



# Asymptotically Equivalent Generalized Difference Sequences of Fuzzy Real Numbers Defined by Orlicz Function

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**Abstract :** The aim of this article is to introduce the classes of strong  $\Delta^m$ -lacunary asymptotically equivalent,  $\Delta^m$ -lacunary asymptotically statistically equivalent, strong  $\Delta^m$ -almost asymptotically equivalent sequences of fuzzy real numbers in terms of Orlicz function. We have established some relations between the classes of the sequences.

**Keywords :** asymptotically equivalent; lacunary; statistically convergent; fuzzy real number; Cesàro summable; Orlicz function.

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## 1 Introduction and Preliminaries

Marouf [1] introduced the idea of asymptotically equivalent sequences and asymptotic regular matrices and thereafter Patterson [2] extends these concepts by introducing asymptotically statistically equivalent analog and natural regularity conditions for nonnegative summability matrices. Savas [3, 4] introduced and studied the concepts of asymptotically equivalent and  $\lambda$ -statistical convergence and generalized the idea of  $I$ -asymptotically lacunary statistical equivalent sequences. Recently Esi and Esi [5] introduced the idea of asymptotically  $S_{\lambda, L}^{\Delta}(F)$ -

statistical equivalent by combining  $\Delta$ -asymptotically equivalent and  $\lambda_L^\Delta$ -statistical convergence of fuzzy real numbers and Bilgin [6] studied  $f$ -asymptotically lacunary equivalent sequence. Furthermore, the study of asymptotically equivalent sequences from different point of view is found in [3, 6–11] and some others.

The study of Orlicz sequence spaces was initiated with a specific purpose in Banach space theory. Lindberg [12] got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to  $c_0$  or  $\ell_p$  ( $1 \leq p < \infty$ ). Subsequently Lindenstrauss and Tzafriri [13] investigated Orlicz sequence spaces in more details and they proved that every Orlicz sequence space  $\ell_M$  contains a subspace isomorphic to  $\ell_p$  ( $1 \leq p < \infty$ ).

An Orlicz function  $M$  is a function  $M : [0, \infty) \rightarrow [0, \infty)$  which is continuous, non-decreasing and convex with  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If the convexity of  $M$  is replaced by sub-additive property,  $M(x + y) \leq M(x) + M(y)$  then it is called a modulus function. It is known that  $M(\lambda x) \leq \lambda M(x)$  for all with  $0 < \lambda < 1$ . Orlicz function can be represented in integral form given by

$$M(x) = \int_0^x p(t)dt,$$

where  $p$  known as the kernel of  $M$  is right differentiable for  $t \geq 0$  and (i)  $p(t) > 0$ , for  $t > 0$  (ii)  $q$  is non-decreasing (iii)  $q(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

**Remark 1.1.** An Orlicz function satisfy the inequality  $M(\lambda u) \leq \lambda M(u)$ , for all  $0 < \lambda < 1$ .

The difference sequence space  $X(\Delta)$  was introduced by Kizmaz [14] as follows:  $X(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in X\}$ , for  $X = \ell_\infty, c$  and  $c_0$ ; where  $\Delta x_k = x_k - x_{k+1}$  for all  $k \in N$ . It was generalized by Et and Çolak [15] as follows:

$$X(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in X\}.$$

The generalized difference operator has the following binomial representation:

$$\Delta^m X_k = \sum_{n=0}^m (-1)^n \binom{m}{n} X_{n+k}.$$

Çolak et al. [16] studied it in terms of fuzzy real numbers. By a lacunary  $\theta = (k_r)$  ( $r = 0, 1, 2, 3, \dots$ ) we mean an increasing sequence of non-negative integers with,  $k_0 = 0$  and  $h_r = k_r - k_{r-1} \rightarrow \infty$  as  $r \rightarrow \infty$ . The interval determined by  $\theta$  is denoted by  $I_r = (k_{r-1}, k_r]$  and the ratio  $\frac{k_r}{k_{r-1}}$  is denoted by  $q_r$ .

Different classes of lacunary sequences have been investigated by Altin et al. [17], Altinok et al. [18], Dutta [19], Mursaleen [20] and others [6–10, 21].

