denotes the intrinsic rate of increase of susceptible prey; b relates to the carrying capacity or crowding effects of the prey; c is the capture rate of the susceptible prey by the predator; d denotes the death rate of predators in the absence of prey; e is the growth rate of predators due to predation of susceptible prey; k denotes the rate of capturing of infected prey by the predators; h is the death rate of infected phytoplankton; ζ is the force of infection between susceptible and infected prey populations; and k_1 is the growth rate of predator due to predation of infected phytoplankton $k_1 \leq k$.

2 Preliminaries

Definition 2.1. The *Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$* is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The *Riemann-Liouville and Caputo fractional derivative of order* $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds,$$

and

$$D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds,$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

The initial value problem related to Definition 2.2 is

$$\begin{cases} D^{\alpha} x(t) = f(t, x(t)), \\ x(t)|_{t=0^+} = x_0, \end{cases} \tag{2.1}$$

where $0 < \alpha < 1$ and $D^{\alpha} = D_0^{\alpha}$.

Now, some stability theorems on fractional-order systems are introduced.

Theorem 2.3 ([31])**.** *The following autonomous system:*

$$\frac{d^{\alpha} x}{dt^{\alpha}} = Ax, \quad x(0) = x_0, \tag{2.2}$$

with $0 < \alpha \leq 1$*,* $x \in \mathbb{R}^n$ *and* $A \in \mathbb{R}^{n \times n}$*, is asymptotically stable if and only if* $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ *is satisfied for all eigenvalues of matrix A. Also, this system is stable if and only if* $|\arg(\lambda)| \geq \frac{\alpha\pi}{2}$ *is satisfied for all eigenvalues of matrix A with those critical eigenvalues satisfying* $|\arg(\lambda)| = \frac{\alpha\pi}{2}$ *having geometric multiplicity of one. The geometric multiplicity of an eigenvalue* λ *of the matrix A is the dimension of the subspace of vectors v for which* $Av = \lambda v$*.*

Theorem 2.4 ([32])**.** *Consider the following commensurate fractional-order system:*

$$\frac{d^{\alpha} x}{dt^{\alpha}} = f(x), \quad x(0) = x_0, \tag{2.3}$$

with $0 < \alpha \leq 1$ *and* $x \in \mathbb{R}^n$*. The equilibrium points of system (3.1) are calculated by solving the following equation:* $f(x) = 0$*. These points are locally asymptotically stable if all eigenvalues* λ_i *of the Jacobian matrix* $J = \frac{\partial f}{\partial x}$ *evaluated at the equilibrium points satisfy:* $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$*.*

3 Main Results

In this section we deal with local stability of the system (1.1). Let

$$\frac{d^{\alpha} S}{dt^{\alpha}} = 0, \frac{d^{\alpha} I}{dt^{\alpha}} = 0, \frac{d^{\alpha} P}{dt^{\alpha}} = 0. \tag{3.1}$$

System (3.1) has a trivial equilibrium $E_0 = (0,0,0)$ and a infected prey and predator-free equilibrium $E_1 = (\frac{1}{\gamma}, 0, 0)$. Let $R_0 = \frac{\zeta}{\gamma h}$ is the basic reproduction number. If $R_0 > 1$, then (3.1) admits a unique predator-free equilibrium $E_2 = (S_2, I_2, 0)$, where

$$S_2 = \frac{1}{\gamma R_0}, I_2 = \frac{\frac{a}{\gamma}(R_0 - 1)}{R_0(a + R_0 h)}.$$

Let $R_0^* = \frac{e}{d(\gamma+\alpha_1)}$. If S_3, is a real positive root of the algebraic equation $d\alpha_1\gamma\beta S^3 + (d\beta(\gamma - \alpha_1)S + (e - d\beta - d\alpha_1)S + d = 0$, then (3.1) admits the equilibrium $E_3 = (S_3, 0, P_3)$ where $P_3 = \frac{\beta}{\alpha_2}S_3(1 - \gamma S_3)$ and $\beta = \frac{ea\alpha_2}{cd}$. Also if S^* is a real positive root of the following equation, then (3.1) admits the equilibrium $E_* = (S^*, I^*, P^*)$:

