



Extension of Integral Equations of Fredholm Type Involving Special Functions

V.B.L. Chaurasia[†] and Vinod Gill^{‡,1}

[†]Department of Mathematics, University of Rajasthan,
Jaipur-302004, Rajasthan, India.
e-mail : drvblc@yahoo.com

[‡]Department of Mathematics, Amity University Rajasthan
Jaipur-302030, India.
e-mail : vinod.gill108@gmail.com

Abstract : Integral equations occur in many fields of mechanics and mathematical physics. But specially Fredholm integral equations arise in the theory of signal processing, most notably as the spectral concentration problem. They also commonly arise in linear forward modeling and inverse problems. In this paper we establish a solution of integral equation of Fredholm type whose kernel involve certain product of special functions by using Riemann-Liouville and Weyl fractional integral operators.

Keywords : Fredholm type integral equations; Riemann-Liouville fractional integral; Weyl fractional integral; Mellin transform; multivariable I -function; general class of polynomials; generalized polynomials; I -function.

2010 Mathematics Subject Classification : 26A33; 33C99; 33E60; 33E70; 33E20; 45B05.

1 Introduction

Fredholm type integral equations involving various polynomials or special functions as their kernel, have been studied by many authors notably Buchman [1], Fox [2], Higgins [3], Love [4, 5], Prabhakar and Kashyap [6], Srivastava and Buchman

¹Corresponding author.

[7], Srivastava [8], Srivastava and Raina [9], Chaurasia and Patni [10], Chaurasia and Kumar [11] and among others. In the present paper, we obtain the solutions of the following Fredholm integral equations

$$\int_0^\infty y^{-\alpha} S_{N_1, \dots, N_k}^{M_1, \dots, M_k} \left[u_1 \left(\frac{x}{y} \right)^p, \dots, u_k \left(\frac{x}{y} \right)^p \right] \times I_{p_2, q_2; p_3, q_3; \dots; p_r, q_r}^{0, n_2; 0, n_3; \dots; 0, n_r} (m', n'; \dots; (m^{(r)}, n^{(r)}))$$

$$\left[\begin{array}{l} z_1 \left(\frac{x}{y} \right)^q \\ \vdots \\ z_r \left(\frac{x}{y} \right)^q \end{array} \left| \begin{array}{l} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2}; \dots; (a_{rj}; \alpha'_{rj}, \dots, \alpha^{(r)}_{rj})_{1, p_r}; (a'_j, \alpha'_j)_{1, p'}; \dots; (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}} \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2}; \dots; (b_{rj}; \beta'_{rj}, \dots, \beta^{(r)}_{rj})_{1, q_r}; (b'_j, \beta'_j)_{1, q'}; \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}} \end{array} \right]$$

$$\times f(y) dy = g(x) \quad (0 < x < \infty) \tag{1.1}$$

and

$$\int_0^\infty y^{-\alpha} S_V^U \left[\lambda \left(\frac{x}{y} \right)^\sigma \right] I_{P, Q}^{M, N} \left[u \left(\frac{x}{y} \right)^p \left| \begin{array}{l} (\alpha'_1, A'_1, a'_1), \dots, (\alpha'_P, A'_P, a'_P) \\ (\beta'_1, B'_1, b'_1), \dots, (\beta'_Q, B'_Q, b'_Q) \end{array} \right. \right]$$

$$\times I_{p, q}^{m, n} \left[z \left(\frac{x}{y} \right)^q \left| \begin{array}{l} (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q) \end{array} \right. \right] f(y) dy = g(x) \quad (0 < x < \infty). \tag{1.2}$$

The multivariable *I*-function is defined and represented in the following manner by Prasad [12]:

$$I[z_1, \dots, z_r] = I_{p_2, q_2; p_3, q_3; \dots; p_r, q_r}^{0, n_2; 0, n_3; \dots; 0, n_r} (m', n'; \dots; (m^{(r)}, n^{(r)}))$$

$$\left[\begin{array}{l} z_1 \\ \vdots \\ z_r \end{array} \left| \begin{array}{l} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2}; (a_{3j}; \alpha'_{3j}, \alpha''_{3j}, \alpha'''_{3j})_{1, p_3}; \dots; (a_{rj}; \alpha'_{rj}, \dots, \alpha^{(r)}_{rj})_{1, p_r}; (a'_j, \alpha'_j)_{1, p'}; \dots; (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}} \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2}; (b_{3j}; \beta'_{3j}, \beta''_{3j}, \beta'''_{3j})_{1, q_3}; \dots; (b_{rj}; \beta'_{rj}, \dots, \beta^{(r)}_{rj})_{1, q_r}; (b'_j, \beta'_j)_{1, q'}; \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}} \end{array} \right]$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \dots \int_{L_r} \psi_1(\xi_1) \dots \psi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r \tag{1.3}$$

where $i = \sqrt{-1}$. For convergence conditions and other details of multivariable *I*-function, see Prasad [12].

