Thai Journal of Mathematics (2003) 1: 49-57

On the Residue of the Generalized Function P^λ

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Abstract: In this paper we study the residue of the generalized function P^λ where

$$P = \left(\sum_{i=1}^{p} x_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} x_j^2\right)^m,$$

$p+q=n$ is a dimension of the Euclidean space $\mathbb{R}^n$, m is a positive integer and λ is a complex number. For the case $m=1$ we obtain the special residue appear in the sense of I.M. Gel'fand and G.E. Shilov. Moreover, for case $m=2$ we obtain the residue of Fourier transform of the Diamond kernel related to the spectrum, see [2, pages 715-723]. And for case $m=4$ obtain the residue of the Fourier transform of the distributional kernel related to the spectrum, see [3].

Keywords: generalized function, Diamond kernel, distributional kernel.
2000 Mathematics Subject Classification: 46F10

1 Introduction

I.M. Gel'fand and G.E. Shilov [1, p253-258] have studied the generalized function P^λ, where

$$P = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \tag{1.1}$$

is quadratic form, λ is a complex number and $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$. They found that P^λ has two sets of singularities, namely $\lambda = -1, -2, \ldots, -k, \ldots$ and $\lambda = -\frac{n}{2}, -\frac{n}{2}-1, \ldots, -\frac{n}{2}-k, \ldots$, where k is a positive integer. For singular point $\lambda = -k$, the generalized function P^λ has a simple pole with residues,

$$\operatorname*{res}_{\lambda=-k} P^\lambda = \frac{(-1)^k}{(k-1)!}\delta_1^{(k-1)}(P) \tag{1.2}$$

for $p+q=n$ is odd with p odd and q even.

[1]Supported by The Royal Golden Jubilee Project grant No. PHD/0221/2543.

Also, for the singular point $\lambda = -\frac{n}{2} - k$ they obtain

$$\operatorname*{res}_{\lambda=-\frac{n}{2}-k} P^\lambda = \frac{\pi^{\frac{n}{2}}(-1)^{\frac{q}{2}} L^k \delta(x)}{2^{2k} k! \Gamma(\frac{n}{2}+k)} \tag{1.3}$$

where

$$L = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2};$$

$p+q=n$ is odd with p odd and q even.

The purpose of this work is studying the residues of the generalized function P^λ which is positive definite and defined by

$$P^\lambda = \left[\left(\sum_{i=1}^{p} x_i^2 \right)^m - \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^m \right]^\lambda \tag{1.4}$$

where λ is a complex number and m is a positive integer. Now (1.4) is generalized function of (1.1).

We obtain

$$\operatorname*{res}_{\lambda=-k} P^\lambda = \frac{(-1)^k}{(k-1)!} \delta_1^{*(k-1)}(P) \tag{1.5}$$

and also

$$\operatorname*{res}_{\lambda=-\frac{n}{2m}-k} < P^\lambda, \varphi >= \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi(-\frac{n}{2m} - k, u) \right]_{u=0} \tag{1.6}$$

where $\delta_1^{*(k-1)}(P)$ defined by (2.15), $\Phi(-\frac{n}{2m} - k, u)$ defined by (3.10) with $\lambda = -\frac{n}{2m} - k$ and $u = r^{2m}$. In particular, for $m = 1$ (1.5) reduces to (1.2) and (1.6) reduces to (1.3).

Moreover, for case $m = 2$ we obtain the residue of the Fourier transform of the Diamond kernel related to the spectrum, See [2, pages 715-723]. And for case $m = 4$ we obtain the residue of the Fourier transform of the distributional kernel related to the spectrum, see [3].

2 Preliminaries

Definition 2.1 Given $\varphi(x)$ be any testing function in the Schwartz space $\mathbf{S}$. Define

$$< P^\lambda, \varphi >= \int_{P>0} P^\lambda \varphi(x)\, dx \tag{2.1}$$

where P^λ is defined by (1.4). Actually $P^\lambda \in \mathbf{S}$ - the space of tempered distribution.

