



Escaping Set of $e^z - 1$

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Abstract : We investigate the set $I(f)$ of points that converge to infinity under iteration of the map $f(z) = e^z - 1$ and show that it is the disjoint union of countably many rays and uncountable union of infinite sets whose points escape to infinity through the sets.

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1 Introduction

Let f be a transcendental entire function. For $n \in \mathbb{N}$ let f^n denote the n -th iterate of f .

The set

$$F(f) = \{z \in \mathbb{C} : \{f^n\}_{n \in \mathbb{N}} \text{ is normal in some neighborhood of } z\}$$

is called the *Fatou set* of f or the *set of normality* of f and its complement $J(f)$ is the Julia set of f . See [1] for an introduction to the properties of these sets. The escaping set of f denoted by $I(f)$ is the set of points in the complex plane that tend to infinity under iteration of f . In general, it is neither an open nor a closed subset of \mathbb{C} and has interesting topological properties. For a transcendental entire

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function f , the escaping set was studied for the first time by Eremenko [2] who proved that

1. $I(f) \neq \emptyset$;
2. $J(f) = \partial I(f)$;
3. $I(f) \cap J(f) \neq \emptyset$;
4. $\overline{I(f)}$ has no bounded components.

For the exponential maps of the form $f(z) = e^z + \lambda$ with $\lambda > -1$, it is known, by Rempe [3], that the escaping set is a connected subset of the plane, and for $\lambda < -1$, it is the disjoint union of uncountably many curves to infinity, each of which is connected component of $I(f)$ [4] (These maps have no critical points and exactly one asymptotic value which is the omitted value λ). For the Eremenko-Lyubich class $\mathcal{B} = \{f : \mathbb{C} \rightarrow \mathbb{C} \text{ transcendental entire} : \text{Sing}(f^{-1}) \text{ is bounded}\}$ (where $\text{Sing}(f^{-1})$ is the set of critical and asymptotic values of f), it is known that $I(f)$ is a subset of the Julia set $J(f)$, see [5]. It was shown in [6] that every escaping point of every exponential map can be connected to ∞ by a curve consisting of escaping points.

This paper has been written by drawing simple approach which perhaps is the previous work of [7] and [6], but with different approach, concept and augmentation. While making the comparative analysis of the result of the same topic the conclusion of the previous work is vast and pinpointing all the general aspects through the appended notes, whereas the conclusion of ours is rather small and very specific but apt in the relevant context. Though difference is clearly visible and authentic.

From now onwards f will denote the map $f(z) = e^z - 1$. In this paper we have given an explicit description of escaping set of this map. We have repeatedly used the fact that those points of the complex plane which land into the left half plane under iteration of this map are not going to escape.

2 Main Results

Theorem 2.1. *For the function $f(z) = e^z - 1$, $I(f)$ is the disjoint union of countably many rays to ∞ in the right half plane and uncountable union of infinite sets in the right half plane whose points escape to infinity through the sets.*

The proof is elementary. It is segregated in several lemmas and corollaries. The main aim is to show that no point in the left half plane belongs to $I(f)$. Using this concept we try to find out the points in the right half plane, which do not belong to $I(f)$. We observe that 0 is a rationally indifferent fixed point and so belongs to $J(f)$ and moreover, points which do not escape seems to be approaching the fixed point 0.

Lemma 2.2. *The set $I_1 = \{z \in \mathbb{C} : \text{Re}(z) < 0 \text{ and } \text{Re}(e^z) < 0\}$ and $I_2 = \{z \in \mathbb{C} : \text{Re}(z) < 0 \text{ and } \text{Re}(e^{z+i\alpha}) < 0, \alpha \in \mathbb{R}^+\}$ do not belong to $I(f)$.*

The above lemma will establish that none of the points in the left half plane belong to $I(f)$.