Srivastava [13] introduced the general class of polynomials, as follows:

$$S_V^U[x] = \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V, K} x^K, \quad V = 0, 1, 2, \dots \tag{1.4}$$

where U is an arbitrary positive integer and the coefficients $A_{V, K}$, ($V, K \geq 0$) are arbitrary constants, real or complex. The generalized polynomials defined by Srivastava [14], is as follows:

$$S_{N_1, \dots, N_k}^{M_1, \dots, M_k} [x_1, \dots, x_k]$$

$$= \sum_{\alpha_1=0}^{[N_1/M_1]} \dots \sum_{\alpha_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!} \dots \frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!} B[N_1, \alpha_1; \dots; N_k, \alpha_k] x_k^{\alpha_1} \dots x_k^{\alpha_k}, \tag{1.5}$$

where $N_{i'} = 0, 1, 2, \dots$ for all $i' = (1, \dots, k)$, M_1, \dots, M_k arbitrary positive integers and the coefficient $B[N_1, \alpha_1; \dots; N_k, \alpha_k]$ are arbitrary constants, real or complex.

Rathie [15] introduced the I -function, in the following manner:

$$I_{p,q}^{m,n}[z] = I_{p,q}^{m,n} \left[z \left| \begin{matrix} (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \psi(s) z^s ds \tag{1.6}$$

where

$$\psi(s) = \frac{\prod_{j=1}^m \{\Gamma(\beta_j - B_j s)\}^{b_j} \prod_{j=1}^n \{\Gamma(1 - \alpha_j + A_j s)\}^{a_j}}{\prod_{j=m+1}^q \{\Gamma(1 - \beta_j + B_j s)\}^{b_j} \prod_{j=n+1}^p \{\Gamma(\alpha_j - A_j s)\}^{a_j}}, \tag{1.7}$$

where $a_j, j = 1, \dots, p$ and $b_j, j = 1, \dots, q$ are not in general, positive integers. Here z may be real or complex but is not equal to zero. An empty product is interpreted as unity; p, q, m and n are integers such that $0 \leq m \leq q; 0 \leq n \leq p; A_j > 0 (j = 1, \dots, p), B_j > 0 (j = 1, \dots, q); \alpha_j (j = 1, \dots, p)$ and $\beta_j (j = 1, \dots, q)$ are complex parameters. If we take $b_j (j = 1, \dots, m)$ and $a_j (j = n + 1, \dots, p)$ unity in (1.6), I -function reduces to \bar{H} -function [16] and when exponents $a_j = b_j = 1$ for all j in (1.6), I -function reduces to the familiar Fox's H -function defined by Fox [17]. Some important characteristics related to H -function are also given in [18]. For a detailed definition of contour, convergence conditions etc. of the I -function, see [15].

The series representation for the I -function [15] is given as: by calculating the residues at the poles of $\Gamma(\beta_j - B_j s)$ for $j = 1, \dots, m$ with $b_j = 1$ for $j = 1, \dots, m$ in (1.7), we obtain the following representation of the I -function in a series form [15] as

$$I_{p,q}^{m,n}[z] = I_{p,q}^{m,n} \left[z \left| \begin{matrix} (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q) \end{matrix} \right. \right] = \sum_{h=1}^m \sum_{k=0}^{\infty} \frac{(-1)^k \psi(\zeta)}{k! B_h} z^\zeta, \tag{1.8}$$

where

$$\zeta = \frac{\beta_h + k}{B_h}$$

and

$$\psi(\zeta) = \frac{\prod_{j=1}^m \{\Gamma(\beta_j - B_j \zeta)\}^{b_j} \prod_{j=1}^n \{\Gamma(1 - \alpha_j + A_j \zeta)\}^{a_j}}{\prod_{j=m+1}^q \{\Gamma(1 - \beta_j + B_j \zeta)\}^{b_j} \prod_{j=n+1}^p \{\Gamma(\alpha_j - A_j \zeta)\}^{a_j}}. \tag{1.9}$$

Let S denote the space of all functions f which are defined on $R^+ = [0, \infty)$ and satisfy

- (i) $f \in C^\infty (R^+)$,
- (ii) $\lim_{x \rightarrow \infty} [x^\gamma f^r(x)] = 0$ for all non-negative integers γ and r ,
- (iii) $f(x) = 0(1)$ as $x \rightarrow 0$,

for correspondence to the space of good functions defined on the whole real line $(-\infty, \infty)$ see Lighthill [19].

