Thai Journal of Mathematics
Volume 15 (2017) Number 2 : 475–482
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Positive Compact Operators on Quaternionic Hilbert Spaces

Massoumeh Fashandi

Faculty of Mathematical Sciences, Ferdowsi University of Mashhad
P.O. Box 1159, Mashhad, 91775 Iran
e-mail : fashandi@um.ac.ir

Abstract : In this paper, some properties of compact operators on quaternionic Hilbert spaces are studied. It is shown that the positiveness of a compact normal operator on a quaternionic Hilbert space is equivalent to positiveness of its eigenvalues. Some results analogous to the ones concerning compact operators on Hilbert spaces are proved in the quaternionic context.

Keywords : quaternionic Hilbert spaces; compact operators; positive operator.
2010 Mathematics Subject Classification : 46E22; 47B07.

1 Introduction and Auxiliary Results

In this paper, $\mathbb{H}$ will stand for the skew field of quaternions, whose elements are in the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and x_3 are real numbers and i, j, k are called imaginary units and obey the following multiplication rules:

$$i^2 = j^2 = k^2 = -1, \;\; ij = -ji = k, \;\; jk = -kj = i, \;\text{ and }\; ki = -ik = j. \tag{1.1}$$

We omit to describe the properties of quaternions and refer the readers to [1] for more pertinent details.

Let H be a linear vector space over the field of quaternions under right scalar multiplication. We suppose that a function $\langle .,. \rangle : \mathsf{H} \times \mathsf{H} \longrightarrow \mathbb{H}$ exists such that for every $u, v, w \in \mathsf{H}$ and $p, q \in \mathbb{H}$ the following properties hold:

(i) $\overline{\langle u, v\rangle} = \langle v, u\rangle$,

Copyright © 2017 by the Mathematical Association of Thailand. All rights reserved.

(ii) $\langle u, u\rangle > 0$ unless $u = 0$,

(iii) $\langle u, vp + wq\rangle = \langle u, v\rangle p + \langle u, w\rangle q$,

this function is called an inner product. The quaternionic norm of $u \in \mathsf{H}$ is defined by $\|u\| = \sqrt{\langle u, u\rangle}$ and in [1], Proposition 2.2, it has been proved that the quaternionic norm satisfies all properties of a norm, including the Cauchy-Schwartz inequality. If H with the metric $d(u, v) = \|u - v\|$ is a complete metric space, then H is said to be a right quaternionic Hilbert space. Similar to the complex Hilbert spaces, every right quaternionic Hilber space admits a Hilbert basis (see Propositions 2.5 and 2.6 of [1]). For making this paper self-contained, we bring these two propositions here.

Proposition 1.1 (Proposition 2.5 of [1]). *Let H be a right quaternionic Hilbert space and let N be a subset of H such that, for $z, z' \in \mathsf{N}$, $\langle z|z'\rangle = 0$ if $z \neq z'$ and $\langle z|z\rangle = 1$. Then conditions $(a) - (e)$ listed below are pairwise equivalent.*

(a) *For every $u, v \in \mathsf{H}$ the series $\sum_{z\in\mathsf{N}}\langle u|z\rangle\langle z|v\rangle$ converges absolutely and it holds:*

$$\langle u|v\rangle = \sum_{z\in\mathsf{N}}\langle u|z\rangle\langle z|v\rangle.$$

(b) $\|u\|^2 = \sum_{z\in\mathsf{N}} |\langle z|u\rangle|^2$ *for every* $u \in \mathsf{H}$.

(c) $\mathsf{N}^\perp := \{v \in \mathsf{H} : \langle v|z\rangle = 0, \forall z \in \mathsf{N}\} = \{0\}$.

(d) $\langle\mathsf{N}\rangle$ *is dense in* H.

The subset N in Proposition 1.1, is called a Hilbert basis.

Proposition 1.2 (Proposition 2.6 of [1]). *Every right quaternionic Hilbert space admits a Hilbert basis, and two Hilbert bases have the same cardinality. Furthermore, if N is a Hilbert basis of H, then every $u \in \mathsf{H}$ can be uniquely decomposed as follows:*

$$u = \sum_{z\in\mathsf{N}} z\langle z|u\rangle,$$

where the series $\sum_{z\in\mathsf{N}} z\langle z|u\rangle$ converges absolutely in H.

