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Chromatic Numbers of Glued Graphs

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Abstract : Let G_1 and G_2 be any two graphs. Assume that $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ are non-trivial connected and such that $H_1 \cong H_2$ with an isomorphism f. The *glued graph of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f, denoted by $G_1 \triangleleft_{H_1 \cong_f H_2} \triangleright G_2$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f. We characterize graph gluing between trees, forests, and bipartite graphs. Furthermore, we give an upper bound of the chromatic number of glued graphs in terms of the chromatic numbers of their original graphs. We also provide a family of glued graphs to guarantee the sharpness of this upper bound.

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1 Introduction

The gluing is a natural graph operation. It is mathematically defined in [1]. We follow West [2] for terminologies and notations not defined here and only consider simple graphs. Let G_1 and G_2 be any graphs, $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be non-trivial connected and such that $H_1 \cong H_2$ with an isomorphism f. We define the *glued graph of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f, denoted by $G_1 \triangleleft_{H_1 \cong_f H_2} \triangleright G_2$, as the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f. If H is the copy of H_1 and H_2 in the glued graph, we refer to H as its *clone* and G_1 and G_2 as its *original graphs*. The *glued graph* $G_1 \triangleleft\triangleright G_2$ *at the clone* H means that there exist a subgraph H_1 of G_1, a subgraph H_2 of G_2, and an isomorphism f such that $G_1 \triangleleft_{H_1 \cong_f H_2} \triangleright G_2$ and H is the copy of H_1 and H_2 in the resulting graph. Unless we define specifically, we denote $G_1 \triangleleft\triangleright G_2$ as an arbitrary graph resulting from gluing G_1 and G_2. Note that ,from the definition of glued graphs, clones must be connected and not a single vertex. The notation $P_n(v_1, \ldots, v_n)$ denote a path of n vertices on the vertex set $\{v_1, \ldots, v_n\}$.

We first note few facts that the copy of both original graphs are subgraphs of their glued graphs. The glue operation does not create an edge. Also, a glued

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graph of disconnected graphs is still disconnected. A graph gluing could give a resulting graph with multiple edges. In section 2, we focus more on graph colorings, we will in particular consider multiple edges as a single edge in any glued graphs .

Proposition 1.1. *Let G_1 and G_2 be graphs gluing at a clone H. Then* $|V(G_1 \diamond G_2)| = |V(G_1)| + |V(G_2)| - |V(H)|$, *and* $|E(G_1 \diamond G_2)| = |E(G_1)| + |E(G_2)| - |E(H)|$.

Proof. Since for each vertex and each edge in H are counted twice in the glued graph, we have that $|V(G_1 \diamond G_2)| = |V(G_1)|+|V(G_2)|-|V(H)|$ and $|E(G_1 \diamond G_2)| = |E(G_1)| + |E(G_2)| - |E(H)|$. □

The rest of this section deals with the characterization of graph gluing between trees, forests and bipartite graphs.

Theorem 1.2. *Let T_1 and T_2 be graphs. $T_1 \diamond T_2$ is a tree if and only if T_1 and T_2 are trees.*

Proof. Without loss of generality, we may assume that T_1 is not a tree. Therefore T_1 contains a cycle or T_1 is disconnected. This yields that $T_1 \diamond T_2$ also contains a cycle or it is disconnected. Hence $T_1 \diamond T_2$ is not a tree. Conversely, let T_1 and T_2 be trees, and $T_1 \diamond T_2$ at a clone H. Since a connected subgraph of a tree is a tree, H is also a tree. By Proposition 1.1, we have

$$\begin{aligned} |E(T_1 \diamond T_2)| &= |E(T_1)| + |E(T_2)| - |E(H)| \\ &= |V(T_1)| - 1 + |V(T_2)| - 1 - |V(H)| + 1 \\ &= |V(T_1)| + |V(T_2)| - |V(H)| - 1 \\ &= |V(T_1 \diamond T_2)| - 1. \end{aligned}$$

Moreover, $T_1 \diamond T_2$ is connected because T_1 and T_2 are connected. Therefore $T_1 \diamond T_2$ is a tree. □

In particular, Theorem 1.2 can be applied for connected graphs G_1 and G_2 as follows: *$G_1 \diamond G_2$ has a cycle if and only if G_1 or G_2 has a cycle.* We observe that a glued graph can have a new cycle that is not contained in any original graphs. In a glued graph, cycles in its original graphs are called *original cycles*, otherwise they are *created cycles*.