The Riemann-Liouville fractional integral (of order μ) defined in [20] as follows:

$$D^{-\mu}\{f(x)\} = {}_0D_x^{-\mu}\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_0^x (x - \omega)^{\mu-1} f(\omega) d\omega, \quad (Re(\mu) > 0 : f \in S) \tag{1.10}$$

and the Weyl fractional integral (of order h) defined in [20] as follows:

$$W^{-h}\{f(x)\} = {}_x D_\infty^{-h}\{f(x)\} = \frac{1}{\Gamma(h)} \int_x^\infty (\xi - x)^{h-1} f(\xi) d\xi, \quad (Re(h) > 0 : f \in S). \tag{1.11}$$

2 Solution of the Integral Equation Involving the Product of Generalized Polynomials with Multivariable I -functions

Lemma 2.1. *Let*

- (i) $m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)}$, ($i = 1, \dots, r$), n_ℓ, p_ℓ, q_ℓ ($\ell = 2, \dots, r$) are non-negative integers such that $0 \leq m^{(i)} \leq q^{(i)}$, $0 \leq n^{(i)} \leq p^{(i)}$, $q_\ell \geq 0$ and $0 \leq n_\ell \leq p_\ell$.
- (ii) $Re(\alpha) > Re(\beta)$; $Re\left[\beta + q \sum_{i=1}^r \left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > 0$, ($j = 1, \dots, m^{(i)}$; $q > 0$)
- (iii) $|arg(z_i)| < \frac{1}{2} \pi T_i$, where $T_i > 0$, $i = 1, \dots, r$ and

$$T_i = \sum_{j=1}^{n^{(i)}} \alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} \alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}} \beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}} \beta_j^{(i)} + \left(\sum_{j=1}^{n_2} \alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2} \alpha_{2j}^{(i)} \right) + \left(\sum_{j=1}^{n_3} \alpha_{3j}^{(i)} - \sum_{j=n_3+1}^{p_3} \alpha_{3j}^{(i)} \right) + \dots + \left(\sum_{j=1}^{n_r} \alpha_{rj}^{(i)} - \sum_{j=n_r+1}^{p_r} \alpha_{rj}^{(i)} \right) + \left(\sum_{j=1}^{q_2} \beta_{2j}^{(i)} + \sum_{j=1}^{q_3} \beta_{3j}^{(i)} + \dots + \sum_{j=1}^{q_r} \beta_{rj}^{(i)} \right).$$