Definition 2.2 Given the hyper-surface $P = 0$ where

$$P = \left(\sum_{i=1}^{p} x_i^2 \right)^m - \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^m,$$

$p+q=n$ is a dimension of $\mathbb{R}^n$ and m is a positive integer.

Let $\operatorname{grad} P \neq 0$ that means there is no singular point on $P=0$. Then we define

$$< \delta^{(k)}(P), \varphi >= \int \delta^{(k)}(P)\varphi(x)\,dx \tag{2.2}$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k-derivatives, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ and $dx=dx_1\,dx_2\ldots dx_n$. In a sufficiently small neighborhood U of any point $(x_1,x_2,\ldots,x_n)$ of the hyper-surface $P=0$. We can introduce a new coordinate system such that $P=0$ becomes one of the coordinate hyper-surface. For this purpose we write $P=u_1$ and choose the remaining u_i coordinates (with $i=2,3,\ldots,n$) for which the Jacobian $\mathrm{D}\left(\frac{x}{u}\right)>0$ where

$$\mathrm{D}\left(\tfrac{x}{u}\right)=\frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(P,u_2,\ldots,u_n)}.$$

Thus (2.2) can be written by

$$< \delta^{(k)}(P), \varphi >= (-1)^k\int\left[\frac{\partial^k}{\partial P^k}\{\varphi\,\mathrm{D}\left(\tfrac{x}{u}\right)\}\right]_{P=0} du_2\,du_3\ldots du_n. \tag{2.3}$$

Lemma 2.1 *Given the hyper-surface*

$$P=\left(\sum_{i=1}^{p}x_i^2\right)^m-\left(\sum_{j=p+1}^{p+q}x_j^2\right)^m, \tag{2.4}$$

$p+q=n$ is the dimension of $\mathbb{R}^n$ and m is a positive integer.

If we transform to bipolar coordinates defined by

$$x_1=r\omega_1, x_2=r\omega_2,\ldots,x_p=r\omega_p$$

and

$$x_{p+1}=s\omega_{p+1}, x_{p+2}=s\omega_{p+2},\ldots,x_{p+q}=s\omega_{p+q}$$

where $\sum_{i=1}^{p}\omega_i^2=1$, $\sum_{j=p+1}^{p+q}\omega_j^2=1$. Then $r^2=\sum_{i=1}^{p}x_i^2$ and $s^2=\sum_{j=p+1}^{p+q}x_j^2$ and (2.4) can be written by $P=r^{2m}-s^{2m}$. Then we obtain

$$< \delta^{(k)}(P), \varphi >= \int_0^\infty\left[\left(\frac{1}{2ms^{2m-1}}\frac{\partial}{\partial s}\right)^k\{s^{q-2m}\frac{\psi(r,s)}{2m}\}\right]_{s=r} r^{p-1}dr. \tag{2.5}$$

$$< \delta^{(k)}(P), \varphi >= (-1)^k\int_0^\infty\left[\left(\frac{1}{2mr^{2m-1}}\frac{\partial}{\partial r}\right)^k\{r^{p-2m}\frac{\psi(r,s)}{2m}\}\right]_{r=s} s^{q-1}ds. \tag{2.6}$$

where $\psi(r,s)=\int\varphi\,d\Omega_p\,d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively.