Fuzzy set is a mathematical model of vague qualitative or quantitative data, generated by means of natural language. It is based on the generalization of the classical concepts of set and its characteristic function. Fuzzy sets are considered

with respect to a nonempty base set  $X$  of elements of interest. The essential idea is that each element  $x \in X$  is assigned a membership grade  $\mu(x)$  which takes value from the unit interval  $[0, 1]$  with  $\mu(x) = 0$  for non-membership,  $0 < \mu(x) < 1$  for partial membership and  $\mu(x) = 1$  indicates full membership.

A fuzzy real number  $X$  is a fuzzy set on  $R$ , more precisely a mapping  $X : R \rightarrow I (= [0, 1])$ , associating each real number  $t$  with its grade of membership  $X(t)$ . In general  $X$  satisfies the following conditions:

- (i)  $X$  is normal if there exists  $t \in R$  such that  $X(t) = 1$ .
- (ii)  $X$  is upper-semi-continuous if for each  $\varepsilon > 0$ ,  $X^{-1}([0, a + \varepsilon))$ , is open in the usual topology of  $R$ , for all  $a \in I$ .
- (iii)  $X$  is convex, if  $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$ , where  $s < t < r$ .
- (iv) The closure of the set  $\{t \in R : X(t) > 0\}$ , denoted by  $X^0$  is compact.

The  $\alpha$ -level set of a fuzzy real number  $X$  is defined by

$$[X]^\alpha = \begin{cases} \{t \in R : X(t) \geq \alpha\} & \text{if } 0 < \alpha \leq 1; \\ \overline{\{t \in R : X(t) > \alpha\}} & \text{if } \alpha = 0. \end{cases}$$

The properties (i)-(iv) mentioned above imply that for  $0 < \alpha \leq 1$ , the  $\alpha$ -level set is a non-empty compact convex subset of  $R$ . We denote the class of all upper-semi-continuous, normal, convex fuzzy real numbers is by  $R(I)$  and  $R^*(I)$  denotes the set of all positive fuzzy real numbers. For  $X, Y \in R(I)$ ,  $X \leq Y$  if and only if  $X^\alpha \leq Y^\alpha$  for  $\alpha \in [0, 1]$  and " $\leq$ " is a partial order in  $R(I)$ .

Let  $D$  be the set of all closed bounded intervals  $X = [X^L, X^R], Y = [Y^L, Y^R]$ . Then  $X \leq Y$  implies that  $X^L \leq Y^L$  and  $X^R \leq Y^R$ . We write  $d(X, Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$ . It is easy to verify that  $(D, d)$  is a complete metric space.

Consider the mapping  $\bar{d} : R(I) \times R(I) \rightarrow R$  defined by  $\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X, Y)$ , for  $X, Y \in R(I)$ . Clearly  $\bar{d}$  define a metric on  $R(I)$ . For any  $X, Y, Z \in R(I)$ , the linear structure of  $R(I)$  induces addition  $X+Y$  and scalar multiplication  $\lambda X$ ,  $\lambda \in R$  in terms of  $\alpha$ -level set, defined as  $[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$  and  $[\lambda X]^\alpha = \lambda[X]^\alpha$ , for each  $\alpha \in [0, 1]$ . A subset  $E$  of  $R(I)$  is said to be *bounded above* if there exist a fuzzy real number  $\mu$  such that  $X \leq \mu$  for every  $X \in E$ . We called  $\mu$  as the *upper bound* of  $E$  and it is called *least upper bound* if  $\mu \leq \mu^*$  for all upper bound  $\mu^*$  of  $E$ . A lower bound and greatest lower bound is defined similarly. The set  $E$  is said to be *bounded* if it is both bounded above and bounded below.

The set of real numbers  $R$  can be embedded into  $R(I)$ , since each  $r \in R$  can be regarded as a fuzzy number  $\bar{r}$  given by

$$\bar{r}(t) = \begin{cases} 1 & \text{if } t = r; \\ 0 & \text{if } t \neq r. \end{cases}$$

The additive identity and multiplicative identity of  $R(I)$  are denoted by  $\bar{0}$  and  $\bar{1}$  respectively.