Then

$$\begin{aligned}
 &W^{\beta-\alpha} \left[y^{-\alpha} S_{N_1, \dots, N_k}^{M_1, \dots, M_k} \left[u_1 \left(\frac{x}{y} \right)^p, \dots, u_k \left(\frac{x}{y} \right)^p \right] \times I_{p_2, q_2; p_3, q_3; \dots; p_r, q_r}^{0, n_2; 0, n_3; \dots; 0, n_r; (m', n'); \dots; (m^{(r)}, n^{(r)})} \right. \\
 &\quad \left. \begin{aligned}
 &\left[\begin{array}{l} z_1 \left(\frac{x}{y} \right)^q \\ \vdots \\ z_r \left(\frac{x}{y} \right)^q \end{array} \right] \begin{array}{l} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2}; (a_{3j}; \alpha'_{3j}, \alpha''_{3j}, \alpha'''_{3j})_{1, p_3}; \dots; \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2}; (b_{3j}; \beta'_{3j}, \beta''_{3j}, \beta'''_{3j})_{1, q_3}; \dots; \\ (a_{rj}; \alpha'_{rj}, \dots, \alpha^{(r)}_{rj})_{1, p_r}; (a'_j, \alpha'_j)_{1, p'}; \dots; (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}} \\ (b_{rj}; \beta'_{rj}, \dots, \beta^{(r)}_{rj})_{1, q_r}; (b'_j, \beta'_j)_{1, q'}; \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}} \end{array} \right] \\
 &= y^{-\beta} \sum_{\alpha_1=0}^{[N_1/M_1]} \dots \sum_{\alpha_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1 \alpha_1}}{\alpha_1!} \dots \frac{(-N_k)_{M_k \alpha_k}}{\alpha_k!} \\
 &\times B[N_1, \alpha_1; \dots; N_k, \alpha_k] u_1^{\alpha_1} \left(\frac{x}{y} \right)^{p\alpha_1} \dots u_k^{\alpha_k} \left(\frac{x}{y} \right)^{p\alpha_k} I_{p_2, q_2; p_3, q_3; \dots; p_r, q_r}^{0, n_2; 0, n_3; \dots; 0, n_r; (m', n'); \dots; (m^{(r)}, n^{(r)})} \\
 &\quad \left[\begin{array}{l} z_1 \left(\frac{x}{y} \right)^q \left(1 - \beta - p \sum_{i=1}^k \alpha_i; q, \dots, q \right), (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2}; (a_{3j}; \alpha'_{3j}, \alpha''_{3j}, \alpha'''_{3j})_{1, p_3}; \dots; \\ z_r \left(\frac{x}{y} \right)^q (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2}; (b_{3j}; \beta'_{3j}, \beta''_{3j}, \beta'''_{3j})_{1, q_3}; \dots; (b_{rj}; \beta'_{rj}, \dots, \beta^{(r)}_{rj})_{1, q_r} \\ (a_{rj}; \alpha'_{rj}, \dots, \alpha^{(r)}_{rj})_{1, p_r}; (a'_j, \alpha'_j)_{1, p'}; \dots; (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}} \\ \left(1 - \alpha - p \sum_{i=1}^k \alpha_i; q, \dots, q \right); (b'_j, \beta'_j)_{1, q'}; \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}} \end{array} \right]. \tag{2.1}
 \end{aligned}$$

Proof. To prove Lemma 2.1, we first use the definition of Weyl fractional integral given in (1.11), express generalized polynomial given by (1.5) and multivariable *I*-function in terms of Mellin-Barnes type of contour integrals by (1.3), then we change the order of summations and integrations (which is justified under the stated conditions), evaluate the *t*-integral and reinterpreting the resulting Mellin-Barnes contour integral in term of the multivariable *I*-function, we easily arrive at the desired result. □

Theorem 2.2. *With the set of sufficient conditions (i), (ii) and (iii) of Lemma 2.1,*

$$\begin{aligned}
 &\int_0^\infty y^{-\beta} \sum_{\alpha_1=0}^{[N_1/M_1]} \dots \sum_{\alpha_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1 \alpha_1}}{\alpha_1!} \dots \frac{(-N_k)_{M_k \alpha_k}}{\alpha_k!} B[N_1, \alpha_1; \dots; N_k, \alpha_k] u_1^{\alpha_1} \\
 &\quad \times \left(\frac{x}{y} \right)^{p\alpha_1} \dots u_k^{\alpha_k} \left(\frac{x}{y} \right)^{p\alpha_k} I_{p_2, q_2; p_3, q_3; \dots; p_r+1, q_r+1}^{0, n_2; 0, n_3; \dots; 0, n_r+1; (m', n'); \dots; (m^{(r)}, n^{(r)})}
 \end{aligned}$$

$$\begin{aligned}
 & \left[\begin{array}{l} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{array} \left| \begin{array}{l} \left(1 - \beta - p \sum_{i=1}^k \alpha_i; q, \dots, q\right) (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2}; (a_{3j}; \alpha'_{3j}, \alpha''_{3j}, \alpha'''_{3j})_{1,p_3}; \dots; \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2}; (b_{3j}; \beta'_{3j}, \beta''_{3j}, \beta'''_{3j})_{1,q_3}; \dots; (b_{rj}; \beta'_{rj}, \dots, \beta^{(r)}_{rj})_{1,q_r} \\ (a_{rj}; \alpha'_{rj}, \dots, \alpha^{(r)}_{rj})_{1,p_r}; (a'_j, \alpha'_j)_{1,p'}; \dots; (a_j^{(r)}, \alpha_j^{(r)})_{1,p^{(r)}} \end{array} \right] f(y) dy \\
 & \left(1 - \alpha - p \sum_{i=1}^k \alpha_i; q, \dots, q\right); (b'_j, \beta'_j)_{1,q'}; \dots; (b_j^{(r)}, \beta_j^{(r)})_{1,q^{(r)}} \Big] \\
 & = \int_0^\infty y^{-\alpha} S_{N_1, \dots, N_k}^{M_1, \dots, M_k} \left[u_1 \left(\frac{x}{y}\right)^p, \dots, u_k \left(\frac{x}{y}\right)^p \right] \\
 & \times I_{p_2, q_2; p_3, q_3; \dots; p_r, q_r}^{0, n_2; 0, n_3; \dots; 0, n_r} (m', n'); \dots; (m^{(r)}, n^{(r)}) \left[\begin{array}{l} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{array} \left| \begin{array}{l} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2}; \dots; \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2}; \dots; \\ (a_{rj}; \alpha'_{rj}, \dots, \alpha^{(r)}_{rj})_{1,p_r}; (a'_j, \alpha'_j)_{1,p'}; \dots; (a_j^{(r)}, \alpha_j^{(r)})_{1,p^{(r)}} \\ (b_{rj}; \beta'_{rj}, \dots, \beta^{(r)}_{rj})_{1,q_r}; (b'_j, \beta'_j)_{1,q'}; \dots; (b_j^{(r)}, \beta_j^{(r)})_{1,q^{(r)}} \end{array} \right] D^{\beta-\alpha} \{f(y)\} dy \quad (2.2)
 \end{aligned}$$