Proof. We have the element of volume

$$dx = r^{p-1}s^{q-1}dr\,ds\;d\Omega_q \tag{2.7}$$

where $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Now

$$P = r^{2m} - s^{2m}, \tag{2.8}$$

choose the coordinates to be p, r and the ω_i. Then (2.7) becomes

$$dx = \frac{1}{2m}(r^{2m} - P)^{\frac{q}{2m}-1}r^{p-1}dP\,dr\,d\Omega_p\,d\Omega_q. \tag{2.9}$$

Thus, by (2.3) we obtain

$$< \delta^{(k)}(P), \varphi >= (-1)^k \int \left[\frac{\partial^k}{\partial P^k}\{\frac{1}{2m}(r^{2m} - P)^{\frac{q}{2m}-1}\varphi\}\right]_{P=0} r^{p-1}\,dr\,d\Omega_p\,d\Omega_q. \tag{2.10}$$

Further, if we transform from P to $s = (r^{2m} - P)^{\frac{1}{2m}}$, we have $\frac{\partial}{\partial P} = -\frac{1}{2ms^{2m-1}}\frac{\partial}{\partial s}$. Thus (2.10) becomes

$$< \delta^{(k)}(P), \varphi >= \int \left[\left(\frac{1}{2ms^{2m-1}}\frac{\partial}{\partial s}\right)^k \{s^{q-2m}\frac{\varphi}{2m}\}\right]_{s=r} r^{p-1}dr\,d\Omega_p\,d\Omega_q. \tag{2.11}$$

Write

$$\psi(r,s) = \int \varphi\,d\Omega_p\,d\Omega_q. \tag{2.12}$$

Then (2.11) becomes

$$< \delta^{(k)}(P), \varphi >= \int_0^\infty \left[\left(\frac{1}{2ms^{2m-1}}\frac{\partial}{\partial s}\right)^k \{s^{q-2m}\frac{\psi(r,s)}{2m}\}\right]_{s=r} r^{p-1}dr. \tag{2.13}$$

Similarly

$$< \delta^{(k)}(P), \varphi >= (-1)^k \int_0^\infty \left[\left(\frac{1}{2mr^{2m-1}}\frac{\partial}{\partial r}\right)^k \{r^{p-2m}\frac{\psi(r,s)}{2m}\}\right]_{r=s} s^{q-1}ds. \tag{2.14}$$

Thus, we obtain (2.5) and (2.6) as required. □

Now, we assume that φ vanishes in the neighborhood of the origin; so that these integrals will converge for any k.

Now for $(p-1) + (q-2m) \geq 2mk$ or $k < \frac{1}{2m}(p+q-2m)$, the integral in (2.13) converge for any $\varphi(x) \in \mathbf{S}$. Similarly, for $(q-1) + (p-2m) \geq 2mk$ or $k < \frac{1}{2m}(p+q-2m)$, the integral in (2.14) also converges for any $\varphi(x) \in \mathbf{S}$. Thus we take (2.13) and (2.14) to be the defining equation for $\delta^{(k)}(P)$. If, on the other hand, $k \geq \frac{1}{2m}(p+q-2m)$, we shall define $< \delta_1^{*(k)}(P), \varphi >$ and $< \delta_2^{*(k)}(P), \varphi >$ as the regularization of (2.13) and (2.14) respectively.

We shall say that for $p > 1$ and $q > 1$ the generalized function $\delta_1^{*(k)}(P)$ and $\delta_2^{*(k)}(P)$ are defines by

$$< \delta_1^{*(k)}(P), \varphi >= \int_0^\infty \left[\left(\frac{1}{2ms^{2m-1}} \frac{\partial}{\partial s} \right)^k \{ s^{q-2m} \frac{\psi(r,s)}{2m} \} \right]_{s=r} r^{p-1} dr \qquad (2.15)$$

for $k \geq \frac{1}{2m}(p+q-2m)$ and

$$< \delta_2^{*(k)}(P), \varphi >= (-1)^k \int_0^\infty \left[\left(\frac{1}{2mr^{2m-1}} \frac{\partial}{\partial r} \right)^k \{ r^{p-2m} \frac{\psi(r,s)}{2m} \} \right]_{r=s} s^{q-1} ds \qquad (2.16)$$

for $k \geq \frac{1}{2m}(p+q-2m)$.