A sequence  $X = (X_k)$  fuzzy real number is a function  $X$  from the set of positive integer into  $R(I)$ . The fuzzy number  $X_k$  is called the  $k^{th}$  term of the sequence. The set  $E^F$  of sequences taken from  $R(I)$  is said to be a *sequence space of fuzzy real number* if, for  $(X_k), (Y_k) \in E^F, r \in R$  i.e.  $X_k, Y_k \in R(I)$ , and for all  $k \in N, (X_k) + (Y_k) = (X_k + Y_k) \in E^F$  and  $r(X_k) = (rX_k) \in E^F$ , where

$$r(X_k)(t) = \begin{cases} X_k(r^{-1}t) & \text{if } r \neq 0; \\ \bar{0} & \text{if } r = 0. \end{cases}$$

A sequence  $X = (X_k)$  of fuzzy number is said to be *convergent to a fuzzy number*  $X_0$  if for  $\varepsilon > 0$  there exist a positive integer  $n_0$  such that  $\bar{d}(X_k, X_0) < \varepsilon$ , for every  $k > n_0$ .

A fuzzy real-valued sequence  $(X_k)$  is said to be *bounded* if  $\sup_k d(X_k, \bar{0}) < \infty$ , equivalently, if there exist  $\mu \in R^*(I)$ , such that  $|X_k| \leq \mu$  for all  $k \in N$ .

We denote the class of bounded sequences of fuzzy real numbers by  $\ell_\infty^F$ . Sequences of fuzzy real numbers are studied from different points of view by Nuray and Savas [22], Savas [23], Çolak et al. [24], Jaroslav et al. [25] and many others ([4, 26–29]).

A sequence  $X = (X_k)$  of fuzzy real numbers is said to be *statistically convergent to a fuzzy real number*  $X_0$  if for every  $\varepsilon > 0, \delta(\{k \in N : \bar{d}(X_k, X_0) > \varepsilon\}) = 0$ .

A sequence  $X = (X_k)$  of fuzzy real numbers is said to be  $\Delta$ -*bounded* if the set  $\{\Delta X_k : k \in N\}$  is bounded. We denote the class of  $\Delta$ -bounded sequences of fuzzy real numbers by  $\ell_\infty^F(\Delta)$ .

A sequence  $X = (X_k)$  of fuzzy real numbers is said to be  $\Delta$ -*statistically convergent to a fuzzy real number*  $X_0$ , if for every  $\varepsilon > 0, \delta(\{k \in N : \bar{d}(\Delta X_k, X_0) > \varepsilon\}) = 0$ .

In this article we have introduced the classes of fuzzy real-valued sequences of strong  $\Delta^m$ -lacunary asymptotically equivalent,  $\Delta^m$ -statistical asymptotically equivalent, strong  $\Delta^m$ -Cesàro asymptotically equivalent, strong  $\Delta^m$ -lacunary statistical asymptotically equivalent and established some relations between them in terms of Orlicz function.

## 2 Definitions and Notations

In this section we begin with some known definitions.

**Definition 2.1.** The sequence  $(X_t)$  of fuzzy real numbers is said to be *statistically convergent to a fuzzy real number*  $X_0$ , if

$$\lim_n \frac{1}{n} \{ \text{number of } t \leq n : \bar{d}(X_t, X_0) > \varepsilon \} = 0.$$

**Definition 2.2.** Two sequences  $X = (X_t)$  and  $Y = (Y_t)$  of fuzzy real numbers are said to be *asymptotically equivalent* if

$$\lim_t \bar{d} \left( \frac{X_t}{Y_t}, \bar{1} \right) = 0.$$

**Definition 2.3.** Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be *asymptotically statistically equivalent of multiple  $L$* , provided

$$\lim_n \frac{1}{n} \left\{ \text{number of } t \leq n : \bar{d} \left( \frac{X_t}{Y_t}, L \right) > \varepsilon \right\} = 0.$$

**Definition 2.4.** Let  $\theta = (k_r)$  be a lacunary sequence. Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be *strongly lacunary asymptotically equivalent of multiple  $L$* , if

$$\lim_r \frac{1}{h_r} \sum_{t \in I_r} \bar{d} \left( \frac{X_t}{Y_t}, L \right) = 0.$$

**Definition 2.5.** Let  $\theta = (k_r)$  be a lacunary sequence. Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be *lacunary asymptotically statistical equivalent of multiple  $L$* , provided that for every

$$\lim_r \frac{1}{h_r} \left\{ \text{number of } t \in I_r : \bar{d} \left( \frac{X_t}{Y_t}, L \right) > \varepsilon \right\} = 0.$$

Esi and Esi [5] formulate the following definition for fuzzy real valued sequences.