provided further $f \in S$ and $x > 0$.

Proof. Let Δ denote the first member of the assertion (2.2). Then using Lemma 2.1 and applying (1.11), we have

$$\begin{aligned}
 \Delta &= \int_0^\infty \frac{f(y)}{\Gamma(\alpha - \beta)} \left\{ \int_y^\infty (\xi - y)^{\alpha - \beta - 1} \xi^{-\alpha} S_{N_1, \dots, N_k}^{M_1, \dots, M_k} \left[u_1 \left(\frac{x}{\xi}\right)^p, \dots, u_k \left(\frac{x}{\xi}\right)^p \right] \right. \\
 & \quad \left. \times I \left[z_1 \left(\frac{x}{\xi}\right)^q, \dots, z_r \left(\frac{x}{\xi}\right)^q \right] d\xi \right\} dy
 \end{aligned}$$

the change in the order of integration is assumed to be permissible just as in the proof of Lemma 2.1.

$$\begin{aligned}
 \Delta &= \int_0^\infty \xi^{-\alpha} S_{N_1, \dots, N_k}^{M_1, \dots, M_k} \left[u_1 \left(\frac{x}{\xi}\right)^p, \dots, u_k \left(\frac{x}{\xi}\right)^p \right] \\
 & \times I \left[z_1 \left(\frac{x}{\xi}\right)^q, \dots, z_r \left(\frac{x}{\xi}\right)^q \right] \left\{ \int_0^\xi \frac{(\xi - y)^{\alpha - \beta - 1}}{\Gamma(\alpha - \beta)} f(y) dy \right\} d\xi. \quad (2.3)
 \end{aligned}$$

Finally, invoking (1.10) into the equation (2.3), we obtain

$$\Delta = \int_0^\infty \xi^{-\alpha} S_{N_1, \dots, N_k}^{M_1, \dots, M_k} \left[u_1 \left(\frac{x}{\xi}\right)^p, \dots, u_k \left(\frac{x}{\xi}\right)^p \right]$$

$$\times I \left[z_1 \left(\frac{x}{\xi} \right)^q, \dots, z_r \left(\frac{x}{\xi} \right)^q \right] D^{\beta-\alpha} \{f(\xi)\} d\xi, \tag{2.4}$$

which is precisely the right-hand member of (2.2). This completes the proof of Theorem 2.1. \square

3 Solution of the Integral Equation Involving the Product of General class of Polynomials and *I*-functions

Lemma 3.1. *Assuming the following that*

- (i) *p, q, m and n are integers such that $0 \leq m \leq q; 0 \leq n \leq p; A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$); α_j ($j = 1, \dots, p$) and β_j ($j = 1, \dots, q$) are complex parameters,*
- (ii) *$Re(\alpha) > Re(\beta)$; $Re \left[\beta + \sigma K + p\zeta + q \frac{\beta_j}{B_j} \right] > 0$; where $j = (1, \dots, m)$; $q > 0$,*
- (iii) *$|arg(z)| < \frac{1}{2} \pi T$, where*

$$T = \sum_{j=1}^m |b_j B_j| + \sum_{j=1}^n |a_j A_j| - \sum_{j=m+1}^q |b_j B_j| - \sum_{j=n+1}^p |a_j A_j| > 0,$$