It is said that $T : \mathsf{H} \longrightarrow \mathsf{H}$ is a right linear operator if for all $u, v \in \mathsf{H}$ and $p \in \mathbb{H}$,

$$T(up + v) = (Tu)p + Tv.$$

Such an operator is called bounded if there exists $K \geq 0$ such that, for all $u \in \mathsf{H}$,

$$\|Tu\| \leq K\|u\|.$$

As in the complex case, the norm of a bounded right linear operator T is defined by

$$\|T\| = \sup\left\{\frac{\|Tu\|}{\|u\|} : 0 \neq u \in \mathsf{H}\right\}. \tag{1.2}$$

The set of all bounded right linear operators on H is denoted by $\mathfrak{B}(\mathsf{H})$, which is a complete normed space with the norm defined by (1.2) (see [1]; Proposition 2.11, for more properties of $\mathfrak{B}(\mathsf{H})$). For every $T \in \mathfrak{B}(\mathsf{H})$, there exists a unique operator $T^* \in \mathfrak{B}(\mathsf{H})$, which is called the adjoint of T, such that, for all $u, v \in \mathsf{H}$, $\langle Tu, v\rangle = \langle u, T^*v\rangle$. Many properties of the adjoint operator, are stated and proved in Theorem 2.15 and Remark 2.16 of [1], including $\|T\| = \|T^*\|$. Self adjoint, normal and positive operators are defined in the same manner of complex case (see Definition 2.12 of [1]).

Definition 1.3 (Definition 4.1 of [1])**.** Let H be a right quaternionic Hilbert space and T be a right linear operator on H. For $q \in \mathbb{H}$, the associated operator $\Delta_q(T)$ is defined by:

$$\Delta_q(T) = T^2 - T(q + \overline{q}) + I|q|^2.$$

The *spherical resolvent set* of T is the set $\rho_S(T) \subset \mathbb{H}$ consisting of all quaternions q satisfying all the following conditions:

(a) $\ker(\Delta_q(T)) = \{0\}$.

(b) $Ran(\Delta_q(T))$ is dense in H.

(c) $\Delta_q(T)^{-1} : Ran(\Delta_q(T)) \to D(T^2)$ is bounded.

The *spherical spectrum* $\sigma_S(T)$ of T is defined as the complement of $\rho_S(T)$ in $\mathbb{H}$. A partition for $\sigma_S(T)$ was introduced in [1], as follows:

(i) The *spherical point spectrum* of T:

$$\sigma_{pS} = \{q \in \mathbb{H}; \ker(\Delta_q(T)) \neq \{0\}\}.$$

(ii) The *spherical residual spectrum* of T:

$$\sigma_{rS}(T) = \{q \in \mathbb{H}; \ker(\Delta_q(T)) = \{0\}, \overline{Ran(\Delta_q(T))} \neq \mathsf{H}\}.$$

(iii) The spherical continuous spectrum of T:

$$\sigma_{cS}(T) = \{q \in \mathbb{H}; \ker(\Delta_q(T)) = \{0\}, \overline{Ran(\Delta_q(T))} = \mathsf{H}, \Delta_q(T)^{-1} \notin \mathfrak{B}(\mathsf{H})\}.$$

The *spherical spectral radius* of T, denoted by $r_S(T)$, is defined by:

$$r_S(T) = \sup\{|q| \in \mathbb{R}^+ : q \in \sigma_S(T)\}.$$

The *eigenvector* of T with *eigenvalue* q is an element $u \in \mathsf{H} - \{0\}$, for which $Tu = uq$.

The following proposition summarizes some properties of $\Delta_q(T)$ and its kernel that can be proved easily, so we omit the proof.

Proposition 1.4. *Let* H *be a right quaternionic Hilbert space,* $T \in \mathfrak{B}(\mathsf{H})$ *and* $q \in \mathbb{H}$*, then* (*i*) $\Delta_q(T) \in \mathfrak{B}(\mathsf{H})$.

(*ii*) $(\Delta_q(T))^* = \Delta_q(T^*)$, *and if* T *is self adjoint then so is* $\Delta_q(T)$.

(*iii*) $\Delta_q(T)$ *is a normal operator, whenever* T *is a normal operator. In this case,*

$$\ker \Delta_q(T) = \ker \Delta_q(T^*).$$

(*iv*) $\ker \Delta_q(T)$ *is an invariant subspace for* T*, i.e.* $T(\ker \Delta_q(T)) \subseteq \ker \Delta_q(T)$.

(*v*) $T(\ker \Delta_q(T)^\perp) \subseteq \ker \Delta_q(T)^\perp$ *for a normal operator* T.