$$\begin{aligned}
&\big[1 - \tfrac{\alpha_2 h}{k} + S(\tfrac{\alpha_2\zeta}{k} + \alpha_1 - \tfrac{\alpha_1\alpha_2 h}{k}) + \tfrac{\alpha_1\alpha_2 d\zeta}{k}S^2\big] \\
&\quad\times\big[\tfrac{chd}{ek} + (a - \tfrac{cd\zeta}{ek})S - a\gamma S^2\big] \\
&-\tfrac{1}{k_1}\big[d - e - \tfrac{\alpha_2 hd}{k} + S(\tfrac{\alpha_2\zeta d}{k} + \alpha_1 d - \tfrac{\alpha_1\alpha_2 hd}{k}) + \tfrac{\alpha_1\alpha_2 d\zeta}{k}S^2\big] \\
&\quad\times\big[\tfrac{ck_1 h}{ek} + (a\gamma + \zeta - \tfrac{ck_1\zeta}{ek})S\big] = 0,
\end{aligned} \tag{3.2}$$

where

$$I^* = \frac{1}{k_1}\big[d - \frac{keS^*}{(1+\alpha_1 S^*)(k + \alpha_2(\zeta S^* - h))}\big]$$

and

$$P^* = \frac{1}{k}\big[\zeta S^* - h\big].$$

Now, in order to investigate the local behavior of model system (1.1) around each of the equilibrium points, the Jacobian matrix J of the equilibria point $E = (S, I, P)$ is computed as

$$J(E) = (J_{ij}),$$

where

$$J_{11} = a(1-\gamma(S+I)) - \frac{cP}{(1+\alpha_1 S)(1+\alpha_2 P)} - \zeta I + S\big[-a\gamma + \frac{\alpha_1 cP}{(1+\alpha_1 S)(1+\alpha_2 P)},$$

$$J_{12} = -(a\gamma+\zeta)S, J_{13} = -\tfrac{cS}{(1+\alpha_1 S)(1+\alpha_2 P)}, J_{21} = \zeta I, J_{22} = \zeta S - kP - h, J_{23} = -kI,$$

$$J_{31} = \frac{eP}{(1+\alpha_1 S)(1+\alpha_2 P)}, J_{32} = k_1 P, J_{33} = -d + k_1 I + \frac{eS}{(1+\alpha_1 S)(1+\alpha_2 P)}.$$

The local stability of system (1.1) around each of the equilibria is obtained by computing the variational matrix corresponding to each equilibrium. Now we consider the asymptotically stability of system (1.1) at the equilibrium point E_0. The Jacobian matrix of (1.1) at equilibrium point E_0 is

$$J(E_0)=\begin{pmatrix} a & 0 & 0\\ 0 & -h & 0\\ 0 & 0 & -d \end{pmatrix}, \tag{3.3}$$

The eigenvalues corresponding to the equilibrium E_0 are

$$\lambda_1=a,\lambda_2=-h,\lambda_3=-d.$$

Then we have $\lambda_1>0,\lambda_2<0$ and $\lambda_3<0$. Whence it follows that the equilibrium E_0 of system (1.1) is unstable. Thus the stable manifold of the origin $W^s(E_0)$ is two-dimensional and the unstable manifold of the origin $W^u(E_0)$ is one-dimensional.
The Jacobian matrix of (1.1) at equilibrium point $E_1=(\frac{1}{\gamma},0,0)$ is

$$J(E_1)=\begin{pmatrix} -a & -a-\frac{\zeta}{\gamma} & -\frac{c}{\gamma+\alpha_1}\\ 0 & \frac{\zeta}{\gamma}-h & 0\\ 0 & 0 & -d+\frac{e}{\gamma+\alpha_1} \end{pmatrix}. \tag{3.4}$$

The eigenvalues corresponding to the equilibrium E_1 are

$$\lambda_1=-a,\lambda_2=\frac{\zeta}{\gamma}-h,\lambda_3=-d+\frac{e}{\gamma+\alpha_1}.$$

If $R_0<1$ and $R_0^*<1$, then all the eigenvalues corresponding to the equilibrium E_1 are real and negative. Consequently, we have the following theorem:

Theorem 3.1. *The equilibrium E_1 of system (1.1) is locally asymptotically stable if $R_0<1$ and $R_0^*<1$ holds.*