then

$$\begin{aligned} & W^{\beta-\alpha} \left\{ y^{-\alpha} S_V^U \left[\lambda \left(\frac{x}{y} \right)^\sigma \right] I_{P,Q}^{M,N} \left[u \left(\frac{x}{y} \right)^p \left| \begin{matrix} (\alpha'_1, A'_1, a'_1), \dots, (\alpha'_p, A'_p, a'_p) \\ (\beta'_1, B'_1, b'_1), \dots, (\beta'_q, B'_q, b'_q) \end{matrix} \right. \right] \right. \\ & \quad \left. \times I_{p,q}^{m,n} \left[z \left(\frac{x}{y} \right)^q \left| \begin{matrix} (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q) \end{matrix} \right. \right] \right\} \\ & = y^{-\beta} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^{\infty} \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta \left(\frac{x}{y} \right)^{p\zeta + \sigma K} \\ & \quad \times I_{p+1, q+1}^{m, n+1} \left[z \left(\frac{x}{y} \right)^q \left| \begin{matrix} (1-\beta-\sigma K-p\zeta, q, 1), (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q), (1-\alpha-\sigma K-p\zeta, q, 1) \end{matrix} \right. \right]. \tag{3.1} \end{aligned}$$

Proof. The Lemma 3.1 can be easily established by using the same technique as used in Lemma 2.1. \square

Theorem 3.2. *Under the sufficient conditions (i), (ii) and (iii) of Lemma 3.1,*

$$\int_0^\infty y^{-\beta} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^{\infty} \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta \left(\frac{x}{y} \right)^{p\zeta + \sigma K}$$

$$\begin{aligned}
 & \times I_{p+1,q+1}^{m,n+1} \left[z \left(\frac{x}{y} \right)^q \left| \begin{matrix} (1-\beta-\sigma K-p\zeta, q, 1), (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q), (1-\alpha-\sigma K-p\zeta, q, 1) \end{matrix} \right. \right] f(y) dy \\
 & = \int_0^\infty y^{-\alpha} S_V^U \left[\lambda \left(\frac{x}{y} \right)^\sigma \right] I_{P,Q}^{M,N} \left[u \left(\frac{x}{y} \right)^p \left| \begin{matrix} (\alpha'_1, A'_1, a'_1), \dots, (\alpha'_p, A'_p, a'_p) \\ (\beta'_1, B'_1, b'_1), \dots, (\beta'_q, B'_q, b'_q) \end{matrix} \right. \right] \\
 & \quad \times I_{p,q}^{m,n} \left[z \left(\frac{x}{y} \right)^q \left| \begin{matrix} (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q) \end{matrix} \right. \right] D^{\beta-\alpha} \{f(y)\} dy, \tag{3.2}
 \end{aligned}$$

provided further $f \in \mathbb{S}$ and $x > 0$.

Proof. Theorem 3.1 is established with the help of Lemma 3.1 and the equation (1.12), on proceeding on similar lines as indicated in the proof of Theorem 2.1. \square

4 Use of Other Methods

One-dimensional Fredholm integral equation (1.2) involving the product of general class of polynomials with the product of I -functions in the kernel can also be solved by the application of Mellin transforms.

Theorem 4.1. *if $f \in \mathbb{S}$, $D^{\alpha-\beta}\{f(x)\}$ exists $q > 0, x > 0, |arg(z)| < \frac{1}{2} \pi T, T > 0$ (T is given in Lemma 3.1), $Re(\alpha) > Re(\beta) > 0$, then the solution of the integral equation (1.2) is given by*

$$\begin{aligned}
 f(x) = & \frac{q}{2\pi i} x^{\alpha-1} \lim_{\gamma \rightarrow \infty} \int_{c-i\gamma}^{c+i\gamma} x^{-s} \left\{ \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^\infty \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta \right. \\
 & \left. \times \theta \left[\frac{-s-p\zeta-\sigma K}{q} \right] z^{\frac{-(s+p\zeta+\sigma K)}{q}} \right\}^{-1} \varphi(s) ds, \tag{4.1}
 \end{aligned}$$

provided further that

$$\max_{1 \leq j \leq n} \{Re[(\alpha_j - 1)/A_j]\} < -Re \left(\frac{p\zeta + \sigma K + s}{q} \right) < \min \left\{ Re \left(\frac{\beta_j}{B_j} \right) \right\}, \tag{4.2}$$