**Definition 2.6.** Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be  *$\Delta$ -asymptotically equivalent* if

$$\lim \bar{d} \left( \frac{\Delta X_t}{\Delta Y_t}, \bar{1} \right) = 0.$$

**Definition 2.7.** Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be  *$\Delta$ -statistically asymptotically equivalent of multiple  $L$* , if for every  $\varepsilon > 0$ ,

$$\lim_n \frac{1}{n} \left\{ \text{number of } t \leq n : \bar{d} \left( \frac{\Delta X_t}{\Delta Y_t}, L \right) > \varepsilon \right\} = 0.$$

We write it as  $X \overset{S^L(\Delta)}{\sim} Y$ . For  $L = \bar{1}$  it is simply  $\Delta$ -statistically asymptotically equivalent.

**Definition 2.8.** Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be *strong  $\Delta$ -Cesàro asymptotically equivalent to multiple  $L$* , if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \bar{d} \left( \frac{\Delta X_t}{\Delta Y_t}, L \right) = 0.$$

It is written as  $X \overset{C^L(\Delta)}{\sim} Y$ . For  $L = \bar{1}$ , it is simply strong  $\Delta$ -Cesàro asymptotically equivalent.

We introduced the following definitions.

**Definition 2.9.** Let  $M$  be an Orlicz function. A sequences  $(X_t)$  of fuzzy real numbers is said to be *strong  $\Delta^m$ -Cesàro summable to  $L$  with respect to  $M$*  if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n M \{ \bar{d}(\Delta^m X_t, L) \} = 0.$$

**Definition 2.10.** Let  $M$  be an Orlicz function. A sequences  $(X_t)$  of fuzzy real numbers is said to be  *$\Delta^m$ -strong almost convergent to  $L$  with respect to  $M$  uniformly in  $p$*  if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n M \{ \bar{d}(\Delta^m X_{t+p}, L) \} = 0.$$

**Definition 2.11.** Let  $M$  be an Orlicz function and  $\theta = (k_r)$  be a lacunary sequence. A sequences  $(X_t)$  of fuzzy real numbers is said to be *strong  $\Delta^m$ -lacunary convergent to  $L$  with respect to  $M$*  if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{t \in I_r} M \{ \bar{d}(\Delta^m X_t, L) \} = 0.$$

**Definition 2.12.** Let  $M$  be an Orlicz function. Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be *strong  $\Delta^m$ -asymptotically equivalent of multiple  $L$  with respect to  $M$*  if

$$\lim_t M \left\{ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} = 0.$$

We denote it as  $X \overset{(M^L, \Delta^m)}{\sim} Y$ . For  $L = \bar{1}$  it is simply strong  $\Delta^m$ -asymptotically equivalent.

**Definition 2.13.** Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be *strong  $\Delta^m$ -asymptotically statistically equivalent of multiple  $L$*  if

$$\lim_n \frac{1}{n} \left\{ \text{number of } t \leq n : \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} = 0.$$

We denote it as  $X \overset{(S^L, \Delta^m)}{\sim} Y$ . For  $L = \bar{1}$  it is simply strong  $\Delta^m$ -asymptotically statistically equivalent.

**Definition 2.14.** Let  $M$  be an Orlicz function and  $\theta = (k_r)$  be a lacunary sequence. Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be *strong  $\Delta^m$ -lacunary asymptotically equivalent of multiple  $L$ , with respect to  $M$* , if for every  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{t \in I_r} M \left\{ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} = 0.$$

We write it as  $X \overset{N_{\theta}^L(\Delta^m, M)}{\sim} Y$ . For  $L = \bar{1}$  it is simply strong  $\Delta^m$ -lacunary asymptotically equivalent.

**Definition 2.15.** Let  $M$  be an Orlicz function. Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be *strong  $\Delta^m$ -Cesàro asymptotically equivalent to multiple  $L$  with respect to  $M$* , if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n M \left\{ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} = 0.$$

We write it as  $X \overset{C_1^L(\Delta^m, L)}{\sim} Y$ . For  $L = \bar{1}$ , it is simply  $\Delta^m$ - strong Cesàro asymptotically equivalent.

**Definition 2.16.** Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be  *$\Delta^m$ -lacunary statistically asymptotically equivalent of multiple  $L$  with respect to  $M$* , if for every  $\varepsilon > 0$ ,

$$\lim_r \frac{1}{h_r} \left\{ \text{number of } k \in I_r : \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) > \varepsilon \right\} = 0.$$

We write it as  $X \overset{S_{\theta}^L(\Delta)}{\sim} Y$ . For  $L = \bar{1}$  it is simply  $\Delta^m$ -lacunary statistical asymptotically equivalent.