Theorem 1.3. *Let G_1 and G_2 be graphs. If $G_1 \diamond G_2$ contains a created cycle, then both G_1 and G_2 contain a cycle.*

Proof. Let G_1 and G_2 be graphs glued at the clone H. By contrapositive, we assume that G_1 is acyclic. Without loss of generality, we may assume that both G_1 and G_2 are connected. If G_2 does not contain a cycle. G_1 and G_2 are trees. By Theorem 1.2, $G_1 \diamond G_2$ is a tree which is acyclic. On the other hand, let G_2 contain a cycle. Suppose that $G_1 \diamond G_2$ contains a created cycle, say C. Then there exists a path in C which is not a subgraph of G_2. Let P_1 be a u,v-path in C whose all edges are in $G_1 \setminus G_2$ and $|E(P_1)|$ is maximum. Then both u and v are in $V(G_1) \cap V(G_2)$, and therefore they are in $V(H)$. Since H is connected, there is a u, v-path P_2 in H. Because $H \subseteq G_2$, $P_2 \neq P_1$. Hence $P_1 \cup P_2$ is a closed walk, so it contains a cycle. But $P_1 \cup P_2 \subseteq G_1$, so G_1 contains a cycle, a contradiction. Therefore $G_1 \triangleleft\triangleright G_2$ does not contain a created cycle. ☐

Corollary 1.4. *Let G_1 and G_2 be graphs. $G_1 \triangleleft\triangleright G_2$ is a forest if and only if G_1 and G_2 are forests.*

Proof. Let G_1 and G_2 be graphs. Without loss of generality, we may assume that G_1 is not a forest. So G_1 contains a cycle. Since $G_1 \subseteq G_1 \triangleleft\triangleright G_2$, the glued graph $G_1 \triangleleft\triangleright G_2$ contains a cycle. Hence $G_1 \triangleleft\triangleright G_2$ is not a forest. Conversely, suppose $G_1 \triangleleft\triangleright G_2$ is not a forest. So $G_1 \triangleleft\triangleright G_2$ contains a cycle, say C. If C is an original cycle, then it is done. Suppose C is a created cycle. By Theorem 1.3, both G_1 and G_2 contain a cycle. Hence G_1 and G_2 are not forests. ☐

Theorem 1.5. *Let B_1 and B_2 be graphs. $B_1 \triangleleft\triangleright B_2$ is a bipartite graph if and only if B_1 and B_2 are bipartite.*

Proof. Necessity. Without loss of generality, we may assume that B_1 is not bipartite. Then B_1 contains an odd cycle. Since $B_1 \subseteq B_1 \triangleleft\triangleright B_2$, we have that $B_1 \triangleleft\triangleright B_2$ contains an odd cycle and hence $B_1 \triangleleft\triangleright B_2$ is not a bipartite graph. Sufficiency. Assume B_1 and B_2 are bipartite. Let X_i and Y_i be bipartition of B_i for $i = 1, 2$. Let H be the clone of arbitrary $B_1 \triangleleft\triangleright B_2$. Because H is a subgraph of B_1 and B_2, the clone H is bipartite. Let X_H and Y_H be bipartition of H. We may assume that X_H is a subset of X_1 and X_2 and Y_H is a subset of Y_1 and Y_2. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. To show that X and Y are bipartition of $B_1 \triangleleft\triangleright B_2$, let u and v be adjacent vertices in $B_1 \triangleleft\triangleright B_2$. So both u and v are adjacent in B_1 or B_2. We may assume that u and v are in B_1. Because B_1 is a bipartite graph, u and v are not in the same partition of B_1. Hence $u \in X_1 \subset X$ and $v \in Y_1 \subset Y$ or vise versa. This yields that u and v are not in the same partition in $B_1 \triangleleft\triangleright B_2$. Therefore $B_1 \triangleleft\triangleright B_2$ is a bipartite graph. ☐