Proof. On replacing f by $D^{\alpha-\beta}\{f\}$ in (3.2) and applying (1.2), we have

$$\begin{aligned}
 g(x) = & \int_0^\infty y^{-\beta} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^\infty \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta \left(\frac{x}{y} \right)^p \\
 & \times I_{p+1,q+1}^{m,n+1} \left[z \left(\frac{x}{y} \right)^q \left| \begin{matrix} (1-\beta-\sigma K-p\zeta, q, 1), (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q), (1-\alpha-\sigma K-p\zeta, q, 1) \end{matrix} \right. \right] D^{\alpha-\beta} \{f(y)\} dy. \tag{4.3}
 \end{aligned}$$

Multiplying both sides of (4.3) by x^{s-1} and integrating with respect to x from 0 to ∞ , we have

$$\begin{aligned} \varphi(s) &= \int_0^\infty x^{s-1} g(x) dx \\ &= \int_0^\infty y^{-\beta} D^{\alpha-\beta} \{f(y)\} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^\infty \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta \left(\frac{1}{y}\right)^{p\zeta+\sigma K} \\ &\quad \times \left(\int_0^\infty x^{s+p\zeta+\sigma K-1} I_{p+1,q+1}^{m,n+1} \left[z \left(\frac{x}{y}\right)^q \middle| \begin{matrix} (1-\beta-\sigma K-p\zeta,q,1), (\alpha_1, A_1, a_1), \dots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \dots, (\beta_q, B_q, b_q), (1-\alpha-\sigma K-p\zeta,q,1) \end{matrix} \right] dx \right) dy, \end{aligned} \tag{4.4}$$

where we have assumed the absolute (and uniform convergence of the integrals involved, with a view to justifying the inversion of the order of integration. Now, evaluate the inner integral in (4.4) by a simple change of variables in the familiar results ([17, 21]), it reduces to

$$\begin{aligned} \varphi(s) &= \frac{1}{q} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^\infty \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta z^{-\left(\frac{s+p\zeta+\sigma K}{q}\right)} \\ &\quad \times \theta \left(\frac{-s-p\zeta-\sigma K}{q} \right) \frac{\Gamma(\beta-s)}{\Gamma(\alpha-s)} \int_0^\infty y^{s-\beta} D^{\alpha-\beta} \{f(y)\} dy. \end{aligned} \tag{4.5}$$

Inverting (4.5) by applying Mellin inversion theorem ([22], p.46), we get

$$\begin{aligned} D^{\alpha-\beta} \{f(y)\} &= \frac{q}{2\pi i} \lim_{\gamma \rightarrow \infty} \int_{c-i\gamma}^{c+i\gamma} \left\{ \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^\infty \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta \right. \\ &\quad \times \left. \frac{\Gamma(\beta-s)}{\Gamma(\alpha-s)} \theta \left[\frac{-s-p\zeta-\sigma K}{q} \right] z^{-\left(\frac{s+p\zeta+\sigma K}{q}\right)} \right\}^{-1} y^{\beta-s-1} \varphi(s) ds. \end{aligned} \tag{4.6}$$

Operating upon both sides by $D^{\alpha-\beta}$, (4.6) gives us

$$\begin{aligned} f(y) &= \frac{q}{2\pi i} D^{\beta-\alpha} \left[\lim_{\gamma \rightarrow \infty} \int_{c-i\gamma}^{c+i\gamma} y^{\beta-s-1} \left\{ \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^\infty \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta \right. \right. \\ &\quad \times \left. \left. \frac{\Gamma(\beta-s)}{\Gamma(\alpha-s)} \theta \left(\frac{-s-p\zeta-\sigma K}{q} \right) z^{-\left(\frac{s+p\zeta+\sigma K}{q}\right)} \right\}^{-1} \varphi(s) ds \right] \end{aligned} \tag{4.7}$$

which finally yields

$$\begin{aligned} f(x) &= \frac{q}{2\pi i} x^{\alpha-1} \lim_{\gamma \rightarrow \infty} \int_{c-i\gamma}^{c+i\gamma} x^{-s} \left\{ \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K} \lambda^K \sum_{h=1}^M \sum_{k=0}^\infty \frac{(-1)^k \psi(\zeta)}{k! B'_h} u^\zeta \right. \\ &\quad \times \left. \left(\frac{-s-p\zeta-\sigma K}{q} \right) z^{-\left(\frac{s+p\zeta+\sigma K}{q}\right)} \right\}^{-1} \varphi(s) ds \end{aligned} \tag{4.8}$$