In particular, for $m = 1$, $\delta_1^{*(k)}(P)$ reduces to $\delta_1^{(k)}(P)$ and $\delta_2^{*(k)}(P)$ reduces to $\delta_2^{(k)}(P)$, See [1, p250].

3 Main Results

Theorem 3.1 *Let P^λ be hyper-surface given by (1.4) for $p \geq 2m$ and $q \geq 2m$. Then the residues of P^λ at the singular point $\lambda = -k$ and $\lambda = -\frac{n}{2m} - k$ are*

$$\operatorname*{res}_{\lambda=-k} P^\lambda = (-1)^{k-1} \frac{\delta_1^{*(k-1)}(P)}{(k-1)!} \qquad (3.1)$$

and

$$\operatorname*{res}_{\lambda=-\frac{n}{2m}-k} < P^\lambda, \varphi >= \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi(-\frac{n}{2m} - k, u) \right]_{u=0} \qquad (3.2)$$

where $\delta_1^{*(k-1)}(P)$ defined by (2.15), $\Phi(-\frac{n}{2m} - k, u)$ defined by (3.10) with $\lambda = -\frac{n}{2m} - k$ and $u = r^{2m}$. In particular, for $k = 0$ we have

$$\operatorname*{res}_{\lambda=-\frac{n}{2m}} P^\lambda = \frac{1}{m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(1-\frac{n}{2m})\pi^{\frac{n}{2}}}{\Gamma(1-\frac{p}{2m})\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \delta(x). \qquad (3.3)$$

Proof. From (2.1),

$$\begin{aligned} < P^\lambda, \varphi > &= \int_{P>0} P^\lambda \varphi(x)\, dx \\ &= \int_{P>0} (r^{2m} - s^{2m})^\lambda \varphi\, r^{p-1} s^{q-1}\, dr\, ds\, d\Omega_p\, d\Omega_q \end{aligned} \qquad (3.4)$$

by (2.8) and changing to bipolar coordinate. Write

$$\psi(r,s) = \int \varphi\, d\Omega_p\, d\Omega_q. \qquad (3.5)$$

We obtain

$$< P^{\lambda}, \varphi >= \int_0^{\infty} \int_0^{r} (r^{2m} - s^{2m})^{\lambda} \psi(r,s)\, r^{p-1} s^{q-1}\, ds\, dr. \tag{3.6}$$

Since $\varphi \in \mathcal{D}$- the space of infinitely differentiable function with compact support. Then $\psi(r,s)$ as defined in (3.5) is infinitely differentiable function of r^{2m} and s^{2m} with compact support. Thus $\psi(r,s) \in \mathcal{D}$. Put $u = r^{2m}, v = s^{2m}$ in the integrand of (3.6), writing

$$\psi(r,s) = \psi_1(u,v). \tag{3.7}$$

Then (3.6) becomes

$$< P^{\lambda}, \varphi >= \frac{1}{4m^2} \int_0^{\infty} \int_0^{u} (u-v)^{\lambda} \psi_1(u,v)\, u^{\frac{p}{2m}-1} v^{\frac{q}{2m}-1}\, dv\, du. \tag{3.8}$$

Write $v = ut$, thus (3.8) becomes

$$< P^{\lambda}, \varphi >= \frac{1}{4m^2} \int_0^{\infty} u^{\lambda + \frac{1}{2m}(p+q)-1} du \int_0^1 (1-t)^{\lambda} t^{\frac{q}{2m}-1} \psi_1(u,ut)\, dt. \tag{3.9}$$

Write

$$\Phi(\lambda, u) = \frac{1}{4m^2} \int_0^1 (1-t)^{\lambda} t^{\frac{q}{2m}-1} \psi_1(u,ut)\, dt. \tag{3.10}$$

Thus for $q \geq 2m$, $\Phi(\lambda,u)$ is regular for all λ except at the singularities

$$\lambda = -1, -2, -3, \ldots, -k, \ldots,$$

where it has simple poles. At these poles we have

$$\operatorname*{res}_{\lambda=-k} \Phi(\lambda,u) = \frac{1}{4m^2} \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \{ t^{\frac{q}{2m}-1} \psi_1(u,ut) \} \right]_{t=1} \tag{3.11}$$

like the function $< x_+^{\lambda}, \varphi >$, See [1, p50].