Proof. Observe that $Re(e^z) < 0$ implies $(4k-3)\pi/2 < y < (4k-1)\pi/2$, $k \in \mathbb{Z}$ and $Re(e^{z+i\alpha}) < 0$, $\alpha \in \mathbb{R}^+$ implies $\frac{(4k-3)\pi}{2} - \alpha < y < \frac{(4k-1)\pi}{2} - \alpha$, $\alpha \in \mathbb{R}^+$, $k \in \mathbb{Z}$. We show that $|f^k(z)| \leq 2$ for all $z \in I_1$, $k \in \mathbb{N}$. Suppose on the contrary that there exists $n \in \mathbb{N}$ such that $|f^n(z)| > 2$. This will imply that $e^{Ref^{n-1}(z)} > 1$, and so $Ref^{n-1}(z) > 0$. Further this implies that $Re(e^{f^{n-2}(z)}) > 1$ and since $|z| \geq Re(z)$ for all $z \in \mathbb{C}$, we get $e^{Ref^{n-2}(z)} > 1$ and so $Ref^{n-2}(z) > 0$. Proceeding on similar lines we will get $Ref(z) > 0$. But $Ref(z) = Re(e^z) - 1$ which is less than -1 by hypothesis, so we arrive at a contradiction and therefore proves the assertion. Thus orbit of each point in I_1 is bounded and so no point in this set can escape to ∞ . On similar lines it can be seen that $|f^k(z)| \leq 2$ for all $z \in I_2$, $k \in \mathbb{N}$ and hence no point in I_2 can escape to ∞ . On combining the two it follows that none of the points in the left half plane can escape to ∞ . \square

Lemma 2.3. For the sets $I_3 = \{z \in \mathbb{C} : z = x + i(2k+1)\pi/2, x \in \mathbb{R}^+, k \in \mathbb{Z}\}$ and $I_4 = \{z \in \mathbb{C} : z = x + i(2k+1)\pi, x \in \mathbb{R}^+, k \in \mathbb{Z}\}$, none of the points belong to $I(f)$.

Proof. Any z in I_3 has the form $z = x + i(2k+1)\pi/2$, for some $k \in \mathbb{Z}$, $x \in \mathbb{R}^+$ and so $f(z) = -1 \pm ie^x$ (according as k is even or odd integer) which belong to left half plane and hence cannot escape to ∞ by Lemma 2.2. So no point in I_3 can belong to escaping set of f .

For any z in I_4 , $f(z)$ lies in the left half plane and hence cannot escape to ∞ . Thus no point in I_4 can escape to ∞ . \square

Lemma 2.4. For the set $I_5 = \{z \in \mathbb{C} : z = x + iy, x \in \mathbb{R}^+, \pi/2 < y < 3\pi/2\}$, no point lies in $I(f)$.

Proof. For any z in I_5 , $f(z)$ lies in the left half plane and so cannot escape to ∞ . Thus no point in I_5 can escape to ∞ . \square

It is clear that all the backward iterates of points in the sets I_1 to I_5 do not belong to $I(f)$. We now show that no points on the line $Re(z) = 0$ escape to infinity.

Lemma 2.5. For the set $I_6 = \{z \in \mathbb{C} : Re(z) = 0\}$, none of the points belong to $I(f)$.

Proof. Any $z \in I_6$ has the form $z = iy$ for some $y \in \mathbb{R}$. It can be seen that $|f(iy)| \leq 2$ and $|f^2(iy)| \leq 2$. In fact, we show that $|f^n(iy)| \leq 2$ for all $n \in \mathbb{N}$. Suppose on the contrary that $|f^k(iy)| > 2$ for some $k \in \mathbb{N}$. Write $z = iy$. Then $|e^{f^{k-1}(z)} - 1| > 2$, which implies that $e^{Ref^{k-1}(z)} > 1$ and so $Ref^{k-1}(z) > 0$ which then implies that $Re(e^{f^{k-2}(z)}) > 1$. Since $|z| \geq Re(z)$ for all $z \in \mathbb{C}$, we

get $|e^{f^{k-2}(z)}| > 1$ and so $\operatorname{Re} f^{k-2}(z) > 0$. Proceeding on similar lines we will get $\operatorname{Re} f(z) > 0$. But $\operatorname{Re} f(z) = \cos y - 1 \leq 0$, so we arrive at a contradiction and hence the assertion gets proved. Since $z \in I_6$ was arbitrary we have orbit of each point in I_6 is bounded and so no point in this set can escape to infinity. \square

Lemma 2.6. *The set $I_7 = \{z \in \mathbb{C} : z = x + i2k\pi, x \in \mathbb{R}^+, k \in \mathbb{Z}\}$ is contained in the escaping set $I(f)$ of f .*