**Definition 2.17.** Two sequences  $(X_t)$  and  $(Y_t)$  of fuzzy real numbers are said to be  *$\Delta^m$ -strong almost asymptotically equivalent of multiple  $L$  with respect to  $M$* , if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n M \left\{ \bar{d} \left( \frac{\Delta^m X_{t+m}}{\Delta^m Y_{t+m}}, L \right) \right\} = 0.$$

We write it as  $X \overset{|AC|^L(\Delta^m, M)}{\sim} Y$ . For  $L = \bar{1}$ , it is simply strongly  $\Delta^m$ -almost asymptotically equivalent.

### 3 Main Results

**Theorem 3.1.** Let  $M$  be an Orlicz function, then

- (i) If  $X \overset{(M^L, \Delta^m)}{\sim} Y$  then  $X \overset{\Delta^m}{\sim} Y$ .
- (ii) If  $X \overset{C_1^L(\Delta^m, M)}{\sim} Y$  then  $X \overset{(S^L, \Delta^m)}{\sim} Y$ .
- (iii) If  $M$  is bounded then  $X \overset{(S^L, \Delta^m)}{\sim} Y \Rightarrow X \overset{C_1^L(\Delta^m, M)}{\sim} Y$ .

*Proof.* (i) Let  $X = (X_t), Y = (Y_t) \in w^F$ . Since  $M$  is continuous and  $M(x) = 0$  iff  $x = 0$ . We have

$$\lim_t M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] = 0 \Rightarrow M \left[ \lim_t \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] = 0 \text{ iff } \lim_t \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) = 0.$$

Thus  $X \overset{\Delta^m}{\sim} Y$ . This completes the proof.

(ii) Let  $X = (X_t), Y = (Y_t) \in w^F$  be such that  $X \overset{C_1^L(\Delta^m, M)}{\sim} Y$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] = 0.$$

We consider  $\bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \geq \varepsilon$  for a given  $\varepsilon > 0$ . Thus we have

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] &\geq \frac{1}{n} \sum_{t \leq n} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &\geq M(\varepsilon) \frac{1}{n} \left\{ \text{number of } t \leq n : \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \geq \varepsilon \right\}. \end{aligned}$$

Now taking limit as  $r \rightarrow \infty$  the result follows,  $X \overset{(S^L, \Delta^m)}{\sim} Y$ .

(iii) Let  $X \overset{(S^L, \Delta^m)}{\sim} Y$  and suppose that  $M$  is bounded. Then we have

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] &= \frac{1}{n} \sum_{t=1; \bar{d}(\frac{\Delta^m X_t}{\Delta^m Y_t}, L) \geq \varepsilon}^n M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &\quad + \frac{1}{n} \sum_{t=1; \bar{d}(\frac{\Delta^m X_t}{\Delta^m Y_t}, L) < \varepsilon}^n M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &\leq \sup M(n) \frac{1}{n} \left\{ \text{number of } t \leq n : \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \geq \varepsilon \right\} + M(\varepsilon). \end{aligned}$$

Thus the result is obtained by taking limit as  $n \rightarrow \infty$ . □

**Lemma 3.2.** Let  $M$  be an Orlicz function and consider  $0 < \delta < 1$ . Then for  $y \neq 0$  and each  $\frac{x}{y} > \delta$  we have  $M \left( \frac{x}{y} \right) \leq 2M(1)\delta^{-1} \left( \frac{x}{y} \right)$ .

**Theorem 3.3.** Let  $\theta = (k_r)$  be a lacunary and  $M$  be an Orlicz function. Then  $X \overset{N_\theta^L(\Delta^m)}{\sim} Y$  implies  $X \overset{N_\theta^L(M, \Delta^m)}{\sim} Y$ .

*Proof.* Let  $X = (X_t), Y = (Y_t) \in w^F$  be such that  $X \overset{N_\theta^L(\Delta^m)}{\sim} Y$ . Then

$$\tau_r = \frac{1}{h_r} \sum_{t \in I_r} \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \rightarrow 0 \text{ as } r \rightarrow \infty.$$



For a given  $\varepsilon > 0$  choose  $0 < \delta < 1$  such that  $f(u) < \varepsilon$  for  $0 \leq u \leq \delta$ .