2 Chromatic Numbers of Glued Graphs

A *k-coloring* of a graph G is a labelling $f : V(G) \to S$, where $|S| = k$. The labels are *colors*; the vertices of one color form a *color class*. A k-coloring is *proper* if adjacent vertices have different labels. A graph is *k-colorable* if it has a proper k-coloring. The *chromatic number of graph G*, $\chi(G)$, is the least k such that G is k-colorable. For any glued graph $G_1 \triangleleft\triangleright G_2$, since G_1 and G_2 are subgraphs $G_1 \triangleleft\triangleright G_2$, the chromatic number of $\chi(G_1 \triangleleft\triangleright G_2)$ is at least $\chi(G_1)$ and $\chi(G_2)$. We therefore get a lower bound for any graphs G_1 and G_2 that

$$\chi(G_1 \triangleleft\triangleright G_2) \geq max\{\chi(G_1), \chi(G_2)\}.$$

It is possible that the chromatic number of a glued graph exceeds both chromatic numbers of it original graphs. For instant, we obtain K_4 by gluing $K_4 \setminus \{e\}$

with K_3 at a clone P_2 where P_2 contains both endpoints of the edge e. So $\chi(K_4 \setminus \{e\} \lhd\!\rhd K_3) = \chi(K_4) = 4$ while $\chi(K_4 \setminus \{e\}) = 3 = \chi(K_3)$. However, for any glued graph with chromatic number $\chi(G_1 \lhd\!\rhd G_2) = 2$ or 3, we have $\chi(G_1 \lhd\!\rhd G_2) = \max\{\chi(G_1),\chi(G_2)\}$. This result is found in Proposition 2.1. On the other hand, when $\chi(G_1 \lhd\!\rhd G_2) \geq 4$, there exist a family of glued graphs each of whose chromatic number greater than both chromatic numbers of its original graphs. This family of glued graphs are given in Theorem 2.3.

Proposition 2.1. *Let G_1 and G_2 be non-trivial graphs. Then*
(i) $\chi(G_1 \lhd\!\rhd G_2) = 2$ if and only if $\chi(G_1) = 2 = \chi(G_2)$, and
(ii) if $\chi(G_1 \lhd\!\rhd G_2) = 3$, then $\chi(G_1) = 3$ or $\chi(G_2) = 3$.

Proof. We note that for any graph G, G is non-trivial bipartite if and only if $\chi(G) = 2$. Together with Theorem 1.5, statement (i) is concluded.

To prove statement (ii), let G_1 and G_2 be non-trivial graphs. Thus $\chi(G_1) \neq 1$ and $\chi(G_2) \neq 1$. Assume that $\chi(G_1 \lhd\!\rhd G_2) = 3$. Since $\chi(G_1 \lhd\!\rhd G_2) \geq max\{\chi(G_1), \chi(G_2)\}$, we have $\chi(G_1) \leq 3$ and $\chi(G_2) \leq 3$. By (i), we have $\chi(G_1) \neq 2$ or $\chi(G_2) \neq 2$. Therefore $\chi(G_1) = 3$ or $\chi(G_2) = 3$. □

In general, to determine the chromatic number of a glued graph between G_1 and G_2, ones intuitively believe that a set of $\chi(G_1) + \chi(G_2)$ colors should be enough to provide a proper coloring for $G_1 \lhd\!\rhd G_2$. However, this intuition is not always true as provided in Figure 1.

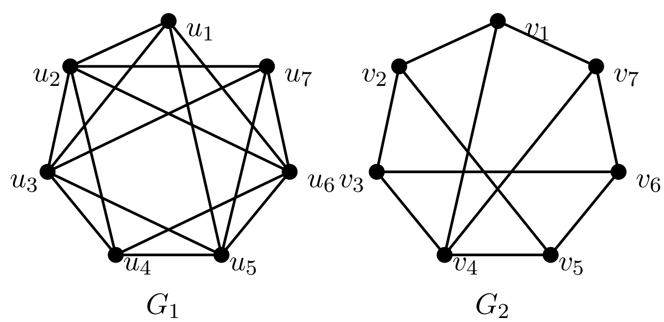

Figure 1: $\chi(G_1) = 3 = \chi(G_2)$ and $\chi(G_1 \lhd\!\rhd G_2) = 7$.

Though the sum of the chromatic numbers of the original graphs of a glued graph cannot be an upper bound, the product of the chromatic numbers of its original graphs is large enough to be an upper bound. We prove this fact in Theorem 2.2. Furthermore, this bound is sharp as provided by a family of glued graphs in Theorem 2.3.

Theorem 2.2. *Let G_1 and G_2 be non-trivial graphs. Then*
$\chi(G_1 \lhd\!\rhd G_2) \leq \chi(G_1)\chi(G_2)$.