Proof. Any z in I_7 has the form $z = x + i2k\pi$, for some $k \in \mathbb{Z}$, $x \in \mathbb{R}^+$ and all the f^n , $n \in \mathbb{N}$ are positive real valued function on this set. Now $f(z) = f(x) = e^x - 1$. Since $e^x > 1 + x$ for all $x \in \mathbb{R}^+$, it follows that $f^2(z) = f^2(x) > f(x)$ and by induction it can be seen that $f^n(x) > f^{n-1}(x)$ for all $n \in \mathbb{N}$. We now show that $\{f^n(z)\}_{n \in \mathbb{N}}$, $z \in I_7$ is an unbounded set. It suffices to show that $\{f^n(x)\}$ is unbounded for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^+$. Let $x > 0$. Then there exists some $\alpha > 0$ such that $0 < \alpha < x$. Now

$$\begin{aligned} f(x) &= e^x - 1 \\ &= x + \frac{x^2}{2} + \dots \\ &> x + \frac{\alpha^2}{2} \quad (\text{as } \alpha < x). \end{aligned}$$

Again

$$\begin{aligned} f^2(x) &= f(f(x)) \\ &> f(x) + \frac{\alpha^2}{2} \\ &> x + \frac{\alpha^2}{2} + \frac{\alpha^2}{2} \\ &= x + 2\frac{\alpha^2}{2}. \end{aligned}$$

By induction we get $f^n(x) > x + n\frac{\alpha^2}{2}$ which tends to ∞ as $n \rightarrow \infty$. Thus each $x > 0$ escapes to infinity. Hence the set I_7 is contained in the escaping set $I(f)$. Moreover, the backward iterates of points in this set which are countably many curves or rays to infinity also belong to $I(f)$. \square

We now concentrate on the semi-infinite strip $I_8 = \{x + iy : x \in \mathbb{R}^+, 0 < y < \pi/2\}$. Consider the subset $A = \{x + iy : \ln(\frac{\pi}{2\sin y}) < x < \ln(\frac{3\pi}{2\sin y}), 0 < y < \pi/2\}$ of I_8 . For any z in A , $f^2(z)$ lies in the left half plane and so cannot escape to ∞ . Thus the set A and all its backward iterates do not belong to the escaping set $I(f)$.

It now remains to see what happens to the remaining portion say B , left in the semi-infinite strip I_8 . We note that if points in B move to the left half plane, or the set I_5 or A under iteration of f , then they do not belong to $I(f)$. If B contains a domain B_1 such that $f^n(B_1)$ does not take the values of the left half plane for

all n , then by Montel's theorem, B_1 lies in the Fatou component of f . Also since f has no Baker domain as f belongs to the Eremenko-Lyubich class \mathcal{B} , it follows that such a domain B_1 must be bounded, and by the classification of periodic components of the Fatou set of f , points of B_1 cannot escape to infinity. We next deal with the situation when B consists only of points. If there are only finitely many points, then they being isolated cannot belong to $J(f)$ and so this situation cannot arise. If there are infinite points, then either they will be countably infinite or uncountable. If they are countably infinite then orbit of every point must escape to infinity and this escape will be through the points of the set B , for if orbit of any point in B lands outside B then it will stay bounded and hence will belong to the Fatou set which is not possible. Furthermore, if orbit of any point in B converges to some point in B then obviously point will belong to Fatou set $F(f)$ which is again not possible. If the set B is uncountable, then a similar argument given for the countable case will demonstrate that orbit of every point in B will escape to infinity.

A similar argument can be given for the semi-infinite strip $I_9 = \{x + iy : x \in \mathbb{R}^+, 3\pi/2 < y < 2\pi\}$. Thus in the semi-infinite strip $S = \{x + iy : x \in \mathbb{R}^+, 0 \leq y \leq 2\pi\}$, the escaping set is the disjoint union of two rays tending to infinity and uncountable union of infinite sets whose points escape to infinity through the sets. Finally, using periodicity of the map $f(z) = e^z - 1$ we can transfer the behavior of the semi-infinite strip of width 2π in the right half plane to entire right half plane.

Conclusion : The escaping set of the map $f(z) = e^z - 1$ is the disjoint union of countably many rays to infinity in the right half plane and uncountable union of infinite sets in the right half plane whose points escape to infinity through the sets and Theorem 2.1 gets proved. This is in confirmation with the result of Eremenko and Lyubich, see [8, Theorem 10].

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