By using Lemma 3.2 we have

$$\begin{aligned} \frac{1}{h_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] &= \frac{1}{h_r} \sum_{t \in I_r; \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \leq \varepsilon} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &\quad + \frac{1}{h_r} \sum_{t \in I_r; \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) > \varepsilon} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &\leq \frac{1}{h_r} (h_r \varepsilon) + \frac{1}{h_r} 2M(1) \delta^{-1} \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &\leq \frac{1}{h_r} (h_r \varepsilon) + \frac{1}{h_r} 2M(1) \delta^{-1} \tau_r h_r. \end{aligned}$$

Thus the result is obtained by taking limit as  $r \rightarrow \infty$ .  $\square$

**Theorem 3.4.** Let  $\theta = (k_r)$  be a lacunary and  $M$  be an Orlicz function. Then  $X \overset{N_\theta^L(M, \Delta^m)}{\sim} Y \Rightarrow X \overset{N_\theta^L(\Delta^m)}{\sim} Y$  iff  $\lim_{t \rightarrow \infty} \frac{M(t)}{t} = \beta > 0$ .

*Proof.* Let  $X = (X_t), Y = (Y_t) \in w^F$  be such that  $X \overset{N_\theta^L(M, \Delta^m)}{\sim} Y$ . Select  $\beta > 0$  such that  $M(t) \geq \beta t$ , for all  $t \geq 0$ . Now we have

$$\begin{aligned} \frac{1}{h_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] &\geq \frac{1}{h_r} \sum_{t \in I_r} \beta \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &= \frac{1}{h_r} \beta \sum_{t \in I_r} \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right]. \end{aligned}$$

Taking  $r \rightarrow \infty$ , the L.H.S will be zero and hence we have

$$\lim_r \frac{1}{h_r} \beta \sum_{t \in I_r} \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \rightarrow 0.$$

Thus  $X \overset{N_\theta^L(\Delta^m)}{\sim} Y$ .  $\square$

**Theorem 3.5.** Let  $\theta = (k_r)$  be a lacunary and  $M$  be an Orlicz function. Then  $X \overset{|AC|^L(\Delta^m, M)}{\sim} Y$  implies  $X \overset{N_\theta^L(\Delta^m, M)}{\sim} Y$ .

*Proof.* Let  $X = (X_t), Y = (Y_t) \in w^F$  be such that  $X \overset{|AC|^L(\Delta^m, M)}{\sim} Y$ . Then for  $\varepsilon > 0$  there exist  $N > 0$  such that

$$\frac{1}{n} \sum_{t=1}^n M \left[ \bar{d} \left( \frac{\Delta^m X_{t+m}}{\Delta^m Y_{t+m}}, L \right) \right] < \varepsilon \text{ for } n > N \text{ and } m = 0, 1, 2, 3, \dots$$

For  $0 < R \leq r$  such that  $h_r > N$  and then we have

$$\frac{1}{h_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] < \varepsilon$$

Taking limit as  $r \rightarrow \infty$  we have the result  $X^{N_{\theta}^L(\Delta^m, M)} Y$ . □

**Theorem 3.6.** *Let  $\theta = (k_r)$  be a lacunary and  $M$  be an Orlicz function. If  $\lim_r \inf q_r > 1$  then  $X^{C_1^L(\Delta^m, M)} Y \Rightarrow X^{N_{\theta}^L(\Delta^m, M)} Y$ .*

*Proof.* Let  $X, Y \in w^F$  be such that  $X^{C_1^L(\Delta^m, M)} Y$ . Suppose  $1 < \lim_r \inf q_r$  then there exist  $\delta > 0$  such that  $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$  for sufficiently large  $r$  and  $\frac{h_r}{k_r} = \frac{k_r - k_{r-1}}{k_r} \leq \frac{\delta}{1 + \delta}$ . Now we have for  $k_{r-1} < n \leq k_r$

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] &> \frac{1}{k_r} \sum_{t=1}^{k_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &> \frac{1}{k_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &= \frac{h_r}{k_r} \frac{1}{h_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right]. \end{aligned}$$

Taking limit as  $r \rightarrow \infty$  and using the fact that  $X^{C_1^L(\Delta^m, M)} Y$  we have

$$\frac{1}{h_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \rightarrow \infty.$$