Thus $\operatorname{res}_{\lambda=-k} \Phi(\lambda,u)$ is a functional concentrated on the space of $P = 0$ cone. Now (3.9), can also be written as

$$< P^{\lambda}, \varphi >= \int_0^{\infty} u^{\lambda + \frac{1}{2m}(p+q)-1} \Phi(\lambda,u)\, du. \tag{3.12}$$

Even at the regular points of $\Phi(\lambda,u)$ the integral of (3.12) have poles at

$$\lambda = -\frac{n}{2m}, -\frac{n}{2m} - 1, \ldots, -\frac{n}{2m} - k, \ldots$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$ and n is odd.

At these point we have

$$\operatorname*{res}_{\lambda=-\frac{n}{2m}-k} < P^{\lambda}, \varphi >= \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi(-\frac{n}{2m} - k, u) \right]_{u=0}. \tag{3.13}$$

These the residues of $< P^\lambda, \varphi >$ at $\lambda = -\frac{n}{2m} - k$ is a functional concentrated on the vertex of the cone.

Now consider the singular point $\lambda = -k$. By (3.11) and (3.12) and also See [1, p255-256]. We obtain

$$\operatorname*{res}_{\lambda=-k} \frac{(-1)^{k-1}}{(k-1)!} < \delta_1^{*(k-1)}(P), \varphi > \tag{3.14}$$

where

$$< \delta_1^{*(k-1)}(P), \varphi >= \frac{1}{4m^2}\int_0^\infty \left[\left(\frac{1}{2ms^{2m-1}}\frac{\partial}{\partial s}\right)^{k-1} \{s^{q-2m}\frac{\psi(r,s)}{2m}\}\right]_{s=r} r^{p-1}dr. \tag{3.15}$$

Summarizing, we have the following. For odd n for even n if $k < \frac{1}{2m}n$, the generalized function P^λ has simple poles at $\lambda = -k$ for positive integral values of k, where the residues are

$$\operatorname*{res}_{\lambda=-k} P^\lambda = \frac{(-1)^{k-1}}{(k-1)!}\delta_1^{*(k-1)}(P). \tag{3.16}$$

Now consider the singular point at $\lambda = -\frac{n}{2m} - k$, from (3.13) we have

$$\operatorname*{res}_{\lambda=-\frac{n}{2m}-k} < P^\lambda, \varphi >= \frac{1}{k!}\left[\frac{\partial^k}{\partial u^k}\Phi(-\frac{n}{2m} - k, u)\right]_{u=0}$$

for n odd with p odd and q even. Thus, for $k = 0$ we have

$$\operatorname*{res}_{\lambda=-\frac{n}{2m}} < P^\lambda, \varphi >= \Phi(-\frac{n}{2m}, 0).$$

By (3.10), we obtain

$$\begin{aligned}\operatorname*{res}_{\lambda=-\frac{n}{2m}} < P^\lambda, \varphi > &= \frac{1}{4m^2}\int_0^1 (1-t)^{-\frac{n}{2m}} t^{\frac{q}{2m}-1}\psi_1(0,0)dt \\ &= \frac{\psi_1(0,0)}{4m^2}\int_0^1 (1-t)^{-\frac{n}{2m}} t^{\frac{q}{2m}-1}dt \\ &= \frac{\psi_1(0,0)}{4m^2}\frac{\Gamma(\frac{q}{2m})\Gamma(-\frac{n}{2m}+1)}{\Gamma(-\frac{p}{2m}+1)},\end{aligned}$$

since

$$\int_0^1 (1-t)^{-\frac{n}{2m}} t^{\frac{q}{2m}-1}dt = \frac{\Gamma(\frac{q}{2m})\Gamma(-\frac{n}{2m}+1)}{\Gamma(-\frac{p}{2m}+1)}.$$