Now we consider the asymptotically stability of system (1.1) at the equilibrium point E_2. The Jacobian matrix of (1.1) at equilibrium point E_2 is

$$\begin{aligned} J(E_2)&=\begin{pmatrix} -\frac{a}{R_0} & -\frac{a\gamma+\zeta}{\gamma R_0} & -\frac{cS_2}{1+\alpha_1 S_2}\\ \zeta I_2 & 0 & -kI_2\\ 0 & 0 & -d+\frac{eS_2}{1+\alpha_1 S_2}+k_1 I_2 \end{pmatrix},\\ &=\begin{pmatrix} -\frac{a}{R_0} & -\frac{a\gamma+\zeta}{\gamma R_0} & -\frac{c}{\gamma R_0+\alpha_1}\\ \zeta\frac{a(R_0-1)}{\gamma R_0(a+R_0h)} & 0 & -k\frac{a(R_0-1)}{\gamma R_0(a+R_0h)}\\ 0 & 0 & -d+\frac{e}{\gamma R_0+\alpha_1}+k_1\frac{a(R_0-1)}{\gamma R_0(a+R_0h)}, \end{pmatrix}, \end{aligned} \tag{3.5}$$

with the characteristic equation

$$\begin{aligned} Q(\lambda)=det(\lambda-J(E_2))=[\lambda-(-d+\tfrac{e}{\gamma R_0+\alpha_1}+k_1\tfrac{a(R_0-1)}{\gamma R_0(a+R_0h)})]\\ \times(\lambda^2+\tfrac{a}{R_0}\lambda+\tfrac{a\zeta(a\gamma+\zeta)(R_0-1)}{\gamma^2R_0^2(a+R_0h)}). \end{aligned} \tag{3.6}$$

If $1 < R_0 < \frac{4\zeta(a\gamma+\zeta)+a^2\gamma^2}{4\zeta(a\gamma+\zeta)-ha^2\gamma^2}, h < \frac{\zeta(a\gamma+\zeta)}{a\gamma^2}$ and $k_1 < \min\{\frac{\gamma R_0(a+R_0h)}{R_0-1}(d-\frac{e}{\gamma R_0+\alpha_1}), k\}$ hold, then the eigenvalues corresponding to the equilibrium E_2 are real and negative as follows:

$$\lambda_1 = -d + \frac{e}{\gamma R_0+\alpha_1} + k_1\frac{a(R_0-1)}{\gamma R_0(a+R_0h)},$$

$$\lambda_{2,3} = \frac{1}{2}\big(-\frac{a}{R_0} \pm \sqrt{\frac{a^2}{R_0^2} - 4\frac{a\zeta(a\gamma+\zeta)(R_0-1)}{\gamma^2R_0^2(a+R_0h)}}\big).$$

Consequently, we have the following theorem:

Theorem 3.2. *The equilibrium E_2 of system (1.1) is locally asymptotically stable if $1 < R_0 < \frac{4\zeta(a\gamma+\zeta)+a^2\gamma^2}{4\zeta(a\gamma+\zeta)-ha^2\gamma^2}, h < \frac{\zeta(a\gamma+\zeta)}{a\gamma^2}$ and $k_1 < \min\{\frac{\gamma R_0(a+R_0h)}{R_0-1}(d-\frac{e}{\gamma R_0+\alpha_1}), k\}$ holds.*

If $h < \frac{\zeta(a\gamma+\zeta)}{a\gamma^2}$,$R_0 > \max\{\frac{4\zeta(a\gamma+\zeta)+a^2\gamma^2}{4\zeta(a\gamma+\zeta)-ha^2\gamma^2}, 1\}$ and $k_1 < \min\{\frac{\gamma R_0(a+R_0h)}{R_0-1}(d-\frac{e}{\gamma R_0+\alpha_1}), k\}$ then $\lambda_1 < 0$ and the eigenvalues $\lambda_{2,3}$ corresponding to the equilibrium E_2 are complex as follows

$$\lambda_{2,3} = \frac{1}{2}\big(-\frac{a}{R_0} \pm i\sqrt{-\frac{a^2}{R_0^2} + 4\frac{a\zeta(a\gamma+\zeta)(R_0-1)}{\gamma^2R_0^2(a+R_0h)}}\big). \tag{3.7}$$