Proof. Let G_1 and G_2 be non-trivial graphs gluing at a connected clone H. Assume $\chi(G_1) = p$ and $\chi(G_2) = q$. Let $\gamma_1 : V(G_1) \to A = \{1, 2, \dots, p\}$ and $\gamma_2 : V(G_2) \to B = \{1, 2, \dots, q\}$ be proper colorings of G_1 and G_2, respectively. Define $\beta : V(G_1 \lhd\rhd G_2) \to A \times B$ by for all $v \in V(G_1 \lhd\rhd G_2)$,

$$\beta(v) = \begin{cases} (\gamma_1(v), 1) & \text{if} \quad v \in V(G_1 \setminus H), \\ (\gamma_1(v), \gamma_2(v)) & \text{if} \quad v \in V(H), \\ (1, \gamma_2(v)) & \text{if} \quad v \in V(G_2 \setminus H). \end{cases}$$

Let v, $u \in V(G_1 \lhd\rhd G_2)$ be such that v and u are adjacent by an edge e. Then $e \in E(G_1)$ or $E(G_2)$. In case $e \in E(G_1)$, we have $\beta(v) = (\gamma_1(v), a)$ where $a = 1$ or $\gamma_2(v)$, and $\beta(u) = (\gamma_1(u), b)$ where $b = 1$ or $\gamma_2(u)$. Since $\gamma_1(v) \neq \gamma_1(u)$, it follows that $\beta(v) \neq \beta(u)$ Similarly, suppose $e \in E(G_2)$, we have $\beta(v) = (c, \gamma_2(v))$ where $c = 1$ or $\gamma_1(v)$, and $\beta(u) = (d, \gamma_2(u))$ where $d = 1$ or $\gamma_1(u)$. Since $\gamma_2(v) \neq \gamma_2(u)$, $\beta(v) \neq \beta(u)$. Hence β is a proper coloring of the glued graph $G_1 \lhd\rhd G_2$. Therefore $\chi(G_1 \lhd\rhd G_2) \leq |A \times B| = pq = \chi(G_1)\chi(G_2)$. □

Theorem 2.3. *Let p and q be integers such that $p, q \geq 2$ but $pq \neq 4$. Then there exist G_1 and G_2 with a glued graph $G_1 \lhd\rhd G_2$ such that $\chi(G_1) = p$, $\chi(G_2) = q$, and $\chi(G_1 \lhd\rhd G_2) = pq$.*

Proof. Let p and q be integers at least 2 but $pq \neq 4$. We construct separately a family of glued graphs satisfying the required property when $p = q$ and $p \neq q$.

Case 1 $p = q$**:** Let G_1 be defined by $V(G_1) = \{u_1, u_2, \dots, u_{p^2}\}$ and u_i and u_j are adjacent if and only if $i \not\equiv j \pmod p$. Now let $\gamma_1 : V(G_1) \to \{1, 2, \dots, p\}$ be a coloring of G_1 defined by

$$\gamma_1(u_i) = l \ \text{ where } l \equiv i \pmod p \text{ and } l \in \{1, 2, \dots, p\}.$$

For u_i, $u_j \in V(G_1)$, assume that u_i and u_j are adjacent. Then $i \not\equiv j \pmod p$. Suppose $\gamma_1(u_i) = l$ and $\gamma_1(u_j) = k$, so $i \equiv l \pmod p$ and $j \equiv k \pmod p$. Hence $l \equiv i \not\equiv j \equiv k \pmod p$. Then $\gamma_1(u_i) \neq \gamma_1(u_j)$. Therefore γ_1 is proper and also $\chi(G_1) \leq p$. Since the set of vertices $\{u_1, u_2, \dots, u_p\}$ forms a p-clique, $\chi(G_1) \geq p$. Hence $\chi(G_1) = p$.

We next define graph G_2 by $V(G_2) = \{v_1, v_2, \dots, v_{p^2}\}$ and v_i and v_j are adjacent if and only if $i = j + 1$ or $i \equiv j \pmod p$. Let $\gamma_2 : V(G_2) \to \{1, 2, \dots, p\}$ be a coloring of G_2. For each $i \in \{1, 2, \dots, p^2\}$, write $i = ap + b$ where $a, b \in \mathbb{Z}, a \geq 0, 0 < b \leq p$, define $\gamma_2(v_i)$ by

$$\gamma_2(v_i) = l \text{ where } l \equiv a + b \pmod p \text{ and } l \in \{1, 2, \dots, p\}.$$

Figure 2 illustrates G_1 and G_2 for $p = q = 3$. If we let clones $H_1 = P_9(u_1, \dots, u_9)$ and $H_2 = P_9(v_1, \dots, v_9)$ with the isomorphism f where $f(u_i) = v_i$, then $\chi(G_1) = \chi(G_2) = 3$ while $\chi(G_1 \lhd_{H_1 \cong_f H_2} \rhd G_2) = \chi(K_9) = 9$.