Now

$$\begin{aligned}\psi_1(0,0) = \psi(0,0) &= \int \varphi(0)\, d\Omega_p\, d\Omega_q \quad \text{by (3.5)} \\ &= \Omega_p\, \Omega_q\, \varphi(0)\end{aligned}$$

and

$$\Omega_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \quad \text{and} \quad \Omega_q = \frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}.$$

Thus

$$\operatorname*{res}_{\lambda=-\frac{n}{2m}} < P^\lambda, \varphi >= \frac{1}{4m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(-\frac{n}{2m}+1)}{\Gamma(-\frac{p}{2m}+1)} \left(\frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}\right) \left(\frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}\right) \varphi(0). \tag{3.17}$$

Now, for $p \geq 2m$ and p is even then $\Gamma(-\frac{p}{2m} - 1) = \infty$. Thus $\operatorname{res}_{\lambda=-\frac{n}{2m}} < P^\lambda, \varphi > = 0$. From (3.17),

$$\begin{aligned} \operatorname*{res}_{\lambda=-\frac{n}{2m}} < P^\lambda, \varphi > &= \frac{1}{4m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(-\frac{n}{2m}+1)}{\Gamma(-\frac{p}{2m}+1)} \frac{4\pi^{\frac{p+q}{2}}}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} < \delta(x), \varphi > \\ &= \frac{1}{m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(-\frac{n}{2m}+1)}{\Gamma(-\frac{p}{2m}+1)} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} < \delta(x), \varphi > . \end{aligned}$$

Thus

$$\operatorname*{res}_{\lambda=-\frac{n}{2m}} P^\lambda = \frac{1}{m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(1-\frac{n}{2m})\pi^{\frac{n}{2}}}{\Gamma(1-\frac{p}{2m})\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \delta(x). \tag{3.18}$$

In particular, if $m = 1$ then (3.18) reduces to

$$\operatorname*{res}_{\lambda=-\frac{n}{2}} P^\lambda = \frac{(-1)^q \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \delta(x),$$

which appeared in [1,eq.(23),p258].

Moreover, for equations (3.1) and (3.2), if $m = 2$ we obtain

$$\operatorname*{res}_{\lambda=-k} (-1)^k \widehat{K_{2k,2k}}(x) = \frac{1}{(2\pi)^{n/2}} \operatorname*{res}_{\lambda=-k} P^\lambda = \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!} \delta_1^{*(k-1)}(P)$$

and

$$\operatorname*{res}_{\lambda=-\frac{n}{4}-k} < P^\lambda, \varphi >= \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi(-\frac{n}{4} - k, u)\right]_{u=0}$$

where $(-1)^k K_{2k,2k}(x)$ is the Fourier transform of the Diamond kernel, see [2, pages 715-723]. And if $m = 4$ we obtain

$$\operatorname*{res}_{\lambda=-k} (-1)^k \widehat{K_{2k,2k,2k,2k}}(x) = \operatorname*{res}_{\lambda=-k} P^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{*(k-1)}(P)$$

and

$$\operatorname*{res}_{\lambda=-\frac{n}{8}-k} < P^\lambda, \varphi >= \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi(-\frac{n}{8} - k, u)\right]_{u=0}$$

where $(-1)^k K_{2k,2k,2k,2k}(x)$ is the Fourier transform of the distributional kernel of the operator $\oplus^k$, defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4\right]^k, \qquad p + q = n, \text{see [3].}\square$$

Acknowledgements

The authors would like to thank The Thailand Research Fund for financial support.

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(Received 10 June 2003)

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