Thus from theorem (2), if $|\arg(\lambda_{2,3})| = \tan^{-1}\big(\frac{\sqrt{-\frac{a^2}{R_0^2}+4\frac{a\zeta(a\gamma+\zeta)(R_0-1)}{\gamma^2R_0^2(a+R_0h)}}}{\frac{a}{R_0}}\big) > \frac{\alpha\pi}{2}$ hold then the equilibrium E_2 is asymptotically stable. Consequently, we have the following theorem:

Theorem 3.3. *If $k_1 < \min\{\frac{\gamma R_0(a+R_0h)}{R_0-1}(d-\frac{e}{\gamma R_0+\alpha_1}), k\}$, $R_0 > \max\{\frac{4\zeta(a\gamma+\zeta)+a^2\gamma^2}{4\zeta(a\gamma+\zeta)-ha^2\gamma^2}, 1\}$ and $h < \frac{\zeta(a\gamma+\zeta)}{a\gamma^2}$ hold. Then the equilibrium E_2 of system (1.1) is locally asymptotically stable for all*
$\alpha < \frac{2}{\pi}\tan^{-1}\big(\frac{\sqrt{-\frac{a^2}{R_0^2}+4\frac{a\zeta(a\gamma+\zeta)(R_0-1)}{\gamma^2R_0^2(a+R_0h)}}}{\frac{a}{R_0}}\big).$

4 Numerical Simulation and Dissolution

In this section, to verify the effectiveness of the obtained results, some numerical simulations for the fractional-order system (1.1) have been conducted. For solving the fractional-order system (1.1) we used the numerical method discussed in [33]. All the differential equations are solved using the method proposed in the previous section. In all numerical runs, the solution has been approximated at $\delta = \Delta t = 0.01$. We consider the following cases:

Case I

In this case we set $\alpha_1 = 2, \alpha_2 = 4, a = 0.4, b = 0.006, c = 0.5, \zeta = 0.23, k = 1, h =$

$0.1, d = 0.4, e = 0.01, k_1 = 0.8$. In figure 1 we plot the numerical solution of system (1.1) with the initial condition $S_0 = 3, I_0 = 4, P_0 = 2$ for various values of α.

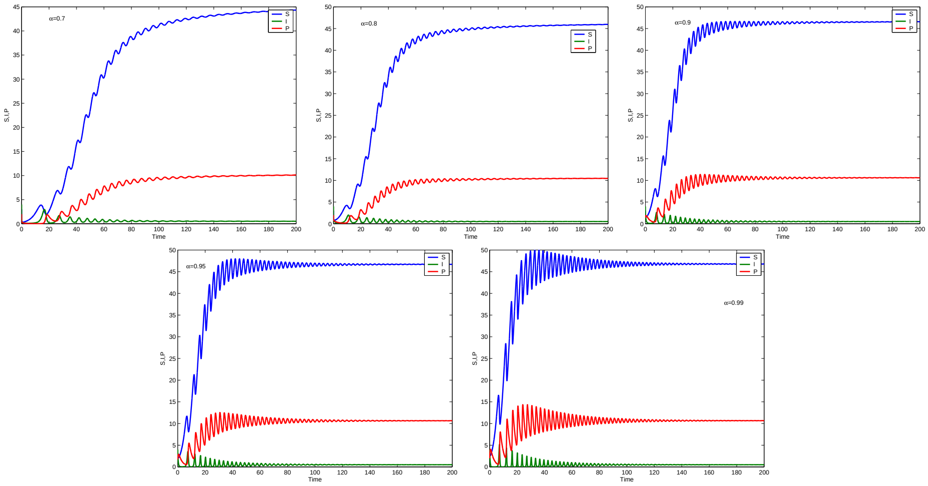
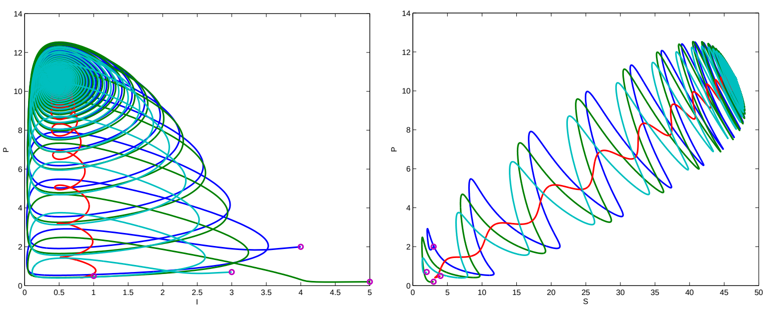