For v_i, $v_j \in V(G_2)$, to show that γ_2 is proper, assume that v_i and v_j are adjacent. Then $i = j + 1$ or $i \equiv j \pmod p$. Let $j = ap + b$ where $a \geq 0$ and $0 < b \leq p$. So $\gamma_2(v_j) \equiv a + b \pmod p$.

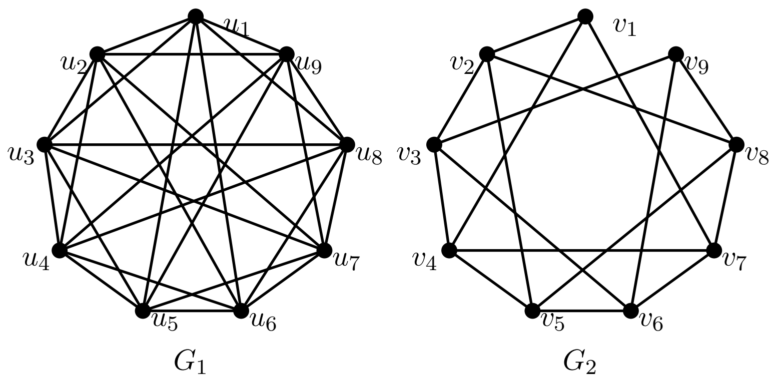

Figure 2: G_1 and G_2 defined in Case 1 when $p = q = 3$.

Case 1.1 $i = j + 1 = ap + b + 1$**:** If $b < p$, then $b + 1 \leq p$, consequently, $\gamma_2(v_i) \equiv a + b + 1 \pmod p \not\equiv a + b \pmod p \equiv \gamma_2(v_j)$. Suppose $b = p$. Then $i = ap + p + 1 = p(a + 1) + 1$. So $\gamma_2(v_i) \equiv a + 2 \pmod p$. Since $p^2 \neq 4$, we have $p \neq 2$. This yields $\gamma_2(v_i) \equiv a + 2 \pmod p \not\equiv a + p \pmod p \equiv \gamma_2(v_j)$.

Case 1.2 $i \equiv j \pmod p$**:** Without loss of generality, we may assume $i > j$. Then $i = j + np = ap + b + np = p(a + n) + b$ where $n \in \mathbb{N}$. Since $1 \leq i \leq p^2$, $1 \leq n \leq p - 1$. Hence $\gamma_2(v_i) \equiv a + n + b \pmod p \not\equiv a + b \pmod p \equiv \gamma_2(v_j)$.

Therefore, by both cases, γ_2 is a proper coloring. This yields $\chi(G_2) \leq p$. Moreover, since the set of vertices $\{v_1, v_{1+p}, v_{1+2p}, \ldots, v_{1+(p-1)p}\}$ forms a p-clique, $\chi(G_2) \geq p$. Hence $\chi(G_2) = p$.

Now consider $H_1 = P_{p^2}(u_1, \ldots, u_{p^2}) \subseteq G_1$ and $H_2 = P_{p^2}(v_1, \ldots, v_{p^2}) \subseteq G_2$. Let an isomorphism $f : H_1 \to H_2$ be defined by $f(u_i) = v_i$ for all $i \in \{1, 2, \ldots, p^2\}$. Let $G = G_1 \triangleleft_{H_1 \cong_f H_2} \triangleright G_2$ and $V(G) = \{w_i : i = 1, \ldots, p^2$ where w_i corresponds to u_i and $v_i\}$. Note that $|V(G)| = p^2$. Let $w_i,\ w_j \in V(G)$. If $i \equiv j \pmod p$, w_i and w_j are adjacent in G_2. Otherwise, w_i and w_j are adjacent in G_1. It follows that w_i and w_j are adjacent in G. Hence G is a complete graph. That is, $\chi(G_1 \triangleleft_{H_1 \cong_f H_2} \triangleright G_2) = p^2$.