as the solution of the integral equation (1.2). □

5 Special Cases

- (i) If we set $p = 0$ and $n_2 = n_3 = \dots = n_{r-1} = 0 = p_2 = p_3 = \dots = p_{r-1} = q_2 = q_3 = \dots = q_{r-1}$ the results in (2.1) and (2.2) reduce to the known results obtained by Chaurasia and Patni [10].
- (ii) Taking $p = 0$ and $a_j = b_j = 1$ for all j , in (3.1), (3.2) and (1.2), the I -function reduces to H -function and the result given by Chaurasia and Patni [10] is obtained.
- (iii) On specializing the parameters in (2.1), (2.2), (3.1), (3.2) and (1.2), we get the results recently obtained by Chaurasia and Kumar [11].
- (iv) By suitably specializing the various parameters in the I -functions and in the General class of polynomials, our results can be reduces to a large number of integral equations and solutions involving a product of \overline{H} -functions, H -functions, G -functions, Hermite polynomials, Jacobi polynomials, Laguerre polynomials and their various special cases.

Acknowledgements : The authors are grateful to the worthy referee for his invaluable suggestions and comments for the improvement of the paper.

References

- [1] R.G. Buchman, An inversion integral, Proc. Amer. Math. Soc. 13 (1962) 675-677.
- [2] C. Fox, A formal solution of certain dual integral equations, Trans. Amer. Math. Soc. 119 (1965) 389-398.
- [3] T.P. Higgins, A hypergeometric function transform, J. Soc. Industr. Appl. Math. 12 (1970) 142-148.
- [4] E.R. Love, Some integral equations hypergeometric functions, Proc. Edinburgh Math. Soc. 15 (2) (1967) 169-198.
- [5] E.R. Love, A hypergeometric integral equation in Fractional Calculus and Its Applications, edited by B. Ross, Springer-Verlag, New York (1975), 272-288.
- [6] T.R. Prabhakar, N.K. Kashyap, A new class of hypergeometric integral equations, Indian J. pure Appl. Math. 11 (1980) 92-97.
- [7] H.M. Srivastava, R.G. Buchman, Convolution integral equations with special function kernels, Halsted Press, John Wiley and Sons, New York, 1977.
- [8] H.M. Srivastava, Some remarks on a generalization of the Stieltjes transform, Publ. Math., Debrecen. 23 (1976) 119-122.
- [9] H.M. Srivastava, R.K. Raina, On certain method of solving class of integral equations of Fredholm type, J. Austral. Math. Soc. Series A 52 (1992) 1-10.

- [10] V.B.L. Chaurasia, R. Patni, Integral equations of Fredholm type with special functions, *Indian J. Pure Appl. Math.* 32 (2001) 747-758.
- [11] V.B.L. Chaurasia, D. Kumar, On the solutions of integral equations of Fredholm type with special functions, *Tamsui Oxf. J. Math. Sci.* 28 (1) (2012) 49-61.
- [12] Y.N. Prasad, Multivariable I -function, *Vijnana Parishad Anusandhan Patrika*, 29 (1986) 231-235.
- [13] H.M. Srivastava, A contour integral involving Fox's H -function, *Indian J. Math.* 14 (1972) 1-6.
- [14] H.M. Srivastava, The Weyl fractional integral of a general class of polynomials, *Boll. Un. Mat. Ital. B* 2 (6) (1983) 219-228.
- [15] A.K. Rathie, A new generalization of generalized hypergeometric functions, *Le Mathematiche*. 52 (1977) 297-310.
- [16] A.A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynmann integrals: II. A generalization of the H -function, *J. Phys. A: Math. Gen.* 28 (1987) 4119-4128.
- [17] C. Fox, The G and H -functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.* 98 (1961) 395-429.
- [18] R.K. Saxena, In memorium of Charles Fox, *Fractional Calculus and Applied Analysis* 12 (3) (2009) 337-344.
- [19] M.J. Lighthill, Introduction to Fourier analysis and generalized functions, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, Cambridge, London and New York, 1958.
- [20] H. Laurent, Sur Le calcul derives a indeces quelconques, *Nouv. Ann. Math.* 3 (3) (1884) 240-252.
- [21] A.M. Mathai, R.K. Saxena, The H -function with Applications in Statistics and Disciplines, Wiley, New York, 1978.
- [22] E.C. Titchmarsh, Introduction to the theory of Fourier integrals, Second Edition, Oxford University Press, London, Clarendon Press, Oxford, 1948

(Received 6 March 2013)

(Accepted 5 August 2015)