Figure 1: The numerical solution of system (1.1) at $\alpha_1 = 2, \alpha_2 = 4, a = 0.4, b = 0.006, c = 0.5, \zeta = 0.23, k = 1, h = 0.1, d = 0.4, e = 0.01, k_1 = 0.8$ for various values of α

In figures 2-3, we display the phase plane of system (1.1) with the initial conditions $[S_0, I_0, P_0] = [3, 4, 2], [3, 5, 0.2], [4, 1, 0.5], [2, 3, 0.7]$ for $\alpha = 0.95, 0.75$ respectively.



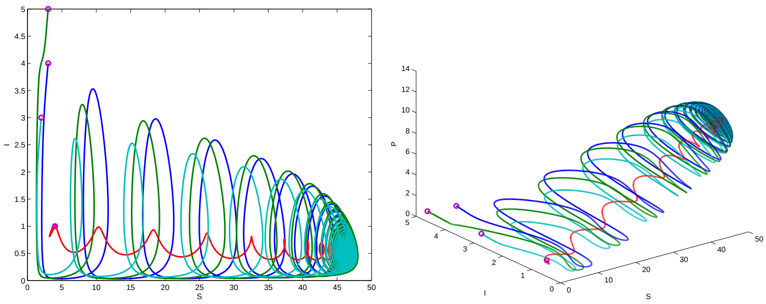


Figure 2: Phase plane of system (1.1) at $\alpha_1 = 2, \alpha_2 = 4, a = 0.4, b = 0.006, c = 0.5, \zeta = 0.23, k = 1, h = 0.1, d = 0.4, e = 0.01, k_1 = 0.8$ and $\alpha = 0.95$.

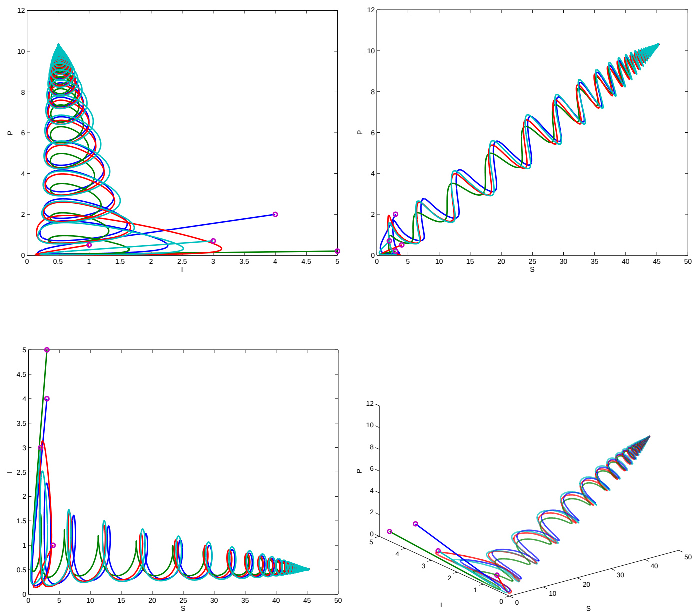


Figure 3: Phase plane of system (1.1) at $\alpha_1 = 2, \alpha_2 = 4, a = 0.4, b = 0.006, c = 0.5, \zeta = 0.23, k = 1, h = 0.1, d = 0.4, e = 0.01, k_1 = 0.8$ and $\alpha = 0.75$.

Case II

In this case we set $\alpha_1 = 2, \alpha_2 = 4, a = 0.4, b = 0.006, c = 0.5, \zeta = 0.23, k = 1, h = 0.1, d = 0.4, e = 0.01, k_1 = 0.008$. In figure 4 we plot the numerical solution of system (1.1) with the initial condition $S_0 = 3, I_0 = 4, P_0 = 2$ for various values of α.