Two quaternions p and q are said to be conjugated to each other, if $p = sqs^{-1}$, for some non-zero quaternion s. The set of all quaternions conjugated to q is called the conjugacy class of q and is denoted by $\mathbb{S}_q$. Obviously, $\overline{q} \in \mathbb{S}_q$ (see [1] for more properties of conjugacy classes).

Before bringing the next result we need to remind Proposition 4.5 of [1].

Proposition 1.5 (Proposition 4.5 of [1])**.** *Let* H *be a right quaternionic Hilbert space and* $T \in \mathfrak{B}(\mathsf{H})$*. Then* $\sigma_{ps}(T)$ *coincides with the set of all eigenvalues of* T.

Proposition 1.6. *Let* H *be a right quaternionic Hilbert space and* $T \in \mathfrak{B}(\mathsf{H})$ *be a normal operator. If* q_1 *and* q_2 *are two eigenvalues of* T*, so that* $q_1 \notin \mathbb{S}_{q_2}$*, then* $\ker \Delta_{q_1}(T) \perp \ker \Delta_{q_2}(T)$.

Proof. Let u and v be non-zero elements in $\ker \Delta_{q_1}(T)$ and $\ker \Delta_{q_2}(T)$, respectively. Note that since T is normal, by part (iii) of Proposition 1.4, we have $\ker \Delta_{q_2}(T) = \ker \Delta_{q_2}(T^*)$. Thus, considering $Tu = uq_1$ and $T^*v = v\overline{q_2}$, we have

$$q_2\langle v|u\rangle = \langle v\overline{q_2}|u\rangle = \langle T^*v|u\rangle = \langle v|Tu\rangle = \langle v|uq_1\rangle = \langle v|u\rangle q_1.$$

If $\langle u|v\rangle \neq 0$ then q_1 and q_2 are conjugated to each other, contradicting the assumption. So $\langle u|v\rangle = 0$, i.e. $\ker \Delta_{q_1}(T) \perp \ker \Delta_{q_2}(T)$. □

Remark 1.7. *According to Theorem 4.8.(b) of [1], the spherical spectrum of a self-adjoint operator* $T \in \mathfrak{B}(\mathsf{H})$ *is a subset of real numbers. Therefore, if* q_1 *and* q_2 *are two eigenvalues of a self-adjoint operator* T*, then they are not conjugated to each other and so, by Proposition 1.6,* $\ker \Delta_{q_1}(T) \perp \ker \Delta_{q_2}(T)$.

2 Positive Compact Operators

During this section, H will stand for a right quaternionic Hilbert space and each operator will be right linear. Similar to the complex case, a compact operator on H, is an operator $T : \mathsf{H} \longrightarrow \mathsf{H}$, for which $\overline{T(B)}$ is a compact set of H, where B is any bounded subset of H. The set of all compact operators on H will be denoted by $\mathfrak{B}_0(\mathsf{H})$. Simple examples of compact operators are the finite rank operators; those with finite dimensional range. One can see more properties of compact operators in [2] and of compact normal operators in [3]. Here we remind those we need later.

Theorem 2.1 (Theorem 2 of [2]). $\mathfrak{B}_0(\mathsf{H})$ *is a closed two sided ideal of* $\mathfrak{B}(\mathsf{H})$ *and is closed under adjunction.*

Theorem 2.2 (Theorem 1.2 of [3]). *For a compact normal operator T with spherical point spectrum $\sigma_{pS}(T)$, there exists a Hilbert basis $\mathsf{N} \subset \mathsf{H}$ of eigenvectors of T such that:*

$$Tx = \sum_{z \in \mathsf{N}} z\lambda_z \langle z|x\rangle, \tag{2.1}$$

where $\lambda_z \in \mathbb{H}$ is an eigenvalue relative to the eigenvector z and, if $\lambda_z \neq 0$ only a finite number of distinct other elements $z' \in \mathsf{N}$ verify $\lambda_z = \lambda_{z'}$, moreover the values λ_z are at most countable. Also, the set Λ of eigenvalues λ_z with $z \in \mathsf{N}$ has the property that for every $\varepsilon > 0$ there is a finite set $\Lambda_\varepsilon \subset \Lambda$ with $|\lambda| < \varepsilon$ if $\lambda \notin \Lambda_\varepsilon$.