Case 2 $p \neq q$**:** We may assume $p < q$. Define G_1 and γ_1 the same as Case 1. Then $\chi(G_1) = p$. For G_2, let $V(G_2) = \{v_1, v_2, \ldots, v_{pq}\}$ and v_i and v_j are adjacent if and only if $i = j + 1$ or $i \equiv j \pmod p$. Figure 3 illustrates for $p = 3, q = 4$.

We define $\gamma_2 : V(G_2) \to \{1, 2, \ldots, q\}$ as follows: For each $i \in \{1, \ldots, pq\}$, write $i = ap + b$ where $a, b \in \mathbb{Z}, a \geq 0, 0 < b \leq p$,

$$\gamma_2(v_i) = l \text{ where } l \equiv 2 + a - b \pmod q \text{ and } l \in \{1, 2, \ldots, q\}.$$

For $v_i,\ v_j \in V(G_2)$, assume that v_i and v_j are adjacent. Then $i = j + 1$ or $i \equiv j \pmod p$. Let $j = ap + b$ where $a \geq 0,\ 0 < b \leq p$. So $\gamma_2(v_j) \equiv 2 + a - b \pmod q$.

Case 2.1 $i = j + 1 = ap + b + 1$**:** If $b < p$, then $b + 1 \leq p$ and also $\gamma_2(v_i) \equiv 2 + a - b - 1 \pmod q \not\equiv 2 + a - b \pmod q \equiv \gamma_2(v_j) \pmod q$. If $b = p$, $i =$

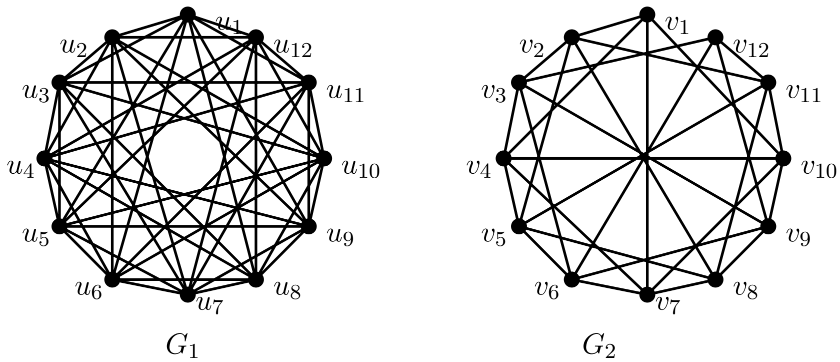

Figure 3: G_1 and G_2 defined in Case 2 when $p = 3$ and $q = 4$.

$ap + p + 1 = p(a + 1) + 1$ and $\gamma_2(v_i) = 2 + a \pmod q$. Since $p < q$, $\gamma_2(v_i) \equiv a + 2 \pmod q \not\equiv 2 + a - p \pmod q \equiv \gamma_2(v_j) \pmod q$.

Case 2.2 $i \equiv j \pmod p$**:** We may assume that $i > j$. So $i = j + np = ap + b + np = p(a + n) + b$ where $n \in \mathbb{N}$. Since $1 \leq i \leq pq$, we have $1 \leq n \leq q - 1$. So $\gamma_2(v_i) \equiv 2 + a + n - b \pmod q \not\equiv 2 + a - b \pmod q \equiv \gamma_2(v_j)$. Hence, by both cases, γ_2 is proper and $\chi(G_2) \leq q$. We can see that the set of vertices $\{v_1, v_{1+p}, v_{1+2p}, \ldots, v_{1+(q-1)p}\}$ forms a q-clique. So $\chi(G_2) \geq q$. Hence $\chi(G_2) = q$.

Finally, let $H_1 = P_{pq}(u_1, \ldots, u_{pq})$ and $H_2 = P_{pq}(v_1, \ldots, v_{pq})$. Define $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i \in \{1, 2, 3, \ldots, pq\}$. Let $G = G_1 \triangleleft_{H_1 \cong_f H_2} \triangleright G_2$ and $V(G) = \{w_i : i = 1, \ldots, pq$ where w_i corresponds to u_i and $v_i\}$. Note that $|V(G)| = pq$. Now, let $w_i,\ w_j \in V(G)$. If $i \equiv j \pmod p$, then w_i and w_j are adjacent in G_2. Otherwise, w_i and w_j are adjacent in G_1. This means that w_i and w_j are adjacent in G. Hence G is a complete graph. Therefore, $\chi(G_1 \triangleleft_{H_1 \cong_f H_2} \triangleright G_2) = pq$. □

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