Thus it proves that  $X^{C_1^L(\Delta^m, M)} Y \Rightarrow X^{N_{\theta}^L(\Delta^m, M)} Y$ . □

**Theorem 3.7.** *Let  $\theta = (k_r)$  be a lacunary and  $M$  be an Orlicz function. If  $\lim_r \sup q_r < \infty$  then  $X^{N_{\theta}^L(\Delta^m, M)} Y \Rightarrow X^{C_1^L(\Delta^m, M)} Y$ .*

*Proof.* Let  $X = (X_t), Y = (Y_t) \in w^F$  be such that  $X^{N_{\theta}^L(\Delta^m, M)} Y$  and  $\lim_r \sup q_r < \infty$ , then there exist  $B > 0$  such that  $q_r < B$ . Now for a given  $\varepsilon > 0$ , we can choose  $n > 0$  and  $K$  such that for every  $r \geq n$

$$H_r = \frac{1}{h_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] < \varepsilon \text{ and } H_r \leq K, \text{ for all } r.$$

Now for  $k_{r-1} < n \leq k_r$  we have

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] &\leq \frac{1}{k_{r-1}} \sum_{t=1}^{k_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\ &= \frac{1}{k_{r-1}} \left[ \sum_{t \in I_1} M \left\{ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} + \dots + \sum_{t \in I_r} M \left\{ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} \right] \\ &= \frac{1}{k_{r-1}} \sum_{t \in I_1} M \left\{ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} + \dots + \frac{1}{k_{r-1}} \sum_{t \in I_n} M \left\{ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{k_{r-1}} \sum_{t \in [I_{n+1}, r]} M \left\{ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right\} \\
& = \frac{1}{k_{r-1}} \{k_1 H_1 + \dots + k_n H_n\} + \frac{1}{k_{r-1}} (k_r - k_n) \varepsilon \\
& \leq \frac{1}{k_{r-1}} \sup_i H_i k_i + \frac{1}{k_{r-1}} (k_r - k_n) \varepsilon < \frac{1}{k_{r-1}} K k_n + \varepsilon B.
\end{aligned}$$

Thus taking limit as  $r \rightarrow \infty$  we have

$$\frac{1}{n} \sum_{t=1}^n M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \rightarrow 0.$$

This shows that  $X \overset{N_\theta^L(\Delta^m, M)}{\sim} Y \Rightarrow X \overset{C_1^L(\Delta^m, M)}{\sim} Y$ .  $\square$

Theorem 3.5 and theorem 3.6 can be combined in to the following way.

**Theorem 3.8.** Let  $\theta = (k_r)$  be a lacunary and  $M$  be an Orlicz function. If  $1 < \liminf q_r \leq \limsup q_r < \infty$ , then  $X \overset{C_1^L(\Delta^m, M)}{\sim} Y \Leftrightarrow X \overset{N_\theta^L(\Delta^m, M)}{\sim} Y$ .

**Theorem 3.9.** Let  $\theta = (k_r)$  be a lacunary and  $M$  be an Orlicz function. Then  $X \overset{N_\theta^L(\Delta^m, M)}{\sim} Y \Rightarrow X \overset{S_\theta^L(\Delta^m)}{\sim} Y$ .

*Proof.* For a given  $\varepsilon > 0$  there exist  $N > 0$  such that

$$\begin{aligned}
\frac{1}{h_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] & \geq \frac{1}{h_r} \sum_{t \in N; \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \geq \varepsilon} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \\
& \geq M(\varepsilon) \frac{1}{h_r} \left\{ \text{number of } t \in I_r : \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \geq \varepsilon \right\}
\end{aligned}$$

The L.H.S will be zero as  $r \rightarrow \infty$  thus we have

$$\frac{1}{h_r} \sum_{t \in I_r} M \left[ \bar{d} \left( \frac{\Delta^m X_t}{\Delta^m Y_t}, L \right) \right] \rightarrow 0$$

This completes the proof.  $\square$

**Theorem 3.10.** Let  $\theta = (k_r)$  be a lacunary and  $M$  be a bounded Orlicz function. Then  $X \overset{N_\theta^L(\Delta^m, M)}{\sim} Y \Leftrightarrow X \overset{S_\theta^L(\Delta^m)}{\sim} Y$ .

*Proof.* The “if part” is similar to theorem 3.8 and the “only if part” can be proved in similar way.  $\square$

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