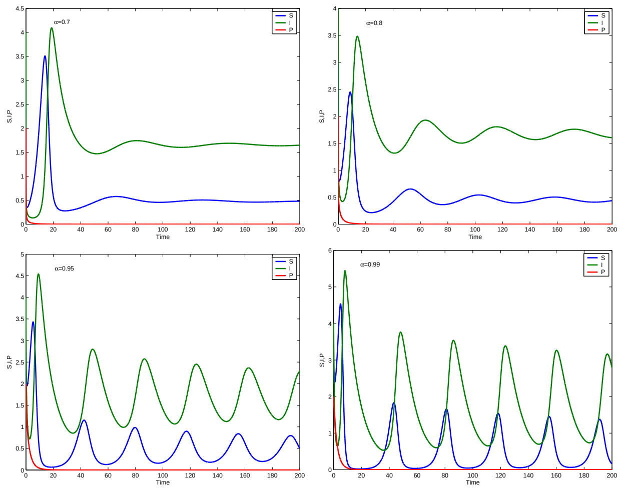


Figure 4: The numerical solution of system (1.1) at $\alpha_1 = 2, \alpha_2 = 4, a = 0.4, b = 0.006, c = 0.5, ; \zeta = 0.23, k = 1, h = 0.1, d = 0.4, e = 0.01, k_1 = 0.008$ for various values of α

[H] In figures 5-6, we display the phase plane of system (1.1) with the initial conditions $[S_0, I_0, P_0] = [3, 4, 2], [3, 5, 0.2], [4, 1, 0.5], [2, 3, 0.7]$ for $\alpha = 0.95, 0.75$ respectively.

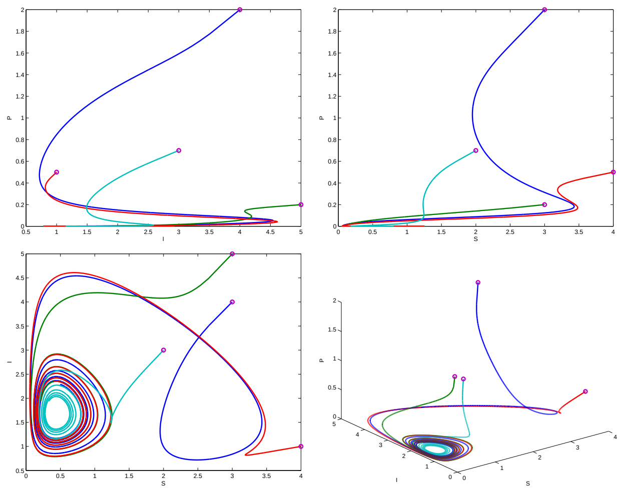


Figure 5: Phase plane of system (1.1) at $\alpha_1 = 2, \alpha_2 = 4, a = 0.4, b = 0.006, c = 0.5, \zeta = 0.23, k = 1, h = 0.1, d = 0.4, e = 0.01, k_1 = 0.008$ and $\alpha = 0.95$.

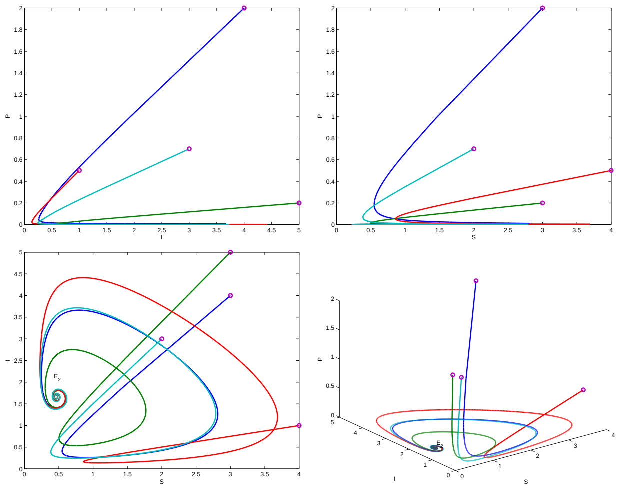


Figure 6: Phase plane of system (1.1) at $\alpha_1 = 2, \alpha_2 = 4, a = 0.4, b = 0.006, c = 0.5, \zeta = 0.23, k = 1, h = 0.1, d = 0.4, e = 0.01, k_1 = 0.008$ and $\alpha = 0.75$.

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