Theorem 2.3 (Theorem 1.4 of [3]). *Let $T \in \mathfrak{B}(\mathsf{H})$. Assume that there exist a Hilbert basis N of H and a map $\mathsf{N} \ni z \mapsto \lambda_z \in \mathbb{H}$ satisfying the following requirements:*

- *(i)* $Tx = \sum_{z \in \mathsf{N}} z\lambda_z \langle z|x\rangle$ *for every* $x \in \mathsf{H}$,
- *(ii) for every $z \in \mathsf{N}$ such that $\lambda_z \neq 0$, only a finite number of distinct other elements $z' \in \mathsf{N}$ verify $\lambda_z = \lambda_{z'}$,*
- *(iii) the set $\Lambda = \{\lambda_z \in \mathbb{H} : z \in \mathsf{N}\}$ is countable at most,*
- *(iv) for every $\varepsilon > 0$, there is a finite set $\Lambda_\varepsilon \subset \Lambda$ with $|\lambda| < \varepsilon$ if $\lambda \notin \Lambda_\varepsilon$.*

Under these conditions T is normal and compact.

The following results regarding positiveness of compact operators, are direct consequences of Theorems 2.2 and 2.3.

Corollary 2.4. *A compact normal operator $T \in \mathfrak{B}(\mathsf{H})$ is positive if and only if all its eigenvalues are non-negative real numbers.*

Proof. If T is a positive operator, for each $x \in \mathsf{H}$ we have $\langle Tx|x\rangle \geqslant 0$. Now using equation (2.1) and considering that $\langle Tz|z\rangle = \overline{\lambda_z}$ for each $z \in \mathsf{N}$, we obtain $\lambda_z \geqslant 0$. Conversely, if all the eigenvalues of T are non-negative real numbers, by Theorem 2.2 and the notations therein, for each $x \in \mathsf{H}$ we have $\langle Tx|x\rangle = \sum_{z \in \mathsf{N}} \lambda_z |\langle z|x\rangle|^2 \geqslant 0$ which proves that T is a positive operator. □

Proposition 2.5. *If $T \in \mathfrak{B}_0(\mathsf{H})$ is a positive operator, then its square root is also a compact operator.*

Proof. By Theorem 2.2 and Corollary 2.4, there is a Hilbert basis N of H consisting of eigenvectors z of T such that the eigenvalues λ_z corresponding to each z is a non-negative real number and such that $Tx = \sum_{z \in \mathsf{N}} z\lambda_z \langle z|x\rangle$. Define

$$Sx = \sum_{z \in \mathsf{N}} z\sqrt{\lambda_z} \langle z|x\rangle. \tag{2.2}$$

According to Theorem 2.2, Theorem 2.3, and Corollary 2.4 the operator S defined by (2.2) is a positive compact operator on H and easy calculations shows that $S^2 = T$. □

Before bringing an immediate consequence of Proposition 2.5, let us recall that the absolute value of $T \in \mathfrak{B}(\mathsf{H})$ is the square root of the positive operator T^*T (see [1] for more properties of the absolute value of $T \in \mathfrak{B}(\mathsf{H})$).

Corollary 2.6. *The absolute value of a compact operator is a compact operator.*

Proof. Let T be a compact operator, by Theorem 2.1, the operators T^* and T^*T are compact. Now, Proposition 2.5 completes the proof. ☐

The following theorem is the quaternionic version of Proposition 2.3 of [4], with the same technique of the proof.

Theorem 2.7. *Let $T \in \mathfrak{B}(\mathsf{H})$ be a positive operator. If there exists a countable Hilbert basis $\mathsf{N} = \{z_n\}_{n\in\mathbb{N}}$ of H such that*

$$\sum_{n=1}^{+\infty}\langle Tz_n|z_n\rangle < +\infty, \tag{2.3}$$

then T is a compact operator.

Proof. Let S be the square root of T. The operator $S \in \mathfrak{B}(\mathsf{H})$ is positive and hence it is also self-adjoint (see Proposition 2.17(b) of [1]). Then $\sum_{n=1}^{+\infty}\|Sz_n\|^2 = \sum_{n=1}^{+\infty}\langle Tz_n|z_n\rangle$, which is finite by the assumption (2.3). Since N is a Hilbert basis for H, by Proposition 1.2, $Sz_n = \sum_{m=1}^{+\infty} z_m\langle z_m|Sz_n\rangle = \sum_{m=1}^{+\infty} z_m q_{mn}$, where $q_{mn} = \langle z_m|Sz_n\rangle$ and the series converges absolutely in H. Thus, for all $n \in \mathbb{N}$,

$$\|Sz_n\|^2 = \sum_{m=1}^{+\infty}|\langle z_m|Sz_n\rangle|^2 = \sum_{m=1}^{+\infty}|q_{mn}|^2.$$

Now, we conclude that $\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}|q_{nm}|^2 = \sum_{n=1}^{+\infty}\|Sz_n\|^2 < +\infty$. For any $N \in \mathbb{N}$, let

$$q_{mn}^N = \begin{cases} q_{mn}, & 1 \le m, n \le N, \\ 0, & \text{otherwise.} \end{cases}$$

Let S_N be the bounded operator in $\mathfrak{B}(\mathsf{H})$ given by

$$S_N z_n = \begin{cases} \sum_{m=1}^{N} z_m q_{mn}^N, & 1 \le n \le N, \\ 0, & n > N. \end{cases}$$

Obviousely, S_N has finite rank and so it is a compact operator. Now, for all $u \in \mathsf{H}$, by Proposition 1.1(b), Proposition 1.2 and the Couchy- Schwarz inequality, we obtain

$$\|(S - S_N)u\| \quad = \quad \|(S - S_N)\sum_{n=1}^{+\infty} z_n\langle z_n|u\rangle\|$$

$$\begin{aligned}
&\leq \sum_{n=1}^{+\infty} \|(S-S_N)z_n\| |\langle z_n|u\rangle| \\
&\leq \left(\sum_{n=1}^{+\infty} |\langle z_n|u\rangle|^2\right)^{1/2} \left(\sum_{n=1}^{+\infty} \|(S-S_N)z_n\|^2\right)^{1/2} \\
&= \|u\| \left(\sum_{n=1}^{+\infty} \|(S-S_N)z_n\|^2\right)^{1/2}.
\end{aligned}$$

But

$$(S-S_N)z_n = \begin{cases} \sum_{m=N+1}^{+\infty} z_m q_{mn}, & 1 \leq n \leq N, \\ \sum_{m=1}^{+\infty} z_m q_{mn}, & n > N. \end{cases} \tag{2.4}$$

Therefore, from (2.4), we have

$$\|(S-S_N)z_n\|^2 = \begin{cases} \sum_{m=N+1}^{+\infty} |q_{mn}|^2, & 1 \leq n \leq N, \\ \sum_{m=1}^{+\infty} |q_{mn}|^2, & n > N. \end{cases}$$

So

$$\begin{aligned}
\sum_{n=1}^{+\infty} \|(S-S_N)z_n\|^2 &= \sum_{n=1}^{N} \sum_{m=N+1}^{+\infty} |q_{mn}|^2 + \sum_{n=N+1}^{+\infty} \sum_{m=1}^{+\infty} |q_{mn}|^2 \\
&\leq \sum_{m=N+1}^{+\infty} \sum_{n=1}^{\infty} |q_{mn}|^2 + \sum_{n=N+1}^{+\infty} \sum_{m=1}^{+\infty} |q_{mn}|^2 \to 0,
\end{aligned}$$

as $N \to +\infty$. Thus, $S_N \to S$ in the norm of $\mathfrak{B}(\mathsf{H})$. Finally, since $\mathfrak{B}_0(\mathsf{H})$ is a closed two-sided ideal of $\mathfrak{B}(\mathsf{H})$, as stated in Theorem 2.1, the operators S and $T = S^2$ are compact. $\square$

Acknowledgements : The author would like to thank the referee for careful reading the manuscipt and useful comments. Also, the author would like to thank Professor John R. Klauder from University of Florida, for his encouragement and comments on the first version of this manuscript.

References

[1] R. Ghiloni, V. Moretti, A. Perotti, Continuous slice functional calculus in quaternionic Hilbert spaces, Rev. Math. Phys. 25 (2013) 1350006.

[2] M. Fashandi, Compact operators on quaternionic Hilbert spaces, Facta Univ. Ser. Math. Inform. 28 (3) (2013) 249-256.

[3] R. Ghiloni, V. Moretti, A. Perotti, Spectral properties of compact normal quaternionic operators, In: S. Bernstein, U. Kähler, I. Sabadini, F. Sommen, editors, Hypercomplex Analysis: New Perspectives and Applications, Trends in Mathematics, Basel, Birkhauser (2014), 133-143.

[4] M.W. Wong, Wavelet Transforms and Localization Operators, Basel, Birkhauser, 2002.

(Received 9 June 2015)
(Accepted 9 May 2016)